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ENGINEERING MECHANICS

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PREFACE

The importance of mechanics in the preparation of young engineers for work in specialized fields cannot be overemphasized. The demand from industry is more and more for young men who are soundly grounded in their fundamental subjects rather than for those with specialized training. There is good reason for this trend: The industrial engineer is continually being confronted by new problems, which do not always yield to routine methods of solution. The man who can successfully cope with such problems must have a sound understanding of the fundamental principles that apply and be familiar with various general methods of attack rather than proficient in the use of any one. It seems evident, then, that university training in such a fundamental subject as mechanics must seek to build a strong foundation, to acquaint the student with as many general methods of attack as possible, to illustrate the application of these methods to practical engineering problems, but to avoid routine drill in the manipulation of standardized methods of solution. Such are the aims of this book.

The content of the book is somewhat wider than can be covered in two courses of three semester hours or five quarter hours each. At the end of the discussion of statics, for example, there is a chapter on the principle of virtual work. The use of this principle results in great simplification in the solution of certain problems of statics, and it seems desirable to acquaint the student with its possibilities. At the end of the discussion of dynamics, there is a short chapter on relative motion, together with applications to engineering problems. These chapters can easily be omitted without introducing any discontinuity if there is insufficient time for them. Where time will not permit their consideration, they at least serve the purpose of indicating to the student that he has not exhausted the possibilities of the subject in his first encounter with it. Also, it is hoped that such material will be of value to those students especially interested in mechanics.

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In many of our engineering schools, statics is given during the second semester of the sophomore year, before the student has studied integral calculus. For this reason Part One of this volume has been so written that, except for one or two sections that can easily be omitted, no knowledge of mathematics beyond the differential calculus is required. However, a free use of mathematics is made within these limits. Statics is probably the first course wherein the student has a chance to make practical use of his training in mathematics, and it seems important that he be not only given the opportunity but encouraged to use it to the full extent of its applicability.

The situation is usually quite different with dynamics. In some schools, for instance, this course does not immediately follow statics but is taken after strength of materials. Thus the students are more mature, and it seems justifiable in Part Two to make free use of the calculus and even some use of elementary differential equations. In this latter respect, however, the solutions are discussed in sufficient detail so that the student without special preparation in differential equations need have no difficulty.

Throughout Part Two the equations of motion are presented and handled as differential equations. Dynamics is not a subject to be handled superficially, and a too-arduous attempt to simplify its presentation can easily result in the fostering of false notions in the mind of the beginner. Besides helping to forestall such possible misconceptions, the use of the differential equation of motion, as such, possesses several other advantages: (1) It makes it possible, at the outset, to place proper emphasis upon the inherent difference between dynamical problems involving known motion and those involving known acting forces. (2) It makes practicable the discussion of certain problems of dynamics (such as vibration problems) which otherwise could be handled only in a very cumbersome manner, if at all. (3) It gives the student a foundation in dynamics upon which he can successfully build if he desires to pursue advanced study or to read current literature on the subject.

Since the student usually has his greatest difficulty in applying the principles and theorems that he has just learned to specific situations, special attention has been given to the selection and treatment of a series of illustrative examples at the end of each article. The purpose of these examples is twofold: (1) They are sometimes used as a medium of presentation of material not included in the text proper. (2) They are designed to set an example to the student in logical methods of approach to the solution of engineering problems. It is hoped that the examples will help the student to bridge the gap between mere cognizance of the general principles and the ability to apply them to concrete problems. Mastery in this respect is the true goal of engineering education. The examples warrant as much attention from the student as the text material proper.

The solution of a problem in mechanics usually consists of three steps: (1) the reduction of a complex physical problem to such a state of idealization that it can be expressed algebraically or geometrically; (2) the solution of this purely mathematical problem; and (3) the interpretation of the results of the solution in terms of the given physical problem. It is too often the case that the student's attention is called only to the second step so that he does not see clearly the connection between this and the true physical problem. By successive development of these three steps in the solution of each illustrative example, it is hoped to lead the student to a realization of the full significance of mechanics, and also to encourage him to approach the solution of his own problems in a similar way.

Many of the illustrative examples are worked out in algebraic form, the answers being given simply as formulas. When numerical data are given, their substitution is made only in the final answer at the end. Such a procedure possesses several advantages, one of which is the training the student gets in reliable methods of checking answers. Two of the most valuable aids in checking the solution of a problem are the "dimensional check" and the consideration of certain limiting cases as logical extremes. The opportunity of making either of these checks is lost when given numerical data are substituted at the beginning of the solution. Another advantage of the algebraic solution is that it greatly enriches the possibilities of the third step in the solution of the problem, namely, significance of results. Finally, the algebraic solution is preferable if proper attention is to be given to numerical calculations, for only by having the result in algebraic form can it be seen with what number of figures any intermediate calculation must be made in order to obtain a desired degree of accuracy in the final result.

Since the first edition of "Engineering Mechanics" appeared in 1937, the authors' "Theory of Structures" and "Advanced Dynamics" have been published, and these later volumes now contain some of the more advanced material that was originally in "Engineering Mechanics." It is hoped that the three volumes taken together represent a fairly complete treatment of engineering mechanics and its applications to problems of modern structures and machines, at the same time leaving the present volume better suited to the undergraduate courses in statics and dynamics as given in our engineering schools today.

In the preparation of this fourth edition, the entire book has been thoroughly revised. In doing this, the authors have had these objectives: (1) simplification of the text proper, (2) improved arrangement of subject matter, and (3) deemphasis of the algebraic treatment of problems. Almost all problems throughout the book are now given with numerical data and numerical answers. Furthermore, the problem sets have been completely revised, and they contain a high percentage of new problems. The problems preceded by an asterisk present special difficulties of solution.

Various textbooks have been used in the preparation of this book, particularly in the selection of problems. In this respect, special acknowledgment is due the book "Collection of Problems of Mechanics," edited by J. V. Mestscherski (St. Petersburg, 1913), in the preparation of which the senior author took part. The authors also take this opportunity to thank their colleagues at Stanford University for many helpful suggestions in regard to this revision, in particular, Prof. Karl Klotter, who read some portions of the revision and made many valuable suggestions for improvement in this edition.

S. Timoshenko
D. H. Young

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Part One

STATICS

1

CONCURRENT FORCES IN A PLANE

1.1. Principles of statics. Statics deals with the conditions of equilibrium of bodies acted upon by forces and is one of the oldest branches of science. Some of its fundamental principles date back to the Egyptians and Babylonians who used them in the solution of problems encountered in building the famous pyramids and old temples. The earliest writings on the subject were left by Archimedes (287–212 B.C.), who formulated the laws of equilibrium of forces acting on a lever and also some principles of hydrostatics. However, the principles from which the subject in its present form is developed were not fully stated until the latter part of the seventeenth century and are mainly the work of Stevinus, Varignon, and Newton, who were the first to use the principle of the parallelogram of forces.

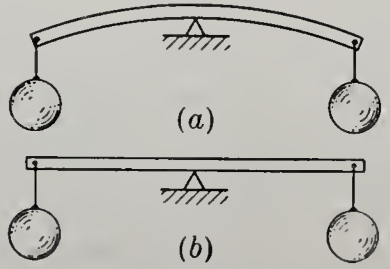

FIG. 1

Rigid Body. We shall be mostly concerned in this book with problems involving the equilibrium of *rigid bodies.* Physical bodies, such as we have to deal with in the design of engineering structures and machine parts, are never absolutely rigid but deform slightly under the action of loads which they have to carry. Consider, for example, the lever shown in Fig. 1*a*. Under the action of the two equal weights at the ends, the bar bends slightly over the fulcrum and the distance of each weight from the fulcrum is decreased by a very small amount. In discussing the equilibrium of the lever (equal weights at equal distances from the fulcrum are in equilibrium), we may safely ignore this deformation and assume that the lever is a

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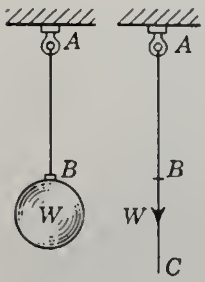
[1] For historical data regarding the development of statics, see Ernst Mach, "Science of Mechanics," Open Court Publishing Company, Chicago, 1902.

rigid body which remains straight, as shown in Fig. 1*b*. That is, we assume that the distance of each weight from the fulcrum is half the length of the bar. This illustrates the significance of the assumption of rigid bodies in dealing with static equilibrium.

If we are interested in the *strength* of the lever in Fig. 1, or the amount of sag, the deformation represented by the bent form becomes important and must be taken into account. Problems in which the effect of small deformations of physical bodies must be taken into account are generally treated in books on *strength of materials* or *theory of elasticity* and for the most part will not be considered further here.

Problems dealing with the conditions of equilibrium of nonrigid bodies, such as liquids and gases, are usually treated in books on *fluid mechanics*. These will also not be considered here, except in so far as they may be involved in determining the pressure or loading exerted on rigid bodies that we do have under consideration.

Force. For the investigation of problems of statics we must introduce the concept of *force*, which may be defined as any action that tends to change the state of rest of a body to which it is applied. There are many kinds of force, such as gravity force with which we are all familiar and the simple push or pull that we can exert upon a body with our hands. Other examples of force are the gravitational attraction between the sun and planets, the tractive effort of a locomotive, the force of magnetic attraction, steam or gas pressure in a cylinder, wind pressure, atmospheric pressure and frictional resistance between contiguous surfaces.



(*a*) (*b*)

FIG. 2

The pull of gravity is one of the most common examples of force with which we shall have to deal. Given a ball that hangs by a string (Fig. 2*a*), we say that the ball pulls on the string with a force W equal to its weight. This force is applied to the string at point B and acts vertically downward.

From the above example, we see that for the complete definition of a force we must know (1) its *magnitude*, (2) its *point of application*, and (3) its *direction*. These three quantities which completely define the force are called its *specifications*.

The magnitude of a force is obtained by comparing it with a certain standard, arbitrarily taken as a unit force. In engineering the unit of force is usually taken as the *pound*, which represents the weight of a certain platinum cylinder kept in the Tower of London and called the *imperial standard pound.* The magnitudes of forces are commonly measured by using various kinds of dynamometers. The essential portion of such an instrument is an elastic spring, which can be calibrated by hanging various known weights on it and marking the corresponding elongations. Having once been calibrated, the dynamometer can then be used for measuring other forces.

The point of application of a force acting upon a body is that point in the body at which the force can be assumed to be concentrated. Physically it will be impossible to concentrate a force at a single point; i.e., every force must have some finite area or volume over which its action is distributed. For example, the force W exerted by the ball upon the string AB in Fig. 2 is in reality distributed over the small cross-sectional area of the string. Likewise, the gravity force which the earth exerts on the ball is distributed throughout the volume of the ball. However, we often find it convenient to think of such distributed force as being concentrated at a single point of application wherever this can be done without sensibly changing the effect of the force on the conditions of equilibrium. In the case of gravity force distributed throughout the volume of a body, the point of application at which the total weight can be assumed to be concentrated is called the *center of gravity* of the body.

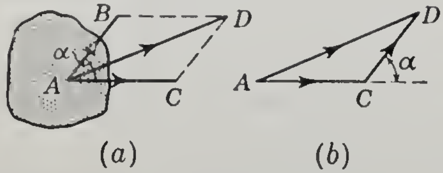
The direction of a force is the direction, along a straight line through its point of application, in which the force tends to move a body to which it is applied. This line is called the *line of action* of the force. The force of gravity, for example, is always directed vertically downward. Again, in the case of a force exerted upon a body by a flexible string, the string defines the line of action of the force. Thus the string AB in Fig. 2 pulls vertically downward on the hook at A.

Any quantity, such as force, that possesses direction as well as magnitude is called a *vector quantity* and can be represented graphically by a segment of a straight line, called a *vector.* For example, in Fig. 2*b*, we can represent the force that the ball exerts on the string by the straight-line segment BC, the length of which, to some convenient scale, shows the magnitude of the force and the vertical downward direction of which, indicated by the arrow, shows the direction of the force. Point B is called the *beginning* of the vector; and point C, the *end.* Either the beginning or the end of a vector may be used to indicate the point of application of the force. With the beginning and the end of a vector indicated by letters (as B and C in Fig. 2*b*) we shall designate the vector by the symbol $\overline{BC}$, which defines it specifically as acting from B toward C.

Parallelogram of Forces. When several forces of various magnitudes and directions act upon a body, they are said to constitute a *system of forces.* The general problem of statics consists of finding the conditions that such a system must satisfy in order to have equilibrium of the body. The various methods of solution of this problem are based on several axioms, called the *principles of statics.* We begin with the principle of the *parallelogram of forces,* first employed indirectly by Stevinus in 1586 and finally formulated by Varignon and Newton in 1687.

PARALLELOGRAM LAW. *If two forces, represented by vectors* $\overline{AB}$ *and* $\overline{AC}$, *acting under an angle* α *(Fig. 3a) are applied to a body at point* A, *their action is equivalent to the action of one force, represented by the vector* $\overline{AD}$, *obtained as the diagonal of the parallelogram constructed on the vectors* $\overline{AB}$ *and* $\overline{AC}$ *and directed as shown in the figure.*

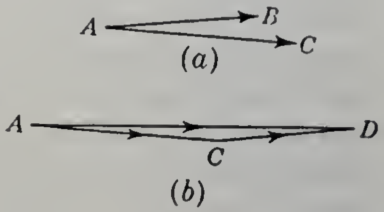
The force $\overline{AD}$ is called the *resultant* of the two forces $\overline{AB}$ and $\overline{AC}$. The forces $\overline{AB}$ and $\overline{AC}$ are called *components* of the force $\overline{AD}$. Thus a force is equivalent to its components, and vice versa.



(*a*) (*b*)

FIG. 3

Instead of constructing the parallelogram of forces, the resultant can be obtained also by constructing the triangle ACD, as shown in Fig. 3*b*. Here we take the vector $\overline{AC}$ and from its end C draw the vector $\overline{CD}$, equal and parallel to the vector $\overline{AB}$. Then the third side $\overline{AD}$ of the triangle gives the resultant, being directed from A, the beginning of the vector $\overline{AC}$, to D, the end of the vector $\overline{CD}$. The vector $\overline{AD}$, when obtained in this way, is called the *geometric sum* of the vectors $\overline{AC}$ and $\overline{CD}$. Thus, the magnitude and direction of the resultant of two forces, applied to a body at point A, may be obtained as the geometric sum of the two vectors representing these forces. Its point of application, of course, is also point A. Since the vectors in Fig. 3*b* do not show the points of application of the forces that they represent, they are called *free vectors.* The triangle ACD is called a *triangle of forces.*



(*b*)

FIG. 4

If two forces $\overline{AB}$ and $\overline{AC}$ act under a very small angle (Fig. 4*a*), the triangle of forces (Fig. 4*b*) becomes very narrow and we conclude that, in the limiting case, where the two forces act along the same line and in the same direction, their resultant is equal to the sum of the forces and acts in the same direction. In the same manner it can be shown that, if two forces act along the same line in opposite directions, their resultant is equal to the difference between the forces and acts in the direction of the larger force. By taking one direction as positive and the other as negative along the common line of action of two forces and considering the forces themselves as positive or negative accordingly, we conclude that the resultant of two *collinear forces* is equal to their *algebraic sum.*

Equilibrium of Collinear Forces. From the principle of the parallelogram of forces, it follows that two forces applied at one point can always be replaced by their resultant which is equivalent to them. Thus, we conclude that two concurrent forces can be in equilibrium only if their resultant is zero. From the discussion of the previous paragraph it follows that this will be the case if we have two forces of equal magnitude acting in opposite directions along the same line. We shall now generalize this conclusion as the second principle of statics.

EQUILIBRIUM LAW. *Two forces can be in equilibrium only if they are equal in magnitude, opposite in direction, and collinear in action.*

In engineering problems of statics we often have to deal with the equilibrium of a body in the form of a prismatic bar on the ends of which two forces are acting, as shown in Fig. 5. Neglecting their own weights, it follows from the principle just stated that either bar can be in equilibrium only when the forces are equal in magnitude, opposite in direction, and collinear in action, which means that they must act along the line joining their points of application. If these points of application can be assumed to be on the central axis of the bar (as is justifiable in many practical cases), the forces must act along this axis. When such central forces are directed as shown in Fig. 5*a*, we say that the bar is in *tension.* When they act as shown in Fig. 5*b*, the bar is said to be in *compression.*

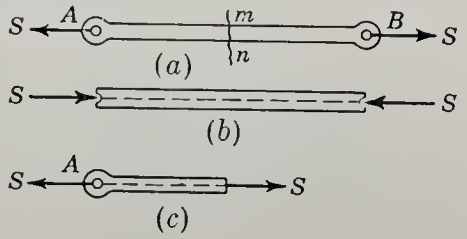


FIG. 5

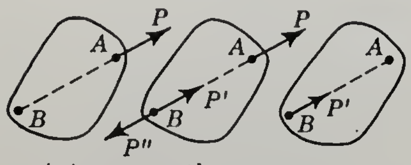
Considering the equilibrium of a portion of the bar AB in Fig. 5*a*, to the left of a section mn, we conclude that to balance the external force S at A the portion to the right must exert on the portion to the left an equal, opposite, and collinear force S, as shown in Fig. 5*c*. The magnitude of this internal axial force which one part of a bar in tension exerts on another part is called the *tensile force in the bar* or simply the *force in the bar,* since in general it may be either a tensile force or a compressive force. Such internal force is actually distributed over the cross-sectional area of the bar, and its *intensity,* i.e., the force per unit of cross-sectional area is called the *stress* in the bar.

We return now to the case of two forces under an angle α (Fig. 3*a*). From the equilibrium law above we conclude that we can hold these two forces in equilibrium by applying, at point A, a force equal and opposite to their resultant. This force is called the *equilibrant* of the two given forces.

Superposition and Transmissibility. When two forces are in equilibrium (equal, opposite, and collinear), their resultant is zero and their combined action on a rigid body is equivalent to that of no force at all. A generalization of this observation gives us the third principle of statics, sometimes called the *law of superposition.*

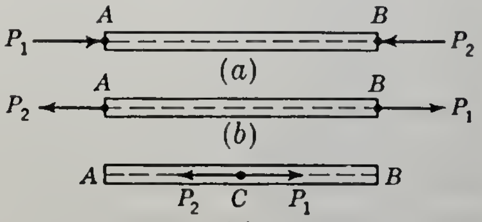
LAW OF SUPERPOSITION. *The action of a given system of forces on a rigid body will in no way be changed if we add to or subtract from them another system of forces in equilibrium.*

Let us consider now a rigid body AB under the action of a force P applied at A and acting along BA as shown in Fig. 6*a*. From the principle of superposition stated above, we conclude that the application at point B of two oppositely directed forces, each equal to and collinear with P, will in no way alter the action of the given force P. That is, the action on the body of the three forces in Fig. 6*b* is identical with the action of the single force P in Fig. 6*a*. Repeating the same reasoning again, we remove, from the system in Fig. 6*b*, the equal, opposite, and collinear forces P and P'' as a system in equilibrium. Thus we obtain the condition shown in Fig. 6*c* where, instead of the original force P applied at A, we have the equal force P' applied at B. This proves that the point of application of a force may be transmitted along its line of action without changing the effect of the force on any rigid body to which it may be applied. This statement is called the *theorem of transmissibility of a force.*



(*a*) (*b*) (*c*)

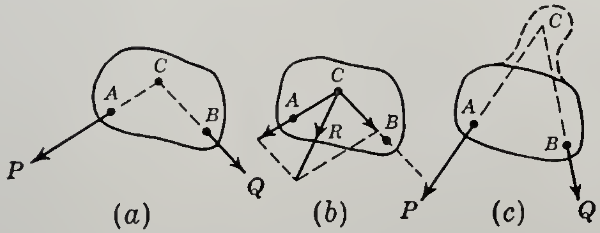
FIG. 6



(*c*)

FIG. 7

By way of an example, let us consider the prismatic bar AB (Fig. 7*a*) which is acted upon by two equal and opposite forces P_1 and P_2, applied at the ends and acting along its axis. As discussed before, the bar is in equilibrium under the action of two such forces and is subjected to compression. Now in accordance with the theorem of transmissibility of a force, we transmit P_1 along AB until its point of application is at B and similarly we transmit P_2 along BA to act at A. The condition of the bar now is represented in Fig. 7*b*, and we see that, while it is still in equilibrium under the action of these forces, the state of compression has been changed to one of tension. Again, imagine that we transmit the point of application of each force to the middle point C of the bar (Fig. 7*c*). The two forces are again in equilibrium, but the bar is now subjected to no internal forces. From this example, we see that, while the transmission of the point of application of a force acting on a body does not change the condition of equilibrium, it may produce a decided change in the internal forces to which the body is subjected. Thus the use of the theorem of transmissibility of a force is limited to those problems of statics in which we are interested only in the conditions of equilibrium of a rigid body and not with the internal forces to which it is subjected.



(*a*) (*b*) (*c*)

FIG. 8

From the theorem of transmissibility of a force it follows that, if two forces P and Q applied to a body at the points A and B (Fig. 8*a*) are acting along lines intersecting at point C, we can transmit the points of application of the forces to point C and replace them by their resultant (Fig. 8*b*). If the intersection point C is outside the boundary of the body (Fig. 8*c*), we assume this point to be rigidly attached to the body by the imaginary extension, indicated in the figure by dotted lines, and then proceed as before.

Action and Reaction. Very often we have to investigate the conditions of equilibrium of bodies that are not entirely free to move. Restriction to the free motion of a body in any direction is called *constraint.* In Fig. 9*a*, for example, we have a ball resting on a horizontal plane such that it is free to move along the plane but cannot move vertically downward. Similarly, the ball in Fig. 2*a* (page 4), although it can swing as a pendulum, is constrained against moving vertically downward by the string AB. In Fig. 10*a*, we have a ball of weight W supported by a string BC and resting against a smooth vertical wall at A. With such constraints, all motion of the ball in the plane of the figure is prevented.[1] There are many other kinds of constraint than those illustrated in Figs. 9*a* and 10*a* but these are typical and will suffice as a basis for our present discussion.

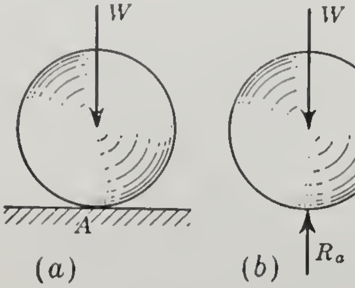


FIG. 9

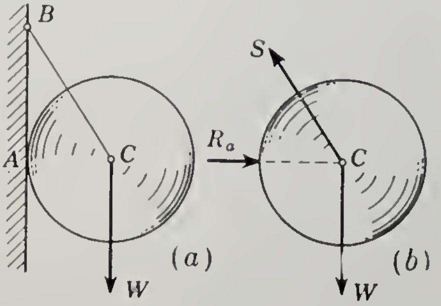


FIG. 10

A body that is not entirely free to move and is acted upon by some applied force (or forces) will, in general, exert *pressures* against its supports. For example, the ball in Fig. 2*a* (page 4) exerts a downward pull on the end of the supporting string as shown in Fig. 2*b*. Similarly, the ball in Fig. 9*a* exerts a vertical push against the surface of the supporting plane at the point of contact A. For the case in Fig. 10*a*, the ball not only pulls downward on the string BC but also pushes to the left against the wall at A. Now in every case, these actions of a constrained body against its supports induce reactions from the supports on the body, and as the fourth principle of statics we take the following statement:

LAW OF ACTION AND REACTION. *Any pressure on a support causes an equal and opposite pressure from the support so that action and reaction are two equal and opposite forces.* This last principle of statics is of course nothing more than Newton's third law of motion stated in a form suitable for the discussion of problems of statics.

Free-body Diagrams. To investigate the equilibrium of a constrained body, we shall always imagine that we remove the supports and replace them by the *reactions* which they exert on the body. Thus, in the case of the ball in Fig. 9*a*, we remove the supporting surface and replace it by the reaction R_a that it exerts on the ball. We know that the point of application of this force must be the point of contact A, and from the law of equilibrium of two forces, we conclude that it must be vertical and equal to the weight W; thus it is completely determined. The sketch in Fig. 9*b* in which the ball is completely isolated from its support and in which all forces acting on it are shown by vectors is called a *free-body diagram.*

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[1] Since there is no tendency for the ball in this case to move upward or to swing away from the wall, we ignore the fact that the constraints as shown may not be able to prevent such motion.

In the case of the ball in Fig. 10*a*, we again remove the supports and isolate the ball as a free body (Fig. 10*b*). Then besides the weight W acting at C, we have two reactive forces to apply, one replacing the string BC and another replacing the wall AB. Since the string is attached to the ball at C and since a string can pull only along its length, we have the reactive force S applied at C and parallel to BC. Its magnitude remains unknown. Regarding the reaction R_a, we have for its point of application the point of contact A. Furthermore, we assume that the surface of the wall is perfectly smooth so that it can withstand only a normal pressure from the ball. Then, accordingly, the reaction R_a will be horizontal and its line of action will pass through C as shown. Again only the magnitude remains unknown and the free-body diagram is completed. The question of finding the magnitudes of S and R_a will not be discussed here, although it is only necessary to so proportion these vectors that their resultant is equal and opposite to the vertical gravity force W.

Proceeding as above with constrained bodies, we shall always obtain two kinds of forces acting on the body: the given forces, usually called *active forces*, such as the gravity force W in Fig. 10*b*, and *reactive forces*, replacing the supports, such as the forces S and R_a in Fig. 10*b*. To have equilibrium of the body, it is necessary that the active forces and reactive forces together represent a system of forces in equilibrium. Thus it is by means of the free-body diagram that we define the system of forces with which we must deal in our investigation of the conditions of equilibrium of any constrained body. The construction of this diagram should be the first step in the analysis of every problem of statics, and it must be evident that any errors or omissions here will reflect themselves on all subsequent work.

The essential problem of statics may now be briefly recapitulated as follows: We have a body either partially or completely constrained which remains at rest under the action of applied forces. We isolate the body from its supports and show all forces acting on it by vectors, both active and reactive. We then consider what conditions this system of forces must satisfy in order to be in equilibrium, i.e., in order that they will have no resultant.

1.2. Composition and resolution of forces. *Composition.* The reduction of a given system of forces to the simplest system that will be its equivalent is called the problem of *composition of forces.* If several forces F_1, F_2, F_3, . . . , applied to a body at one point, all act in the same plane, they represent a system of forces that can be reduced to a single resultant force. It then becomes possible to find this resultant by successive applications of the parallelogram law. Let us consider, for example, the four forces F_1, F_2, F_3, and F_4 acting on a body at point A (Fig. 11a). To find their resultant,

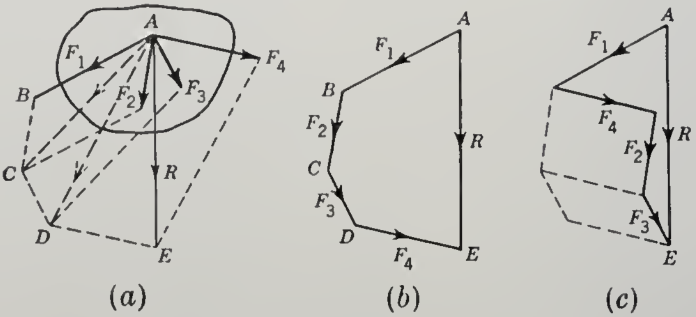


FIG. 11

we begin by obtaining the resultant $\overline{AC}$ of the two forces F_1 and F_2. Combining this resultant with the force F_3, we obtain the resultant $\overline{AD}$ which must be equivalent to F_1, F_2, and F_3. Finally, combining the forces $\overline{AD}$ and F_4, we obtain the resultant R of the given system F_1, . . . , F_4. This procedure may be carried on for any number of given forces acting at one point in a plane.

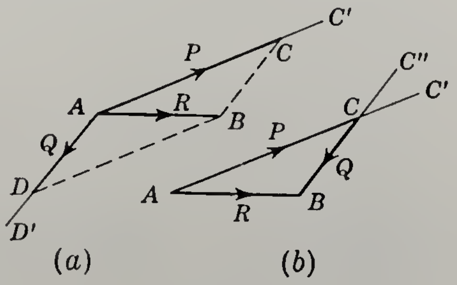
It is evident, in the above case, that exactly the same resultant R will be obtained by successive geometric addition of the free vectors representing the given forces (Fig. 11b). In this case we begin with the vector $\overline{AB}$ representing the force F_1. From the end B of this vector we construct the vector $\overline{BC}$, representing the force F_2, and afterward, the vectors $\overline{CD}$ and $\overline{DE}$, representing the forces F_3 and F_4. The polygon $ABCDE$ obtained in this way is the same as the polygon $ABCDE$ in Fig. 11a, and the vector $\overline{AE}$, from the beginning A of the vector $\overline{AB}$ to the end E of the vector $\overline{DE}$, gives the resultant R which, of course, must be applied at point A in Fig. 11a. The polygon $ABCDE$ in Fig. 11b is called the *polygon of forces* and the resultant is given by the *closing side* of this polygon. It is always directed from the beginning of the first vector to the end of the last vector. Thus, we may say that the resultant of any system of concurrent forces in a plane is obtained as the geometric sum of the given forces. The construction of the polygon of forces, for determining the resultant, is much more direct for a large number of forces than successive applications of the parallelogram law and is preferable in the solution of problems.

It is evident that the resultant R will not depend upon the order in which the free vectors representing the given forces are geometrically added. For instance, in the above example, we can begin with the force F_1, add to it the force F_4 and afterward the forces F_2 and F_3. Proceeding in this way the polygon of forces shown in Fig. 11c will be obtained. The closing side $\overline{AE}$ of the polygon gives the same resultant R as before.

In the particular case where the given forces are all acting along one line, the sides of the polygon of forces will all lie along one line and the geometric summation will be replaced by an algebraic summation. The resultant, in this case, is the algebraic sum of its components.

If the end of the last vector coincides with the beginning of the first, the resultant R is equal to zero and the given system of forces is in equilibrium.

Resolution of a Force. The replacement of a single force by several components which will be equivalent in action to the given force is called the problem of *resolution of a force.* The case in which a single force is to be replaced by two components is the one most commonly encountered. By using the parallelogram law, we can resolve a given force R into any two components P and Q intersecting at a point on its line of action. We shall dicuss two possible cases:



(a) (b)

FIG. 12

1. The directions of both components are given; their magnitudes, to be determined. Imagine, for example, that the force R, represented by the vector $\overline{AB}$ (Fig. 12a), is to be resolved into two components acting along the lines AC' and AD'. We proceed by drawing from point B the dotted lines BC and BD, parallel to the given lines of action of the desired components. The points C and D, where these lines intersect the given lines of action of the components, determine the vectors $\overline{AC}$ and $\overline{AD}$ which completely define the two components P and Q.

We can obtain the same result by using the triangle of force ABC as shown in Fig. 12b. Here the lines AC' and BC'', parallel to the given lines of action of the components, are extended from the beginning A and the end B of the vector $\overline{AB}$ representing the given force R and their point of intersection C determines the vectors $\overline{AC}$ and $\overline{CB}$, representing the components P and Q. These components, applied at any point on the line of action of the force R, will be its equivalent. In the particular case where the two components act at right angles to each other, they are called *rectangular components.*

2. Both the direction and magnitude of one component are given; the direction and magnitude of the other, to be determined. For example, imagine that the force R, represented by the vector $\overline{AB}$, and the component P, represented by the vector $\overline{AC}$ (Fig. 12*a*), have been given. Laying out these two vectors as shown in Fig. 12*b*, the magnitude and direction of the other component Q are given by the vector $\overline{CB}$, obtained by joining the ends C and B of the two given vectors.

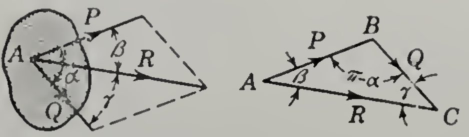
Regarding the resolution of a given force into three coplanar components, acting in three given directions, we see that the magnitude of one of the components can be arbitrarily chosen so that in this case the problem is entirely indeterminate. In the general case of resolution of a force into any number of coplanar components intersecting at one point on its line of action, the problem will be indeterminate unless all but two of the components are completely specified as to both magnitude and direction.

EXAMPLES

1. If two given forces P and Q, acting under the angle α, are applied to a body at A, find formulas for calculating the magnitude of their resultant R and the angles β and γ which its line of action makes with those of the given forces.

Solution. Figure 13*a* shows the parallelogram of forces constructed in the usual manner, while Fig. 13*b* shows the triangle of forces obtained by the geometric addition of their free vectors. From the triangle of forces we find

$$R = \sqrt{P^2 + Q^2 + 2PQ \cos \alpha} \qquad \text{(a)}$$



(*a*) (*b*)

FIG. 13

The magnitude of the resultant R being known from Eq. (a), we may determine the angles β and γ by using the equations

$$\sin \beta = \frac{Q}{R} \sin \alpha \qquad \sin \gamma = \frac{P}{R} \sin \alpha \qquad \text{(b)}$$

It is sometimes convenient to use these formulas for determining the resultant instead of making an accurate construction, to scale, of the triangle of forces.

2. Two very nearly parallel forces P and Q are applied to a rigid body at points A and B, as shown in Fig. 14. Find their resultant R graphically.

Solution. Since the point of intersection of the given forces P and Q is not well defined and does not occur within the limits of the drawing, we begin by adding to the system two equal, opposite, and collinear forces S_1 and S_2 of any convenient magnitudes at points A and B, as shown in the figure. It follows from the law of superposition that two such forces, being in equilibrium, do not change the action of the given forces P and Q. Hence the resultant R_1 of P and S_1 together with the resultant R_2 of Q and S_2, obtained as shown, are statically equivalent to the given forces P and Q and their resultant R will be the one required. To find this resultant R, we transmit R_1 and R_2 along their lines of action to point C, which is a well-defined point, and complete the parallelogram of forces as shown. The vector $\overline{CD}$ represents the required resultant and if all constructions have been made to scale, its magnitude may be measured directly from the drawing.

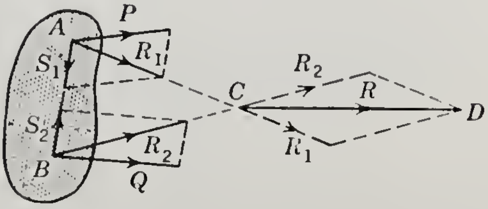


FIG. 14

PROBLEM SET 1.2

1. A man of weight $W = 160$ lb holds one end of a rope that passes over a pulley vertically above his head and to the other end of which is attached a weight $Q = 120$ lb. Find the force with which the man's feet press against the floor. *Ans.* 40 lb.

2. A boat is moved uniformly along a canal by two horses pulling with forces $P = 200$ lb and $Q = 240$ lb acting under an angle $\alpha = 60°$ (Fig. A). Determine the magnitude of the resultant pull on the boat and the angles β and γ as shown in the figure. *Ans.* $R = 382$ lb; $\beta = 33°$; $\gamma = 27°$.

3. What force Q combined with a vertical pull $P = 6$ lb will give a horizontal resultant $R = 8$ lb? *Ans.* 10 lb inclined by 36°52′.

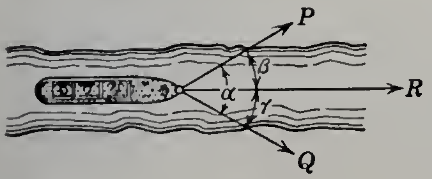


FIG. A

4. To move a boat uniformly along a canal at a given speed requires a resultant force $R = 400$ lb. This is accomplished by two horses pulling with forces P and Q on tow ropes, as shown in Fig. A. If the angles that the tow ropes make with the axis of the canal are $\beta = 35°$ and $\gamma = 25°$, what are the corresponding tensions in the ropes? *Ans.* $P = 195$ lb; $Q = 265$ lb.

5. If, in Fig. A, the horses pull with the forces $P = 240$ lb and $Q = 200$ lb, what must be the angles β and γ to give the resultant $R = 400$ lb? *Ans.* $\beta = 22°22'$; $\gamma = 27°12'$.

6. In level flight, the chord AB of an airplane wing makes an angle $\alpha = 5°$ with the horizontal (Fig. B). The resultant wind pressure on the wing for such conditions is defined by its lift and drag components $L = 1{,}500$ lb and $D = 200$ lb, which are vertical and horizontal, respectively, as shown. Resolve this force into rectangular components X and Y, coinciding with the chord AB and its normal, respectively. *Ans.* $X = 68.5$ lb; $Y = 1511.7$ lb.

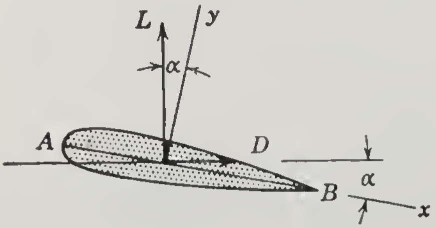


FIG. B

7. A small block of weight $Q = 10$ lb is placed on an inclined plane which makes an angle $\alpha = 30°$ with the horizontal. Resolve the gravity force Q into two rectangular components Q_t and Q_n acting parallel and normal, respectively, to the inclined plane. *Ans.* $Q_t = 5.00$ lb; $Q_n = 8.66$ lb.

8. For the particular position shown in Fig. C the connecting rod BA of an engine exerts a force $P = 500$ lb on the crankpin at A. Resolve this force into two rectangular components P_h and P_v acting horizontally and vertically, respectively, at A. *Ans.* $P_h = 468$ lb; $P_v = 177$ lb.

9. Resolve the force P in Fig. C into two rectangular components P_r and P_t acting along the radius AO and perpendicular thereto, respectively. *Ans.* $P_r = 206$ lb; $P_t = 456$ lb.

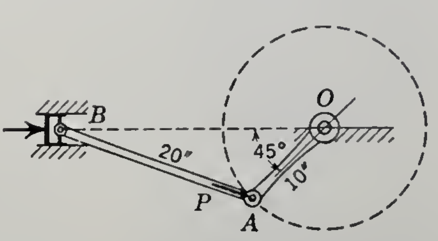


FIG. C

10. Determine graphically the magnitude and direction of the resultant of the four forces shown in Fig. D. *Ans.* $R = 418$ lb; $\theta = 61°45'$.

11. Determine graphically the magnitude and direction of the resultant of the four concurrent forces in Fig. D if each of the 100-lb forces is increased to 150 lb.

12. Determine graphically the magnitude and direction of the resultant of the four concurrent forces in Fig. D if each of the 100-lb forces is reversed in direction.

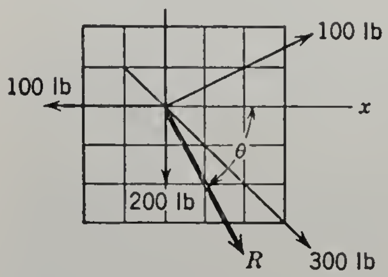


FIG. D

1.3. Equilibrium of concurrent forces in a plane. In Art. 1.2 it was shown that the resultant of any number of concurrent forces in a plane is given by the closing side of the polygon of forces obtained by successive geometric addition of their free vectors. In the particular case where the end of the last vector coincides with the beginning of the first, the resultant vanishes and the system is in equilibrium. The reverse of this statement is a very important one: *If a body known to be in equilibrium is acted upon by several concurrent, coplanar forces, then these forces, or rather their free vectors, when geometrically added must form a closed polygon.* This statement represents the condition of equilibrium for any system of concurrent forces in a plane.

In Fig. 15, we again consider the ball supported in a vertical plane by a string BC and a smooth wall AB. The free-body diagram in which the ball has been isolated from its supports and in which all forces acting upon it, both active and reactive, are indicated by vectors is shown in Fig. 15*b*. The details of making this free-body diagram were already discussed on page 11. The three concurrent forces W, R_a, and S are a system in equilibrium and hence their free vectors must build a closed polygon, in this case, a triangle.

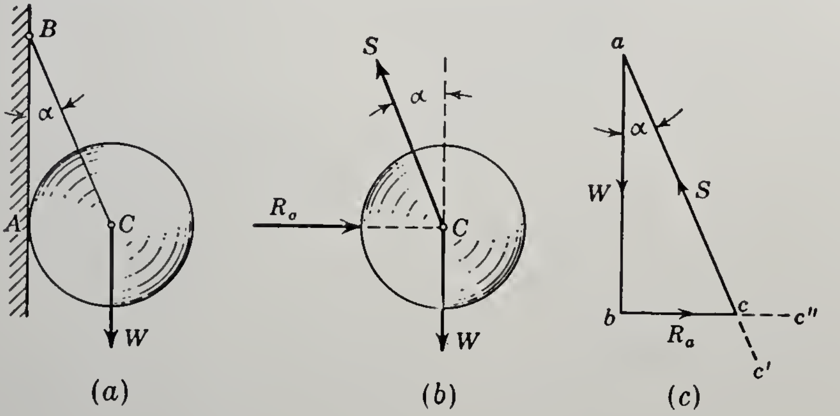


FIG. 15

To construct this closed triangle of forces (Fig. 15*c*), we first lay out the vector $\overline{ab}$, representing to scale the magnitude of the gravity force W. Then from the ends a and b of this vector, we construct the lines ac' and bc'' parallel, respectively, to the known lines of action of the reactive forces S and R_a. The intersection c of these two lines determines the required magnitudes of R_a and S and the arrows show the directions that these forces must have to build a closed triangle. It should be noted that in any closed polygon of forces the vectors must follow one another tail to head around the polygon. The magnitudes of R_a and S may now be scaled from the drawing and the problem is solved.

The foregoing procedure, in which the closed polygon of forces is constructed to scale and the magnitudes of the reactions measured directly from the drawing, is called a *graphical solution* of the problem. In order to make such a solution, of course, numerical values would have to be given for the magnitude of W and the angle α between the string and the wall.

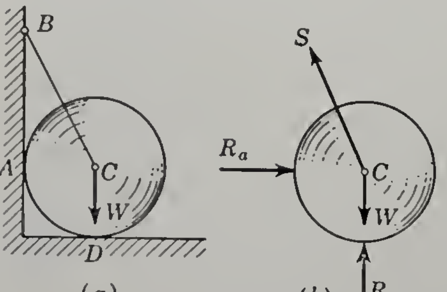
If numerical data are not given, we can still sketch the closed triangle of forces as shown in Fig. 15*c* and then express the magnitudes of R_a and S in terms of W and α by trigonometry. In the present case, for example, we see from the triangle of forces that

$$R_a = W \tan \alpha \qquad S = W \sec \alpha \tag{a}$$

Then for any given numerical data, the magnitudes of R_a and S can be computed from expressions (a). Such an analysis of the problem is called a *trigonometric solution*.

When feasible, the trigonometric solution is preferable to the graphical solution since it is free from the unavoidable small errors associated with graphical constructions and scaling. However, in more complicated problems, the trigonometric method often becomes too involved to be practicable, and we must be satisfied with the less elegant but more straightforward graphical method.

Returning to the free-body diagram in Fig. 15*b*, we see that to balance the applied gravity force W, we need simply an equal, opposite, and collinear force which is called the *equilibrant* of the active forces. We may now consider this equilibrant as represented by the vector $\overline{ba}$ in Fig. 15*c* and proceed to resolve it into components R_a and S parallel to the known lines of action of the reactions. In this way, we get the same magnitudes for R_a and S as before. Recalling that the resolution of a given force into more than two coplanar components is an indeterminate problem (see page 14), we conclude that in dealing with the equilibrium of constrained bodies under the action of concurrent forces in one plane, we cannot determine definitely the magnitudes of more than two reactive forces.



(*a*) (*b*)

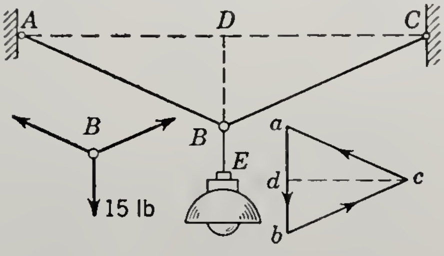
FIG. 16

Suppose, for example, that the ball, otherwise constrained as in Fig. 15*a*, also rests on a horizontal floor at D, as shown in Fig. 16*a*. In such case, the free-body diagram of the ball will be as shown in Fig. 16*b*, and we have three reactive forces R_a, S, and R_d with unknown magnitudes. While the resultant of these three forces must clearly be the equilibrant of W, there is no way to determine their magnitudes definitely and the problem is said to be *statically indeterminate.* Supports in excess of those necessary and sufficient to completely constrain the ball in the plane of the figure are called *redundant constraints.*

EXAMPLES

1. An electric street lamp is suspended from a small ring B supported by two wires AB and CB, the ends A and C of which are on the same level (Fig. 17*a*). Assuming these wires to be perfectly flexible and neglecting their weights, find the force produced in each if the weight of the lamp is 15 lb, the length of each wire, 10 ft, and the sag DB, 4 ft.

Solution. Under the action of the gravity force of the lamp, the wire EB pulls down on the ring B which, in turn, exerts a pull on each of the two wires BA and BC. Hence each of these wires is in tension and exerts an equal and opposite reaction on the ring B, the direction of which must coincide with the axis of the wire. Thus the ring B, considered as a free body, is acted upon by three forces as shown in Fig. 17*b*. Since these three forces are in equilibrium, the vectors representing them must build a closed triangle. To construct this triangle (Fig. 17*c*), we begin with the known vector $\overline{ab}$ representing, to a certain scale, the weight of the lamp, and then draw the sides $\overline{bc}$ and $\overline{ca}$ parallel, respectively, to the wires CB and AB. The lengths of these vectors give the magnitudes of the reactions exerted on the ring B by the wires and consequently the magnitudes of the tensile forces in these wires. If the triangle of forces is constructed to scale, the magnitudes of these forces are obtained by scaling the lengths of the vectors $\overline{bc}$ and $\overline{ca}$.



(*b*) (*a*) (*c*)

FIG. 17

The same magnitudes can be obtained also by calculation. Since the vectors $\overline{bc}$ and $\overline{ca}$, by construction, are parallel, respectively, to the wires CB and AB, we have $\triangle DBC$ similar to $\triangle dbc$, from which

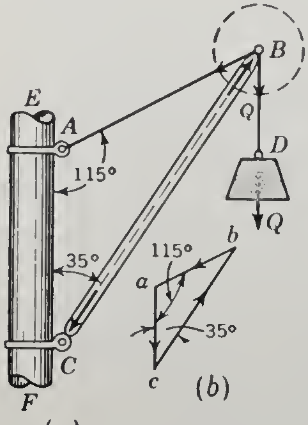
$$ab:bc = 2BD:BC = 8:10$$

and since the force $\overline{ab} = 15$ lb, we find $\overline{bc} = \frac{10}{8} \times 15 = 18.8$ lb.

2. A weight $Q = 500$ lb hanging on a cable BD is supported at point B by a cable AB and a boom BC which is hinged at C (Fig. 18*a*). Neglecting the weights of the cable and boom and assuming an ideal hinge at C, determine the forces transmitted to the mast at points A and C. The angles of $\triangle ABC$ are indicated in the figure.

Solution. We begin by considering the equilibrium of the pin at B. The forces acting on this pin are the active vertical gravity force Q, acting through the cable BD, and the reactions exerted by the cable AB and by the boom BC. Since each of these members is a body acted upon by forces only at its ends and since we are assuming an ideal hinge at C, we conclude that the direction of each of these reactions must coincide with the axis of the member that produces it. The free-body diagram for the pin at B is as shown in the circle around this joint (Fig. 18*a*).

Now having given the magnitude and direction of one of the three forces in equilibrium and the lines of action of the other two, the magnitudes of these latter two forces are obtained by constructing the triangle of forces (Fig. 18*b*). Knowing that the vector $\overline{ac}$, representing the weight Q, acts downward, the arrows on the other two vectors $\overline{cb}$ and $\overline{ba}$ must be directed as shown on the triangle of forces, since all arrows must follow each other tail to head around any closed polygon of forces. Considering the vector $\overline{cb}$ which represents the reaction of the boom on the pin at B, we see that the boom pushes against this pin and hence is in compression. Similarly, the arrow on the vector $\overline{ba}$ indicates tension in the cable AB. In general, if the directions of any unknown reactions are assumed incorrectly in the free-body diagram, they may be corrected after the construction of the polygon of forces.



(*b*)

(*a*)

FIG. 18

We conclude now, since the cable AB is in tension and the boom BC in compression, that the cable pulls on the mast at A with a force equal but opposite to the vector $\overline{ba}$ and that the boom pushes on the mast at C with a force equal but opposite to the vector $\overline{cb}$. These actions at A and C are shown in Fig. 18*a*. Their magnitudes can be obtained either by scaling the lengths of the vectors $\overline{cb}$ and $\overline{ba}$ of the triangle of forces or by trigonometric calculation, which gives, for the force at A, 574 lb and for the force at C, 906 lb.

3. Two smooth spheres, each of radius r and weight Q, rest in a horizontal channel having vertical walls, the distance between which is b (Fig. 19*a*). Find the pressures exerted on the walls and floor at the points of contact A, B, and D. The following numerical data are given: $r = 10$ in., $b = 36$ in., $Q = 100$ lb.

Solution. Since the spheres are smooth, the pressures at the various points of contact must be normal to the surfaces. Removing the supporting walls and floor and replacing them by their reactions R_a, R_b, and R_d, we obtain the free-body diagram for both spheres as shown in Fig. 19*a*. These reactions are equal and opposite to the required pressures exerted by the spheres on the walls and floor. At the point of contact between the two spheres, we have two equal and opposite forces R_1 and R_2 which must act along the line OC joining the centers of the spheres. When considering the equilibrium of the upper sphere, we take only the force R_1 representing the reaction exerted by the lower sphere; likewise when considering the lower sphere, we take only the force R_2. We see now that the upper sphere is in equilibrium under the action of the gravity force Q and the two reactions R_1 and R_a, while the lower sphere is in equilibrium under the action of the four forces, R_2, Q, R_b, and R_d. In each case all forces are in one plane and concurrent at the center of the corresponding sphere on which they act.

We begin by constructing the triangle of forces for the upper sphere. From this triangle (Fig. 19*b*) the reactions R_a and R_1 are determined. Proceeding now to the lower sphere, we have the reaction R_2, equal and opposite to the previously determined force R_1, and the gravity force Q, both of which are completely known. Thus we can complete the polygon of forces (Fig. 19*c*) for this sphere and determine the remaining two unknown reactions R_b and R_d.

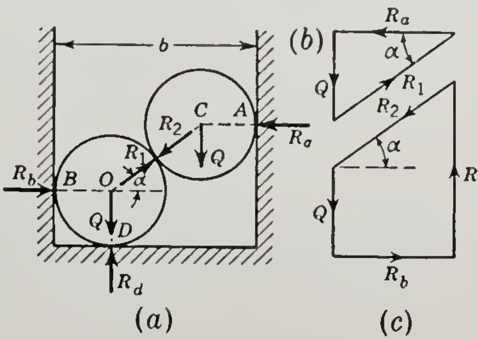


FIG. 19

If the drawing in Fig. 19*a* has been made to scale, the direction of the line OC and consequently of the vectors R_1 and R_2 will be determined graphically. In this event, if the polygons of forces have also been constructed to scale, the magnitudes of the various unknown reactions may be scaled directly from the drawings. Otherwise they may be computed as follows: Referring to Fig. 19*a*, we note that

$$2r + 2r \cos \alpha = b$$

from which

$$\cos \alpha = \frac{b}{2r} - 1 \qquad \text{(b)}$$

Using the value of α determined from Eq. (b), we find from the polygons of forces $R_a = R_b = 1.33Q = 133$ lb and $R_d = 2Q = 200$ lb.

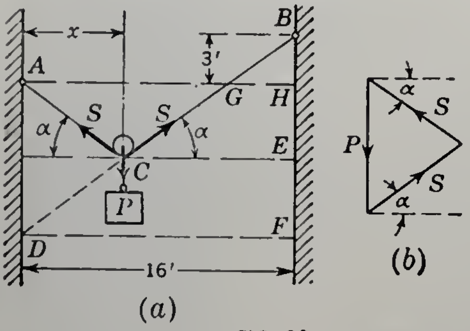


FIG. 20

4. A cord ACB 20 ft long, is attached at points A and B to two vertical walls, 16 ft apart (Fig. 20). A pulley C, so small that we can neglect its radius, carries a suspended load $P = 36$ lb and is free to roll without friction along the cord. Determine the position of equilibrium, as defined by the distance x, that the pulley will assume and also the tensile force in the cord.

Solution. Neglecting friction in the pulley, we conclude that the forces exerted at C by the portions AC and BC of the cord must be equal. These two forces can balance the vertical force P only if they give a vertical resultant. This condition requires that AC and BC be equally inclined to the horizontal. From this it follows that by continuing the line BC down to point D, we obtain an isosceles triangle ACD. Thus, it is evident that $\triangle BFD$ is a 3:4:5 triangle. Now from the similarity of $\triangle BGH$ and $\triangle BDF$ we may write $GH:BH = DF:BF = 4:3$ or, using the given dimensions, $(16 - 2x):3 = 4:3$, from which $x = 6$ ft.

The triangle of forces (Fig. 20*b*) for the three forces in equilibrium at C is similar, by construction, to $\triangle ACD$. Hence, $S:P = 5:6$, from which we conclude that the tensile force in the cord is 30 lb.

PROBLEM SET 1.3

1. An electric-light fixture of weight $Q = 40$ lb is supported as shown in Fig. A. Determine the tensile forces S_1 and S_2 in the wires BA and BC if their angles of inclination are as shown. *Ans.* $S_1 = 29.3$ lb; $S_2 = 20.7$ lb.

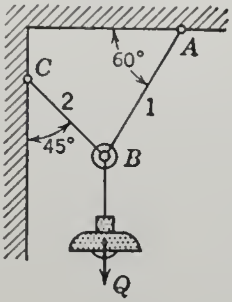


FIG. A

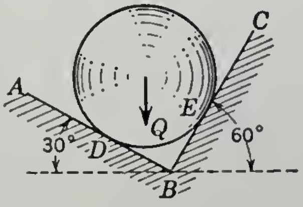


FIG. B

2. A ball of weight $Q = 12$ lb rests in a right-angled trough, as shown in Fig. B. Determine the forces exerted on the sides of the trough at D and E if all surfaces are perfectly smooth. *Ans.* $R_d = 10.4$ lb; $R_e = 6.0$ lb.

3. A circular roller of weight $Q = 100$ lb and radius $r = 6$ in. hangs by a tie rod $AC = 12$ in. and rests against a smooth vertical wall at B, as shown in Fig. C. Determine the tension S in the tie rod and the force R_b exerted against the wall at B. *Ans.* $S = 115.5$ lb; $R_b = 57.7$ lb.

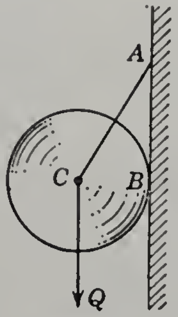


FIG. C

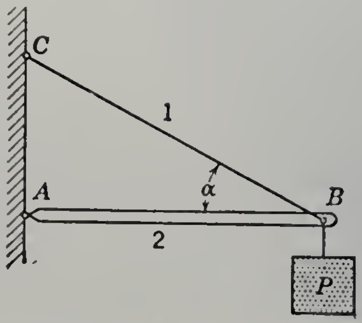


FIG. D

4. What axial forces does the vertical load P induce in the members of the system shown in Fig. D? Neglect the weights of the members themselves and assume an ideal hinge at A and a perfectly flexible string BC. *Ans.* $S_1 = P \csc \alpha$, tension; $S_2 = P \cot \alpha$, compression.

5. What axial forces does the vertical load P induce in the members of the system shown in Fig. E? Make the same idealizing assumptions as in Prob. 4. *Ans.* $S_1 = P \tan \alpha$, tension; $S_2 = P \sec \alpha$, compression.

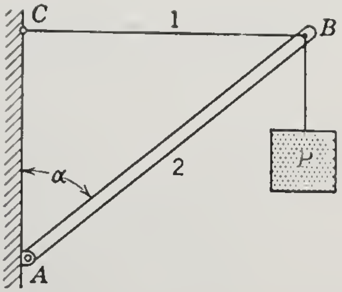


FIG. E

6. A right circular roller of weight W rests on a smooth horizontal plane and is held in position by an inclined bar AC as shown in Fig. F. Find the tension S in the bar AC and the vertical reaction R_b at B if there is also a horizontal force P acting at C. *Ans.* $S = P \sec \alpha$; $R_b = W + P \tan \alpha$.

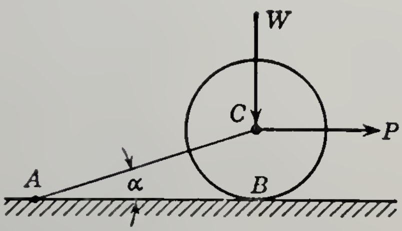


FIG. F

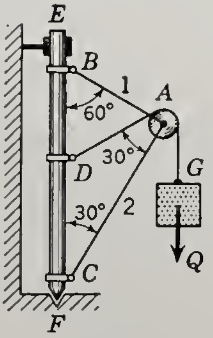


FIG. G

7. A pulley A is supported by two bars AB and AC which are hinged at points B and C to a vertical mast EF (Fig. G). Over the pulley hangs a flexible cable DG which is fastened to the mast at D and carries at the other end G a load $Q = 2$ tons. Neglecting friction in the pulley, determine the forces produced in the bars AB and AC. The angles between the various members are shown in the figure. *Ans.* $S_2 = 3.46$ tons; $S_1 = 0$.

8. Two smooth circular cylinders, each of weight $W = 100$ lb and radius $r = 6$ in., are connected at their centers by a string AB of length $l = 16$ in. and rest upon a horizontal plane, supporting above them a third cylinder of weight $Q = 200$ lb and radius $r = 6$ in. (Fig. H). Find the force S in the string AB and the pressures produced on the floor at the points of contact D and E. *Ans.* $S = 89.4$ lb, tension; $R_d = R_e = 200$ lb.

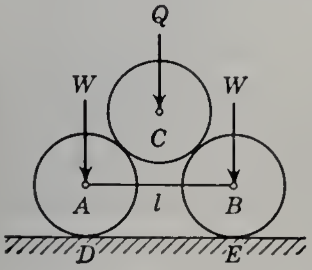


FIG. H

9. Two identical rollers, each of weight $Q = 100$ lb, are supported by an inclined plane and a vertical wall as shown in Fig. I. Assuming smooth surfaces, find the reactions induced at the points of support A, B, and C. *Ans.* $R_a = 86.6$ lb; $R_b = 144$ lb; $R_c = 115$ lb.

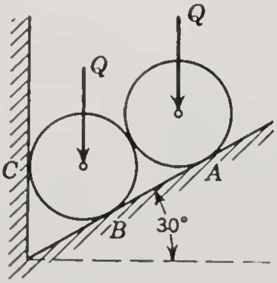


FIG. I

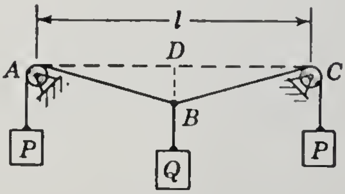


FIG. J

10. A weight Q is suspended from point B of a cord ABC, the ends of which are pulled by equal weights P overhanging small pulleys A and C which are on the same level (Fig. J). Neglecting the radii of the pulleys, determine the sag BD if $l = 12$ ft, $P = 20$ lb, and $Q = 10$ lb. *Ans.* $BD = 1.55$ ft.

11. A weight Q is suspended from a small ring C, supported by two cords AC and BC (Fig. K). The cord AC is fastened at A while the cord BC passes over a frictionless pulley at B and carries the weight P as shown. If $P = Q$ and $\alpha = 50°$, find the value of the angle β. *Ans.* $\beta = 80°$.

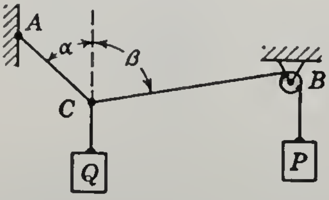


FIG. K

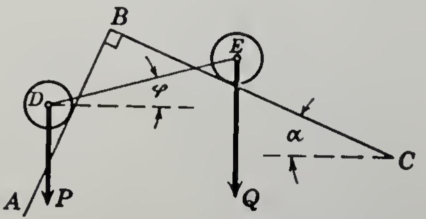


FIG. L

***12.** Two rollers of weights P and Q are connected by a flexible string DE and rest on two mutually perpendicular planes AB and BC, as shown in Fig. L. Find graphically the tension S in the string and the angle φ that it makes with the horizontal when the system is in equilibrium. The following numerical data are given: $P = 60$ lb, $Q = 100$ lb, $\alpha = 30°$. Assume that the string is inextensible and passes freely through slots in the smooth inclined planes AB and BC. *Ans.* $S = 72$ lb; $\varphi = 16°$

1.4. Method of projections. Previously, we have handled all problems of composition, resolution, and equilibrium of concurrent forces in a plane by using the method of geometric addition of their free vectors. These same problems can also be solved by a method of algebraic addition of the *projections* of the given forces on rectangular coordinate axes x and y taken in the plane of action of the forces.

To develop this method, let us consider first the case of two forces F_1 and F_2, applied at point A (Fig. 21) and making with the positive directions of the coordinate axes the angles α_1, β_1 and α_2, β_2, respectively. Their resultant R is obtained from the parallelogram of forces, and the angles that it makes with the x and y axes, respectively, will be denoted by α and β. Considering, now, all forces projected onto the x axis, we find for these projections the values $F_1 \cos \alpha_1$, $F_2 \cos \alpha_2$, and $R \cos \alpha$, and we see that

$$R \cos \alpha = F_1 \cos \alpha_1 + F_2 \cos \alpha_2 \qquad \text{(a)}$$

In the same manner, considering all forces projected onto the y axis, we obtain

$$R \cos \beta = F_1 \cos \beta_1 + F_2 \cos \beta_2 \qquad \text{(b)}$$

Thus from Eqs. (a) and (b) it may be stated that *the projection of the resultant of two forces on any axis is equal to the algebraic sum of the projections of its components on the same axis.*

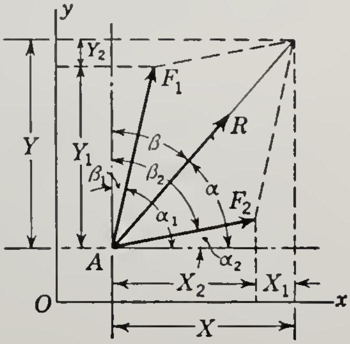


FIG. 21

By successive applications of the principle of the parallelogram of forces, the above conclusion can be obtained for any number of concurrent forces $F_1, F_2, \ldots, F_n$ in a plane. Using, for the projections of the various forces, the following notations

$$X_i = F_i \cos \alpha_i \qquad Y_i = F_i \cos \beta_i$$
$$X = R \cos \alpha \qquad Y = R \cos \beta$$

we obtain

$$\begin{aligned} X &= X_1 + X_2 + \cdots + X_n = \Sigma X_i \\ Y &= Y_1 + Y_2 + \cdots + Y_n = \Sigma Y_i \end{aligned} \qquad (1)$$

where the summations are understood to include all forces in the system. *Thus, the projections, on the coordinate axes, of the resultant of a system of concurrent forces $F_1, F_2, \ldots, F_n$ acting in one plane are equal to the algebraic sums of the corresponding projections of the components.*

Knowing the magnitudes and directions of the various forces, their projections $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ on the rectangular coordinate axes x and y, respectively, may be computed and tabulated in systematic order. The algebraic summations, indicated by Eqs. (1), for determining the projections X and Y of the resultant may then be made and the magnitude and direction of the resultant computed from the following equations:

$$R = \sqrt{X^2 + Y^2} \qquad \tan \alpha = \frac{Y}{X} \qquad (2)$$

The use of Eqs. (2) for determining the resultant of a given system of concurrent forces in a plane is sometimes more advantageous than the method of geometric addition of the vectors representing these forces as discussed in Art. 1.2.

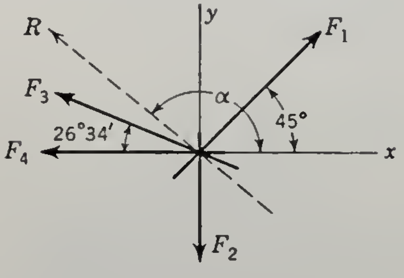
When the given forces $F_1, F_2, \ldots, F_n$ are in equilibrium, their resultant is zero, and from the first of Eqs. (2), it is evident that this condition can be satisfied only if we have $X = 0$ and $Y = 0$, which, referring to Eqs. (1), evidently requires

$$\Sigma X_i = 0 \qquad \Sigma Y_i = 0 \tag{3}$$

These are the two *equations of equilibrium* for a system of concurrent forces in a plane. They are equivalent to the geometric condition of equilibrium that the polygon of forces must close, which has been the basis for the solution of problems in the previous article. The application of Eqs. (3) to the solution of problems of statics will now be illustrated by several examples.

EXAMPLES

1. Using the method of projections, find the magnitude and direction of the resultant R of the four concurrent forces shown in Fig. 22*a* and having the magnitudes shown in Fig. 22*b*.



(*a*)

Force	*(Mag.) lb*	X_i	Y_i
F_1	150	+106.0	+106.0
F_2	100	0	−100.0
F_3	120	−107.2	+ 53.7
F_4	80	− 80.0	0

(*b*)

FIG. 22

Solution. We begin by computing and tabulating the projections X_i and Y_i as shown in the table in Fig. 22*b*. Summing these projections in accordance with Eqs. (1), we obtain

$$X = \Sigma X_i = -81.2 \text{ lb} \qquad Y = \Sigma Y_i = +59.7 \text{ lb}$$

and Eqs. (2) give

$$R = \sqrt{(-81.2)^2 + (+59.7)^2} = 100.7 \text{ lb}$$

$$\alpha = \arctan \frac{Y}{X} = \arctan \frac{+59.7}{-81.2} = 143°41'$$

We see from the signs of X and Y that the resultant lies in the second quadrant; hence the angle α is measured out in the counterclockwise direction from the positive end of the x axis, as shown.

2. A load $P = 1{,}000$ lb is bracketed from a vertical wall by two bars AB and AC hinged together at A and to the wall at B and C as shown in Fig. 23. Using the method of projections, compute the axial forces S_1 and S_2 induced in these bars.

Solution. We first make a free body of the pin A, replacing the bars AB and AC by the reactions S_1 and S_2 directed as shown in the figure. Then choosing coordinate axes x and y as shown, the equations of equilibrium (3) become

$$-S_1 + 0.500P = 0 \qquad +S_2 - 0.866P = 0$$

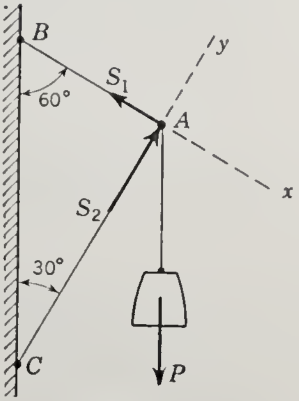


FIG. 23

giving $S_1 = 0.500P = 500$ lb tension and $S_2 = 0.866P = 866$ lb compression.

3. A small ring B carries a vertical load P and is supported by two strings BA and BC, the latter of which carries at its free end a weight $Q = 10$ lb, as shown in Fig. 24. Find the magnitude of the load P and the tension S in the string AB if the angles that the strings AB and BC make with the vertical are as shown in the figure and the system is in equilibrium.

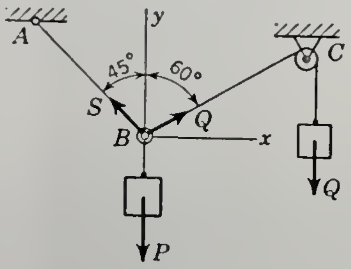


FIG. 24

Solution. Neglecting friction in the pulley at C, it is evident that the tension in the string BC is equal to the load Q. Thus, acting on the ring B, we have three concurrent forces in a plane that are in equilibrium. Taking coordinate axes x and y as shown, Eqs. (3) become

$$\begin{aligned} Q \cos 30^\circ - S \cos 45^\circ &= 0 \\ Q \cos 60^\circ + S \cos 45^\circ &= P \end{aligned}$$

From the first of these equations, we find $S = 5\sqrt{6}$ lb. Then substituting in the second equation, we obtain

$$P = 5(1 + \sqrt{3}) = 13.7 \text{ lb}$$

4. Three bars in one plane, hinged at their ends as shown in Fig. 25*a*, are submitted to the action of a force $P = 10$ lb applied at the hinge B as shown. Determine the magnitude of the force Q that it will be necessary to apply at the hinge C in order to keep the system of bars in equilibrium if the angles between the bars and the lines of action of the forces are as given in the figure.

Solution. We begin with a consideration of the equilibrium of the hinge B. Under the action of the applied force P, the bars AB and BC will be subjected to compression and will accordingly exert reactive forces S_1 and S_2 on the pin and with lines of action coinciding with the axes of the bars. All this is shown on the free-body diagram of hinge B in Fig. 25*b*. Equating to zero the algebraic sum of projections of these forces on the y axis taken perpendicular to AB, we obtain

$$S_2 \cos 45° - P = 0$$

from which $S_2 = P/\cos 45° = \sqrt{2}\,P$. Since this represents the compressive force in the bar BC, we conclude at once that this bar exerts the same force on the hinge C directed as shown in Fig. 25*c*. In addition there is the applied force Q and a reactive force S_3 due to compression in the bar CD. Equating to zero the algebraic sum of the projections of these forces on the x axis perpendicular to CD as shown, we obtain

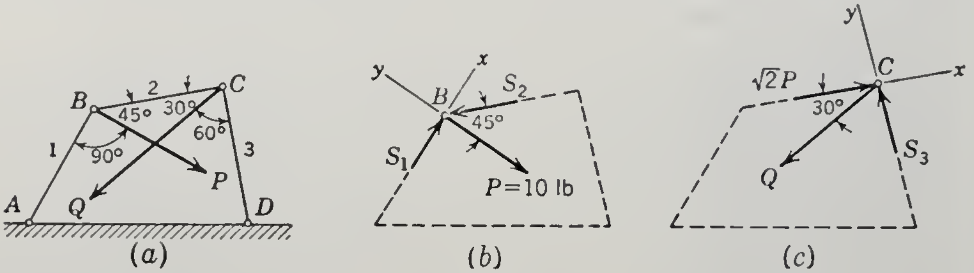


FIG. 25

$$\sqrt{2}\,P - Q \cos 30° = 0$$

from which $Q = 2\sqrt{2}\,P/\sqrt{3} = 1.63P = 16.3$ lb.

We see that by choosing one of the coordinate axes in each case perpendicular to the line of action of an unknown force, we obtain an equation of equilibrium containing only one unknown and thereby gain some simplification. If the forces S_1 and S_3 are required, they can easily be found by writing the other two equations of equilibrium.

It is worthwhile to note that if the system of bars in Fig. 25*a* is disturbed from the configuration of equilibrium shown it will collapse. However, if the hinge C is connected to the foundation by another bar AC in place of the force Q, the system will be stable and the magnitude Q calculated above will simply represent the axial force (tension) in such a bar due to the action of the other applied force P.

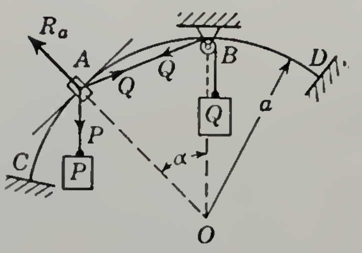


FIG. 26

5. A small ring A can slide without friction along a curved bar CD which has a circular axis of radius a (Fig. 26). Determine the position of equilibrium as defined by the angle α if the loads P and Q are acting as shown in the figure.

Solution. Neglecting friction in the pulley at B, we conclude that the string AB is subjected to a tension numerically equal to Q. Then considering the ring A as a free body, we see that it is acted upon by the three forces P, Q, and the reaction R_a which acts in the radial direction OA. These three forces are in equilibrium; hence the algebraic sum of their projections on any axis must be equal to zero. Projecting them onto an axis in the direction of the tangent to the circle at A (thus excluding the unknown reaction R_a), we obtain

$$Q \cos \frac{\alpha}{2} - P \sin \alpha = 0$$

which may be written in the form

$$Q \cos \frac{\alpha}{2} - 2P \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} = 0$$

From this equation we obtain either

$$\sin \frac{\alpha}{2} = \frac{Q}{2P} \qquad \text{that is, } \alpha = 2 \arcsin \frac{Q}{2P} \tag{c}$$

or

$$\cos \frac{\alpha}{2} = 0 \qquad \text{that is, } \alpha = \pi \tag{d}$$

The value of α given by Eq. (d), corresponds to the case where the ring A is vertically under the center O and hence has no significance in this case; the value of α given by Eq. (c) is the one in which we are interested. For the particular case where P and Q are equal weights, Eq. (c) gives $\alpha = 60°$. Likewise for $Q/P = \frac{3}{4}$, we obtain $\alpha \approx 44°$.

PROBLEM SET 1.4

1. Referring to Fig. A, calculate the tensions S_1 and S_2 in the two strings AB and AC that support the lamp of weight $Q = 40$ lb. Use the method of projections [Eqs. (3)]. *Ans.* $S_1 = 30$ lb; $S_2 = 50$ lb.

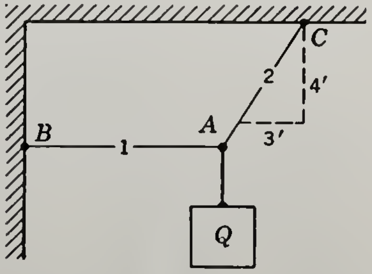


FIG. A

2. A roller of weight $W = 1{,}000$ lb rests on a smooth inclined plane and is kept from rolling down by a string AC as shown in Fig. B. Using the method of projections, find the tension S in the string and the reaction R_b at the point of contact B. *Ans.* $S = 733$ lb; $R_b = 897$ lb.

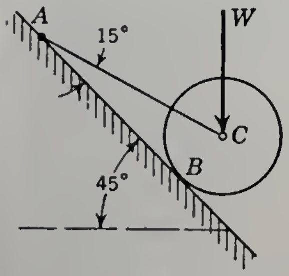


FIG. B

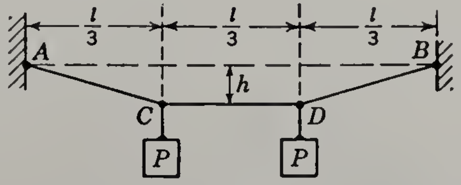


FIG. C

3. Two equal loads P are supported by a flexible string $ACDB$, as shown in Fig. C. Determine the tensile forces S_1 and S_2 in the portions AC and CD, respectively, of the string, if the span $l = 30$ ft and the sag $h = 5$ ft. Neglect the weight of the string. *Ans.* $S_1 = \sqrt{5}P$; $S_2 = 2P$.

4. On the string $ACEDB$ are hung three equal weights Q symmetrically placed with respect to the vertical line through the mid-point E (Fig. D). Determine the value of the angles β if the other angles are as shown in the figure. *Ans.* $\beta = 30°$.

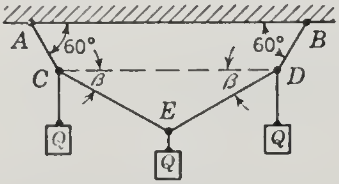


FIG. D

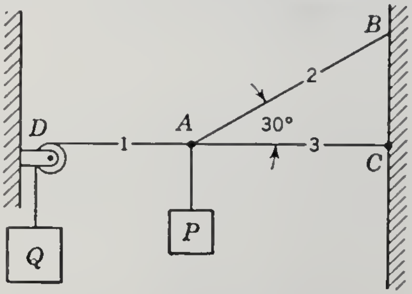


FIG. E

5. In Fig. E, weights P and Q are suspended in a vertical plane by strings 1, 2, 3, arranged as shown. Find the tension induced in each string if $P = 500$ lb and $Q = 1{,}000$ lb. *Ans.* $S_1 = 1{,}000$ lb; $S_2 = 1{,}000$ lb; $S_3 = 134$ lb.

6. To pull up a post, the arrangement shown in Fig. F is used. A cable ABC is fixed to the post at A and to the frame at C having the portion AB vertical and the portion BC inclined thereto by a small angle α. The cable BDE fastened to the ring at B and to the frame at E has the portion BD horizontal and the portion DE inclined to the horizontal by the small angle β. On the ring at D a man pulls vertically downward with his entire weight Q. Determine the vertical pull P applied to the post at A if $\alpha = \beta = 0.1$ radian and $Q = 150$ lb. *Ans.* $P \approx 15{,}000$ lb.

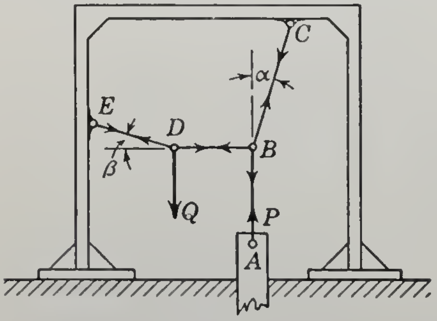


FIG. F

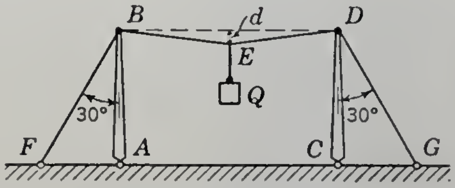


FIG. G

7. Two vertical masts AB and CD are guyed by the wires BF and DG, in the same vertical plane and connected by a cable BD of length l, from the middle point E of which is suspended a load Q (Fig. G). Find the tensile force S in each of the two guy wires BF and DG if the load $Q = 100$ lb, the length $l = 20$ ft, and the sag $d = 1$ ft. *Ans.* $S \approx 1{,}000$ lb.

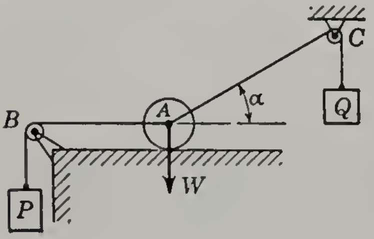


FIG. H

8. A ball of weight W rests upon a smooth horizontal plane and has attached to its center two strings AB and AC which pass over frictionless pulleys at B and C and carry loads P and Q, respectively, as shown in Fig. H. If the string AB is horizontal, find the angle α that the string AC makes with the horizontal when the ball is in a position of equilibrium. Also find the pressure R between the ball and the plane. *Ans.* $\cos \alpha = P/Q$; $R = W - \sqrt{Q^2 - P^2}$.

9. Determine the axial forces S_1 and S_2 induced in the bars AC and BC in Fig. I due to the action of the horizontal applied load at C. The bars are hinged together at C and to the foundation at A and B. *Ans.* $S_1 = 782$ lb, tension; $S_2 = 640$ lb, compression.

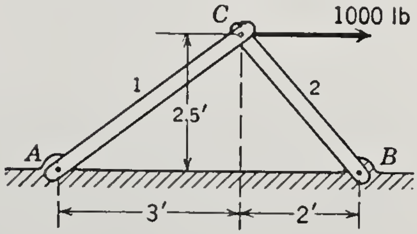


FIG. I

10. Determine the forces produced in the bars of the system shown in Fig. J owing to the horizontal force P applied at the hinge B. *Ans.* $S_1 = 0$; $S_2 = P$, tension; $S_3 = \sqrt{2}P$, compression; $S_4 = P$, tension.

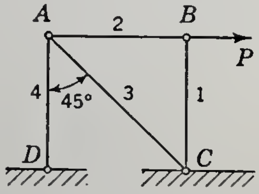


FIG. J

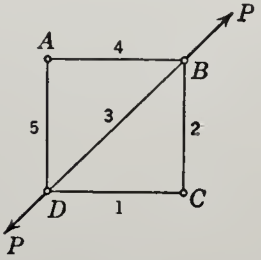


FIG. K

11. A hinged square $ABCD$ (Fig. K) with diagonal BD is submitted to the action of two equal and opposite forces P applied as shown. Determine the forces produced in all bars. *Ans.* $S_1 = S_2 = S_4 = S_5 = 0$; $S_3 = P$, tension.

12. Determine the forces that will be produced in all bars of the frame $ABCD$ (Fig. K) if the external forces P are applied in the same manner to the hinges A and C. *Ans.* $S_1 = S_2 = S_4 = S_5 = P/\sqrt{2}$, tension; $S_3 = P$, compression.

13. In the bar AB of the square frame $ABCD$ (Fig. L) a tensile force P is produced by tightening a turnbuckle F. Determine the forces produced in the other bars. The diagonals AC and BD pass each other freely at E. *Ans.* $S_1 = S_2 = S_3 = P$, tension; $S_4 = S_5 = \sqrt{2}P$, compression.

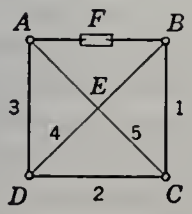


FIG. L

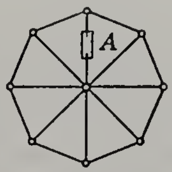


FIG. M

14. By means of a turnbuckle A a tensile force P is produced in one of the radial bars of the hinged regular octagon shown in Fig. M. Determine the

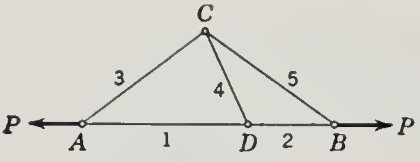


FIG. N

forces produced in the other bars of the system. *Ans.* P, tension in each radial bar; $1.306P$, compression in each outside bar.

15. Determine the axial force induced in each bar of the system shown in Fig. N due to the action of the applied forces P. *Ans.* $S_1 = S_2 = P$, tension; $S_3 = S_4 = S_5 = 0$.

*16. A rigid bar AB with rollers of weights $P = 50$ lb and $Q = 100$ lb at its ends is supported inside a circular ring in a vertical plane as shown in Fig. O. The radius of the ring and the length AB are such that the radii AC and BC form a right angle at C; that is, $\alpha + \beta = 90°$. Neglecting friction and the weight of the bar AB, find the configuration of equilibrium as defined by the angle $(\alpha - \beta)/2$ that AB makes with the horizontal. Find also the reactions R_a and R_b and the compressive force S in the bar AB. *Ans.* $(\alpha - \beta)/2 = 18°26'$; $R_a = 67.1$ lb; $R_b = 134$ lb; $S = 63.3$ lb.

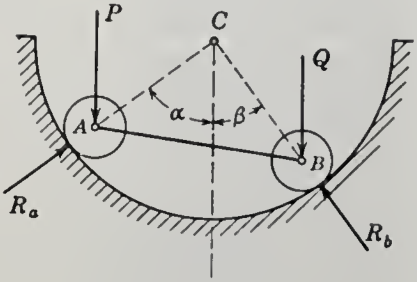


FIG. O

1.5. Equilibrium of three forces in a plane. If three nonparallel forces acting in one plane are in equilibrium, their lines of action must intersect in one point. To prove this statement, let us assume that the three forces P, Q, and S, acting upon a body at points A, B, and C (Fig. 27a) all lie in one plane and are in equilibrium. Prolonging the lines of action of two of the forces, say P and Q, to their point of intersection D and transmitting their points of application to this point, we can replace these two forces by their resultant, which also will act through point D. Since the third force S must hold the resultant of P and Q in equilibrium, and since two forces can be in equilibrium only if they are collinear in action, we conclude that the force S must also pass through the point D. In addition, the free vectors P, Q, and S must build a closed triangle as shown in Fig. 27b. To summarize. *Three nonparallel forces can be in equilibrium only when they lie in one plane, intersect in one point, and their free vectors build a closed triangle.* This statement is called the *theorem of three forces*.

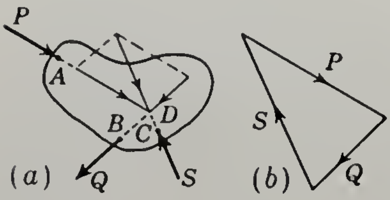
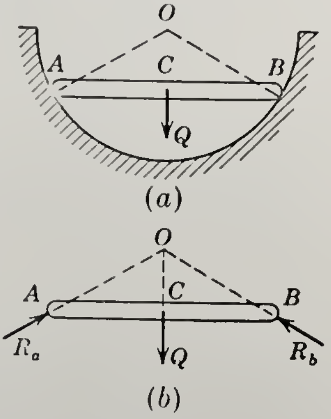


FIG. 27

In engineering problems of statics, especially in discussing the equilibrium of constrained bodies, we very often encounter the case of three forces in one plane that are in equilibrium. In such cases we shall be interested usually in determining the magnitudes and directions of the reactions arising at the points of support. This can be done in each case by observing that the three forces must meet in one point as just stated. Thus, knowing the lines of action of two of the forces and the point of application of the third, the direction of the third can always be determined. Such considerations will always require careful inspection of the various types of constraint encountered at the points of support so that we properly represent the corresponding reactions on our free-body diagram. For this reason, we shall repeat and expand here some earlier remarks about constraints, reactions, and the free-body diagram, before proceeding to further examples and problems.



(a)

(b)

FIG. 28

Consider, first, the case of a prismatic bar AB supported at its ends by a smooth semicircular trough, as shown in Fig. 28*a*, and assume that the bar lies in a vertical plane normal to the axis of the trough. Under the action of the gravity force Q, the ends of the bar press against the wall of the trough and in the case of a *smooth surface* this pressure can be resisted only in the direction normal to the surface at each point of contact. Thus the equal and opposite reactions which the supporting surface can exert on the ends of the bar must be in radial directions, and we obtain the free-body diagram as shown in Fig. 28*b*. We may now conclude from the theorem of three forces that the bar will assume a position of equilibrium such that its center of gravity C is vertically under point O. It is often justifiable to use the notion of an *ideal smooth surface*, and in all cases the reaction from such a surface must be directed along the normal at the point of contact.

As a second example, consider the case of a lever AB supported by a *hinge* C, as shown in Fig. 29*a*. Under the action of applied forces P and Q (assumed such as to maintain equilibrium), the lever exerts pressure on the hinge pin which passes through it. We are concerned with the equal and opposite reactive pressure exerted by the pin on the bar. A detail of the pin and hole is shown in Fig. 29*b*, where we again make the assumption of a perfectly smooth circular cylindrical surface for the pin. Under such conditions, the distributed pressure at all points of contact between pin and inner surface of hole is normal to the surface, as shown. This means that all these distributed pressures must act along radial lines intersecting at the center of the pin. Consequently, the reaction R_c must pass through the center of the pin if it is to produce sensibly the effect of the distributed pressure. This notion of a smooth circular pin or *ideal hinge* will often be used and always the reaction that it produces on a body will pass through its center. Regarding the direction of R_c there is nothing about the physical nature of the constraint to determine this; it can have any line of action through point C. However, for equilibrium, we conclude from the theorem of three forces that R_c must pass through the point of intersection D of the active forces P and Q and the free-body diagram is obtained, as shown in Fig. 29c.

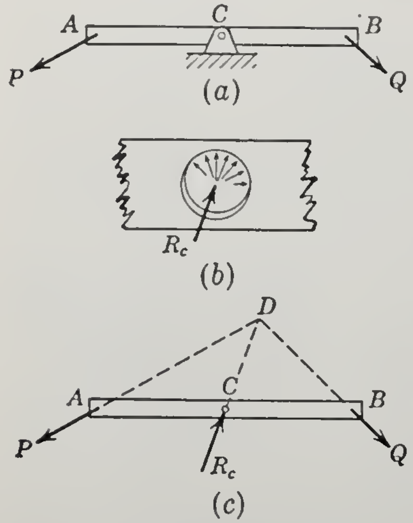


FIG. 29

As a last example, we consider the case of a beam AB supported horizontally by a hinge at A and a small roller at B, as shown in Fig. 30a. Under the action of an applied force P acting in the plane of the figure as shown, the beam exerts pressures on the two supports at its ends. We wish to find the corresponding reactions. To do this, we remove both supports and replace them by reactions, as shown in Fig. 30b. Assuming an ideal hinge at A, the reaction R_a must act through this point, but we do not know immediately in what direction. Consequently we indicate this force by a wavy line as shown, to indicate unknown direction. Now proceeding to point B and assuming a smooth surface for the roller, we conclude that R_b must act in a direction normal to the surface on which the roller rests; in this case, vertically. Such rollers allowing free horizontal movement of one end of a beam are often used for bridge supports to prevent damage due to contraction or expansion of the beam resulting from changes in temperature. This *simple roller* is a common type of constraint and always exerts its reaction normal to the surface on which it rolls. To complete the free-body diagram in Fig. 30b, we

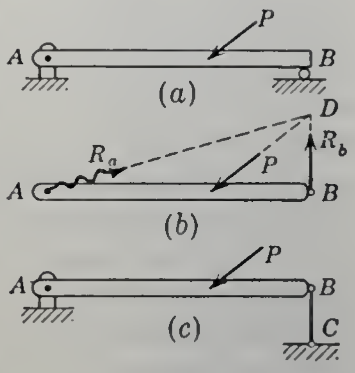


FIG. 30

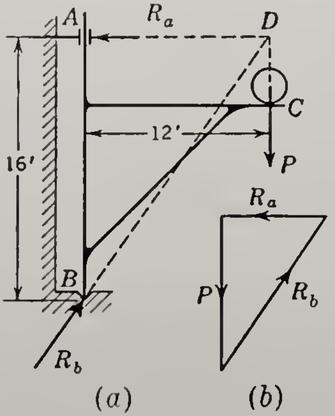
now observe that the known lines of action of P and R_b determine the point of concurrence D of the system of three forces in equilibrium; hence the true line of action AD of the reaction R_a is finally determined.

If the roller at B in Fig. 30*a* is replaced by a vertical bar BC of negligible weight and hinged at both ends (Fig. 30*c*), we shall obtain the same free-body diagram as before. Since forces act on this bar only at its two ends, their lines of action must coincide with the axis BC, as previously explained. Consequently, the reactive forces that the bar exerts on the beam must be directed along BC. Such a *simple strut* (or *tie bar* if it is in tension) is another common type of constraint used as a support.

EXAMPLES

1. The vertical axis AB of a crane is supported by a guide at A and a socket at B as shown in Fig. 31*a*. Determine the reactions R_a and R_b produced at A and B by the load $P = 4$ tons. Friction at the supports should be neglected.

Solution. Considering the entire crane as a free body (Fig. 31*a*), we imagine the supports at A and B removed and replace them by the reactions R_a and R_b which they exert on the crane. Thus we have the case of equilibrium of three forces, P, R_a, and R_b, in a plane and they must intersect in one point. Strictly speaking, the reaction exerted by the guide at A will be distributed over the surface of contact between the guide and the mast, but since the dimensions of this surface are small compared with the dimensions of the entire structure, we can assume this pressure to be concentrated at one point, say, the center of the guide, and further, neglecting friction, it must act perpendicular to AB. Thus by justifiable simplifying assumptions, we obtain the point of application and the direction of the reaction R_a. Extending this line of action to intersect that of the vertical gravity force P, we obtain the point of intersection D, through which the reaction R_b must also pass and its line of action BD is determined.



(*a*) (*b*)

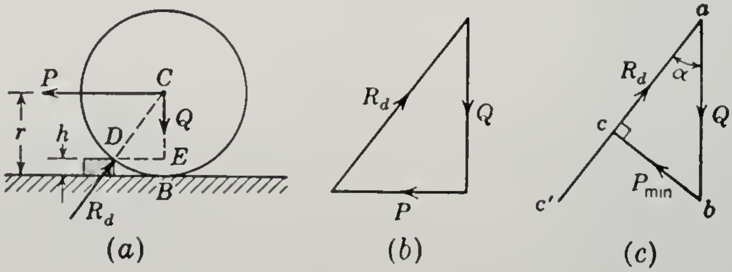
FIG. 31

We now construct the triangle of forces (Fig. 31*b*) with its sides parallel to the known lines of action of the three forces. Noting that the triangle of forces is similar, by construction, to $\triangle ABD$, we conclude that $R_a : 12 = P : 16$, from which $R_a = \frac{3}{4}P = 3$ tons. In the same way we note that $R_b : 20 = P : 16$, from which $R_b = \frac{5}{4}P = 5$ tons. We see further, from the directions of the arrows on the sides of the triangle of forces, that the forces act as shown in Fig. 31*a*.

2. Determine the magnitude of a horizontal force P applied at the center C

of a roller of weight $Q = 1{,}000$ lb and radius $r = 15$ in. which will be necessary to pull it over a 3 in.-curb (Fig. 32*a*).

Solution. When the applied force P is just sufficient to cause motion of the roller to impend, there will be no pressure between the roller and the horizontal plane at B. Thus the roller will be acted upon by only three forces, P, Q, and the reaction R_d as shown in the figure. Since P and Q intersect at the center C of the cylinder, the reaction R_d must pass through this same point also. Thus the lines of action of all forces are known. It should be noted here that the reaction R_d passes through point C regardless of whether or not we neglect friction.



(*a*) (*b*) (*c*)

FIG. 32

The triangle of forces (Fig. 32*b*) will be similar to $\triangle CED$, from which we may write

$$P:Q = DE:CE = 9:12$$

From this we find $P = 3Q/4 = 750$ lb. Also from the similarity of the two triangles, we have

$$R_d:Q = DC:CE = 15:12$$

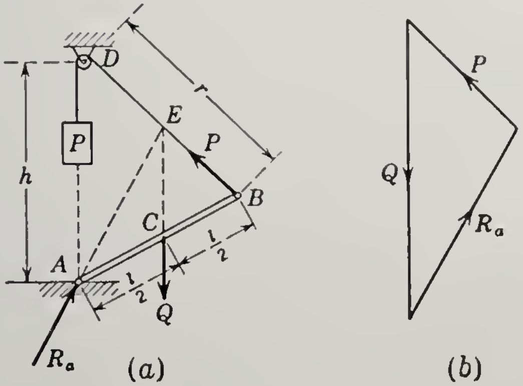
from which $R_d = 5Q/4 = 1{,}250$ lb.

A slight variation of the foregoing problem may be stated as follows: What is the magnitude and direction of the *least force* $P_{\min}$ applied at C that will lift the roller over the curb in Fig. 32*a*? To answer this, we observe that the line of action of R_d remains the same regardless of the direction of P. Hence in Fig. 32*c*, we again lay out the given vector $\bar{Q}$ and through its beginning we construct the line ac' indicating the known direction of R_d. Then dropping a perpendicular from the end of Q to this line, we obtain the vector $\overline{bc}$ representing the least pull $P_{\min}$. We see that this force applied to the roller at C must be in the direction perpendicular to DC. From the triangle of forces (Fig. 32*c*) its magnitude is found to be

$$P_{\min} = Q \sin \alpha = \tfrac{3}{5}Q = 600 \text{ lb}$$

3. A prismatic bar AB of weight Q and length l is hinged at A and supported at B by a string that passes over a pulley D and carries a load P at its free end (Fig. 33*a*). Assuming that the distance h between the hinge A and the pulley D is larger than the length l of the bar, find the configuration of equilibrium of the system as defined by the ratio of lengths r/h.

Solution. We begin with a consideration of the equilibrium of the bar AB which is acted upon by three forces: the gravity force Q applied at the mid-point C, a force P representing the pull of the string at B, and a reaction R_a at the hinge A. Extending the vertical line of action of Q to its intersection E with the string BD, we conclude that the reaction R_a must be directed along the line AE as shown. Further, since C is the mid-point of AB and since CE is parallel to AD, it follows that E is the mid-point of BD; that is, $DE = r/2$.



(a) (b)

FIG. 33

Observing that the triangle of forces (Fig. 33*b*) is similar to $\triangle DAE$, we may write

$$\frac{r}{2}:P = h:Q$$

from which

$$\frac{r}{h} = \frac{2P}{Q} \tag{a}$$

In using Eq. (a), we must observe from the figure that

$$(h - l) < r < (h + l)$$

Hence Eq. (a) can be used only if the ratio P/Q is within the following limits:

$$\frac{h - l}{2h} < \frac{P}{Q} < \frac{h + l}{2h}$$

For all other values of the ratio P/Q, the bar AB will assume a vertical position of equilibrium and Eq. (a) does not apply.

PROBLEM SET 1.5

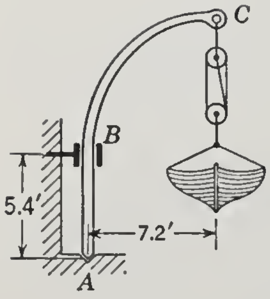


FIG. A

1. A boat is suspended on two identical davits like ABC which is pivoted at A and supported by a guide at B (Fig. A). Determine the reactions R_a and R_b at the points of support A and B if the vertical load transmitted to each davit at C is 960 lb. Friction in the guide at B should be neglected. The dimensions are shown in the figure. *Ans.* $R_a = 1{,}600$ lb; $R_b = 1{,}280$ lb.

2. A prismatic bar AB of weight $Q = 2$ tons is hinged to a vertical wall at A and supported at B by a cable BC (Fig. B). Determine the magnitude and direction of the reaction R_a at the hinge A and the tensile force S in the cable BC. The directions of the bar and the cable are as shown in the figure. *Ans.* $R_a = 1$ ton acting at 60° with the vertical; $S = \sqrt{3}$ tons.

3. A ball of weight Q and radius r is attached by a string AD to a vertical wall AB, as shown in Fig. C. Determine the tensile force S in the string and the pressure R_b against the wall at B if $Q = 8$ lb, $r = 3$ in., $AB = 4$ in. Neglect friction at wall. *Ans.* $S = 10$ lb; $R_b = 6$ lb.

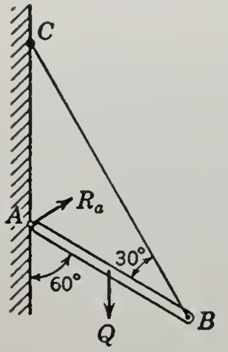


FIG. B

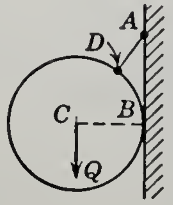


FIG. C

4. A 150-lb man stands on the middle rung of a 50-lb ladder, as shown in Fig. D. Assuming a smooth wall at B and a stop at A to prevent slipping, find the reactions at A and B. *Ans.* $R_a = 206$ lb; $R_b = 50$ lb.

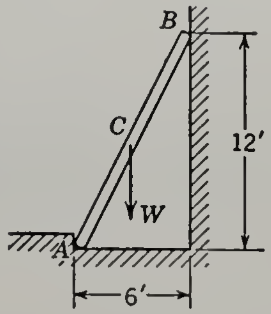


FIG. D

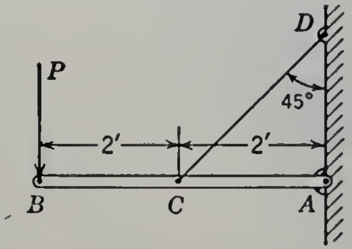


FIG. E

5. A horizontal beam AB is hinged to a vertical wall at A and supported at its mid-point C by a tie rod CD as shown in Fig. E. Find the tension S in the tie rod and the reaction at A due to a vertical load P applied at B. *Ans.* $S = 2.83P$; $R_a = 2.24P$.

6. A horizontal prismatic bar AB, of negligible weight and length l, is hinged to a vertical wall at A and supported at B by a tie rod BC that makes the angle α with the horizontal (Fig. F). A weight P can have any position along the bar as defined by the distance x from the wall. Determine the tensile force S in the tie bar. *Ans.* $S = Px/l \sin \alpha$.

***7.** A prismatic bar AB, of weight Q and length l, is supported at one end B by a string CB of length a and rests at A, vertically below C, against a perfectly smooth vertical wall (Fig. G). Find the position of the bar, as defined by the length x, for which equilibrium will be possible. *Ans.* $x = \sqrt{(a^2 - l^2)/3}$.

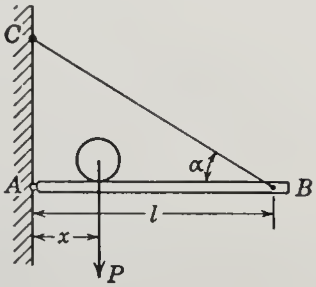


FIG. F

8. A prismatic bar AB of weight $W = 14$ lb and length $l = 8$ ft is hinged to a vertical wall at A and supported at its other end B by a horizontal strut BC (Fig. H). Find the compressive force S induced in the strut and the reaction R_a at A if $\alpha = 25°$. *Ans.* $S = 15$ lb; $R_a = 20.5$ lb.

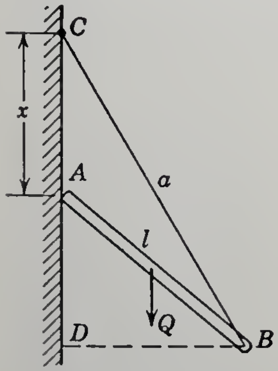


FIG. G

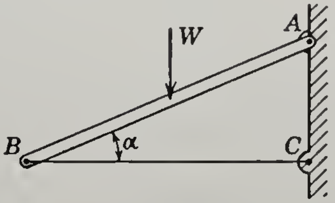


FIG. H

9. A weightless bar AB is supported in a vertical plane by a hinge at A and a tie bar DC, as shown in Fig. I. Determine, graphically, the axial force S induced in the tie bar by the action of a vertical load P applied at B. Angle $ACD = 45°$. *Ans.* $S = 2P$, tension.

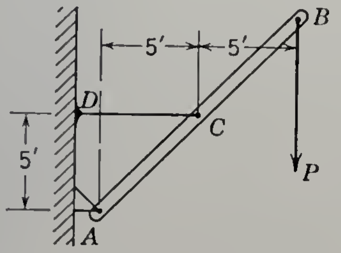


FIG. I

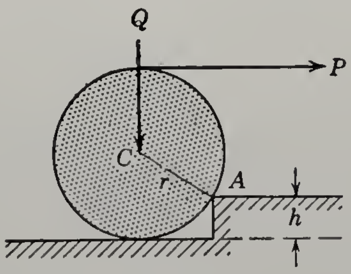


FIG. J

10. A roller of radius $r = 12$ in. and weight $Q = 500$ lb is to be pulled over a curb of height $h = 6$ in. by a horizontal force P applied to the end of a string wound around the circumference of the roller (Fig. J). Find the magnitude of P required to start the roller over the curb. *Ans.* $P = 288$ lb.

11. A bar AB hinged to the foundation at A and supported by a strut CD is subjected to a horizontal 5-ton load at B, as shown in Fig. K. Find graphically the tensile force S in the strut and the reaction R_a at A. *Ans.* $S = 5.55$ tons; $R_a = 5.00$ tons.

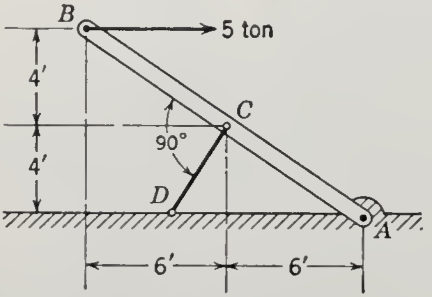


FIG. K

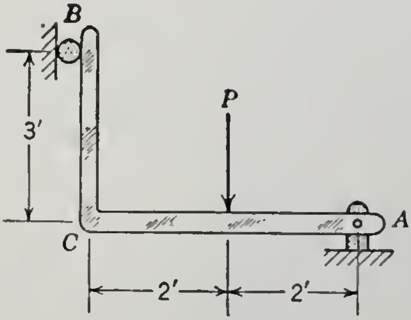


FIG. L

12. Find graphically the reactions R_a and R_b induced at the supports A and B of the right-angle bar ACB supported as shown in Fig. L, and subjected to a vertical load P applied at the mid-point of AC. *Ans.* $R_a = 1.2P$; $R_b = 0.67P$.

1.6. Method of moments. *Moment of a Force with Respect to a Point.* The concept that a force tends to produce rotation about a fixed point in any body to which it is applied is useful in the solution of problems of statics. Consider, for example, the wrench shown in Fig. 34, to the handle of which two equal forces P and Q are applied as shown. It is a matter of common experience that the force P acting at right angles to the handle is more effective in tending to turn the nut to which the wrench is fitted than is the force Q, even though they are of equal magnitude. The effectiveness or importance of a force, as regards its tendency to produce rotation of a body about a fixed point, is called the *moment* of the force with respect to that point, and this moment can be measured by the product of the magnitude of the force and the distance from the point to the line of action of the force. Thus, the magnitude of the moment of the force Q with respect to the point O (Fig. 34) depends not only upon the magnitude of the force itself but also upon the distance OD from the point O to its line of action. The point O is called the *moment center,* and the distance OD is called the *arm of the force.*

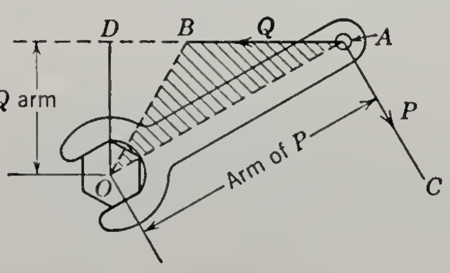


FIG. 34

From the preceding definition, it follows that the moment of the force Q is numerically equal to the doubled area of $\triangle ABO$, constructed on the vector $\overline{AB}$ representing the force, and having its vertex at the moment center. In this calculation the vector $\overline{AB}$ should be measured to the scale used for representing force, while the arm OD should be measured to the scale used for length. Thus it is seen that the unit of moment of force is the unit of length times the unit of force. For example, taking the *pound* as the unit of force and the *foot* as the unit of length, the unit of moment of force will be the *foot-pound*, usually written ft-lb. Another unit of moment of force, very commonly used in engineering, is the *inch-pound*, usually written in.-lb.

Where several concurrent forces in one plane are involved, it will generally be found that some of the forces will tend to produce rotation in one direction around a given moment center while others will tend to produce rotation in the opposite direction around the same center. It is customary, when dealing with several forces in a plane to consider as positive the moments of those forces tending to produce counterclockwise rotation about the moment center and as negative the moments of those forces tending to produce clockwise rotation. Thus, in accordance with this rule, the moment of the force Q with respect to the center O (Fig. 34) is positive while that of the force P, with respect to the same center, is negative.

From the definition of the moment of a force with respect to a point, it follows that the magnitude of this moment does not change (1) if the point of application of the force is transmitted along its line of action or (2) if the moment center is moved along a line parallel to the line of action of the force. It is seen also that the moment of a force which is not zero can become equal to zero only if the arm of the force is zero, *i.e.*, only if the moment center lies on the line of action of the force.

Theorem of Varignon. The moment of the resultant of two concurrent forces with respect to a center in their plane is equal to the algebraic sum of the moments of the components with respect to the same center. To prove this statement, which is called the *theorem of Varignon*, we begin with the case where the moments of both forces P and Q with respect to a center O (Fig. 35a) have the same sign. In the plane of action of the forces, we take any line mn perpendicular to the line OA joining the moment center with the point of concurrence of the forces and construct the perpendiculars Aa, Bb, Cc, and Dd, as shown in the figure. Now the area of $\triangle OAB = \frac{1}{2}OA \cdot ab$, the area of $\triangle OAC = \frac{1}{2}OA \cdot ac$, and the area of $\triangle OAD = \frac{1}{2}OA \cdot ad$. Since

$$ad = ab + bd = ab + ac$$

we conclude that

$$\text{Area } \triangle OAD = \text{area } \triangle OAB + \text{area } \triangle OAC$$

which proves the theorem.

If the signs of the moments of the two forces P and Q are different (Fig. 35*b*), we proceed as in the previous case and from the fact that $ad = ab - db = ab - ac$ we conclude that

$$\text{Area } \triangle OAD = \text{area } \triangle OAB - \text{area } \triangle OAC$$

i.e., the numerical value of the moment of the resultant R is equal to the difference of the numerical values of the moments of the components P and Q, and since, following the assumed rule, the moment of the force Q is negative, it can be concluded that in this case again the moment of the resultant is equal to the algebraic sum of the moments of the components.

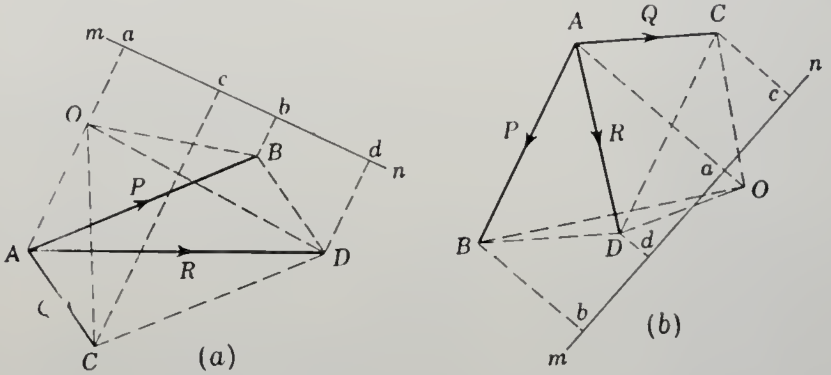


FIG 35

If we have several concurrent forces $F_1, F_2, \ldots, F_n$ acting in a plane, it can be proved by successive applications of the theorem of Varignon that the moment of their resultant R with respect to a given center in the plane of the forces is equal to the algebraic sum of the moments of the components with respect to the same center. The procedure is as follows: Applying the theorem first to the forces F_1 and F_2, the algebraic sum of the moments of these two forces can be represented by the moment of their resultant R_1. Next, combining this partial resultant R_1 with the force F_3, the algebraic sum of the moments of the first three forces F_1, F_2, and F_3 is represented by the moment of their resultant R_2 and so on until the last force F_n has been taken. Thus the above conclusion is reached. Denoting by M_0 the moment of the resultant with respect to the center O and by $(M_0)_1$, $(M_0)_2, \ldots, (M_0)_n$ the moments of the various components with respect to the same center, the above theorem of moments may be expressed analytically by the equation

$$M_0 = (M_0)_1 + (M_0)_2 + \cdots + (M_0)_n = \Sigma(M_0)_i \qquad (4)$$

where the summation is understood to include all forces in the system.

In calculating the moment $(M_0)_i$ with respect to the center O of any force F_i, it is sometimes advantageous to replace the force by its two rectangular components X_i and Y_i parallel, respectively, to the coordinate axes x and y taken through the moment center O (Fig. 36). If x_i and y_i are the coordinates of the point of application A of the force F_i, we conclude from the theorem of Varignon that

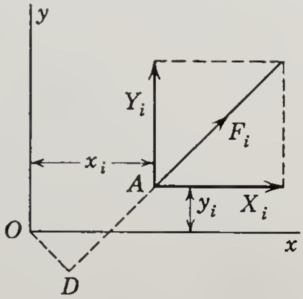


FIG. 36

$$(M_0)_i = F_i \cdot OD = Y_i x_i - X_i y_i \qquad (5)$$

Equilibrium Equations. From Eq. (4) it follows that the algebraic sum of moments of a system of concurrent forces in a plane becomes equal to zero (1) if the center of moments lies on the line of action of the resultant or (2) if the resultant is equal to zero and the forces are in equilibrium. These two conclusions are useful in deciding the question of equilibrium of a system of concurrent forces in a plane.

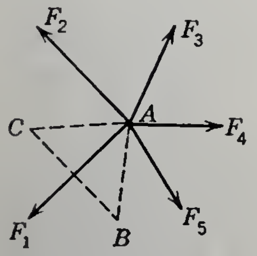


FIG. 37

Consider, for example, the system of forces shown in Fig. 37. It follows from the preceding discussion that, if the algebraic sum of moments of these forces with respect to the center B is zero, either the resultant is zero or else it lies along the line AB. If the algebraic sum of moments with respect to the center C which is not on the line AB is also zero, it follows that the resultant must be zero, since it cannot have both AC and AB as its line of action. Expressing these two conditions of equilibrium analytically, we have

$$\Sigma(M_B)_i = 0 \qquad \Sigma(M_C)_i = 0 \qquad (6)$$

where B and C are any two moment centers in the plane of the forces and not on a straight line with the point of concurrence of the system. These equations are equivalent to Eqs. (3) of Art. 1.4, but their use in the solution of problems of statics often results in considerable simplification. Also it is frequently advantageous, when dealing with the conditions of equilibrium of a system of concurrent forces in a plane, to use one each of Eqs. (3) and (6).

EXAMPLES

1. A prismatic bar AB is hinged at A and supported at B as shown in Fig. 38. Neglecting friction, determine the reaction R_b produced at B owing to the weight Q of the bar.

Solution. Neglecting friction, the reaction R_b must act normal to the axis of the bar, as shown. Since the bar is in equilibrium under the action of the three forces R_b, Q, and a reaction (not shown) at the hinge A, the algebraic sum of the moments of these forces with respect to any center in their plane must accordingly be equal to zero. Taking point A as the moment center (thus eliminating consideration of the unknown reaction at A), we obtain

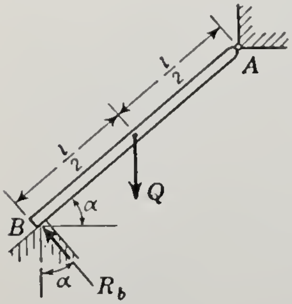


FIG. 38

$$Q\frac{l}{2}\cos\alpha - R_b l = 0$$

from which

$$R_b = \tfrac{1}{2}Q\cos\alpha$$

2. Figure 39 represents the cross section of a retaining wall supporting an earth fill. The earth pressure per foot of length of wall can be replaced by its resultant H = 2 tons/ft acting as shown in the figure. Determine the minimum thickness b of the wall required to prevent overturning about the front edge A if h = 15 ft and the specific gravity of the wall is $2\frac{1}{2}$.

Solution. Let us assume that we are dealing with a unit section of the wall 1 ft in length normal to the plane of the figure. If such a section is made stable, then a similar wall of any length will be equally stable. When conditions are such that overturning of the wall impends the reaction R_a exerted by the foundation on the bottom of the wall will be concentrated at the front edge A as shown in the figure. Thus the 1-ft section of wall is in equilibrium under the action of three forces: the weight

$$Q = 2\tfrac{1}{2} \times 62.4 \times 15b = 2{,}340b \text{ lb}$$

acting at the center of the cross section, the earth pressure

$$H = 2 \text{ tons} = 4{,}000 \text{ lb},$$

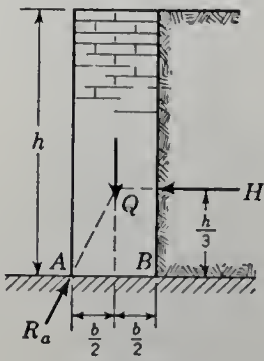


FIG. 39

and the reaction R_a. The algebraic sum of the moments of these forces with respect to any center in the plane of the cross section must be equal to zero. Taking point A as the moment center, we obtain

$$H\frac{h}{3} - Q\frac{b}{2} = 0$$

Substituting the numerical data as given and solving for b gives

$$b = \sqrt{17.09} = 4.13 \text{ ft}$$

3. A slender prismatic bar AB of weight Q and length $2l$ rests on a very small frictionless roller at D and against a smooth vertical wall at A, as shown in Fig. 40. Find the angle α that the bar must make with the horizontal in the condition of equilibrium.

Solution. Isolating the bar AB, we obtain the free-body diagram as shown in the figure. The reaction at A is normal to the wall, that is, horizontal, and the reaction at D is normal to AB. When the bar is in a condition of equilibrium, the three forces Q, R_a, and R_d meet in one point and the algebraic sum of their projections on any axis must be zero [Eqs. (3)]. Likewise, the algebraic sum of moments of all forces with respect to any point in their plane of action must be zero [Eqs. (6)].

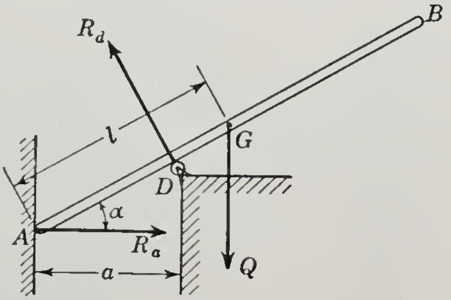


FIG. 40

Projecting the forces onto a vertical axis, we obtain

$$R_d \cos \alpha - Q = 0$$

Taking moments about point A, we obtain

$$\frac{R_d a}{\cos \alpha} - Ql \cos \alpha = 0$$

Eliminating the unknown reaction R_d from these two equations, we have

$$\cos \alpha = \sqrt[3]{\frac{a}{l}}$$

defining the position of equilibrium of the bar.

4. A prismatic circular log of weight $2Q$ and radius r is supported in a horizontal position against a vertical wall by two identical brackets like the one shown in Fig. 41*a*. Each bar AB is hinged to the wall at A and supported at B by a horizontal cable BD. Assuming smooth surfaces at all points of contact, find the value of the angle α that AB should make with the wall to attain a minimum tension S in each cable BD.

Solution. Since each bracket supports half the weight of the log, we assume a coplanar system and consider first the equilibrium of a disk of weight Q supported as shown in Fig. 41*a*. The free-body diagram for this disk is shown in Fig. 41*b*. Since the three concurrent forces Q, R_f, and R_e are in equilibrium, the algebraic sum of their projections on the y axis must be zero. From this condition we have

$$R_e \sin \alpha - Q = 0$$

from which $R_e = Q/\sin \alpha$.

The action of the log on the bar AB is equal and opposite to this as shown on the free-body diagram in Fig. 41*c*. In addition to R_e, we have acting on the bar AB the horizontal force S at B, representing the cable tension, and a reaction R_a at the hinge A. These three coplanar forces are in equilibrium and the algebraic sum of their moments with respect to any point in their plane is zero. Using point A as a moment center, we obtain

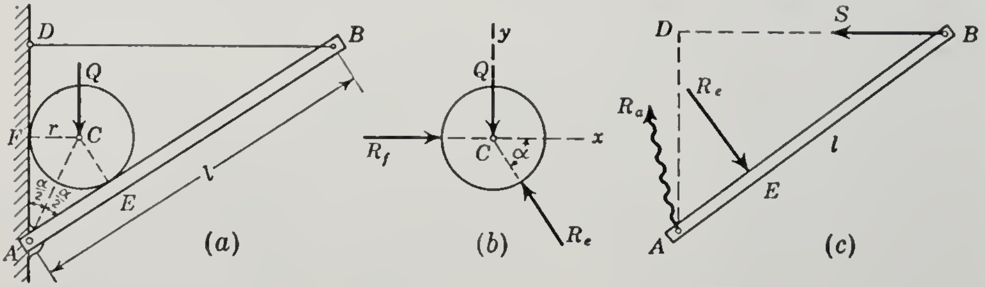


FIG. 41

$$S \cdot AD - R_e \cdot AE = 0$$

from which, using the value for R_e found above,

$$S = R_e \frac{AE}{AD} = \frac{Q}{\sin \alpha} \frac{AE}{AD} \tag{a}$$

Noting from the figure that $AE = r \cot \alpha/2$ and $AD = l \cos \alpha$, expression (a) may be written in the form

$$S = \frac{Qr}{2l \sin^2 (\alpha/2) \cos \alpha} \tag{b}$$

An examination of Eq. (b) shows that for $\alpha = 0$ and for $\alpha = \pi/2$ the tensile force S becomes infinitely large and that for all values of α between these two extremes it will be finite. It follows then that there must be some value of α between 0 and $\pi/2$ for which the value of S is a minimum. To determine this particular value of α, we differentiate expression *(b)* with respect to α and set the derivative $dS/d\alpha = 0$, which gives for determining α the following equation:

$$\sin \alpha \left(4 \sin^2 \frac{\alpha}{2} - 1\right) = 0 \tag{c}$$

From Eq. (c), the solution

$$\sin \frac{\alpha}{2} = \frac{1}{2} \tag{d}$$

is obviously the one corresponding to the minimum value of S as given by Eq. (b). Using the value (d) in Eq. (b), we obtain for the minimum tensile force

$$S_{\min} = \frac{4Qr}{l} \tag{e}$$

Substituting the given numerical data in Eq. (e) we find $S_{\min} = 400$ lb.

PROBLEM SET 1.6

1. If the piston of the engine in Fig. A has a diameter of 4 in. and the gas pressure in the cylinder is 100 psi, calculate the turning moment M exerted on the crankshaft for the particular configuration shown. *Ans.* $M = 8{,}800$ in.-lb.

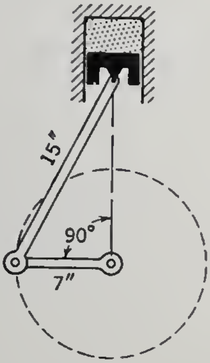


FIG. A

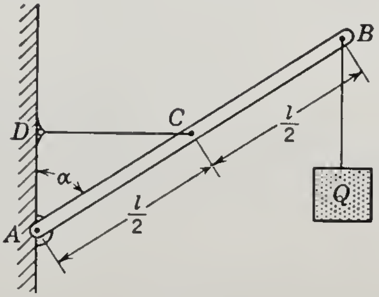


FIG. B

2. A rigid bar AB is supported in a vertical plane and carries a load Q at its free end as shown in Fig. B. Neglecting the weight of the bar itself, compute the magnitude of the tensile force S induced in the horizontal string CD. *Ans.* $S = 2Q \tan \alpha$.

3. A beam AB, hinged at A and supported at B by a vertical bar BC, is subjected to the action of a force P applied as shown in Fig. C. Assuming ideal hinges at A, B, and C, find the force S produced in the bar BC. Neglect the weight of the beam. *Ans.* $S = 0.354P$, compression.

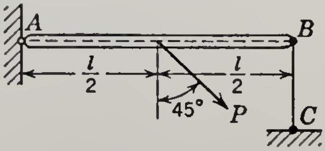


FIG. C

4. Along a ladder supported at A and B, as shown in Fig. D, a vertical load W can have any position as defined by the distance a from the bottom. Neglecting friction, determine the magnitude of the reaction R_b at B. Neglect the weight of the ladder. *Ans.* $R_b = Wa/l \tan \alpha$.

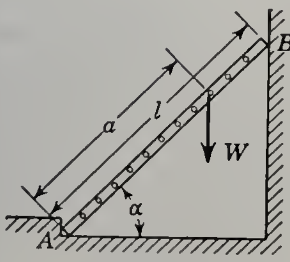


FIG. D

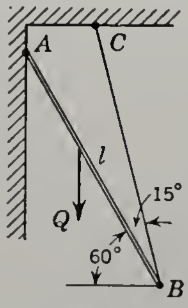


FIG. E

5. A bar AB of length l is supported as shown in Fig. E. At any point along its length a vertical load Q can be applied. Determine the position of this load for which the tensile force S in the cable BC will be a maximum and evaluate same if the various angles are as shown in the figure. In calculation neglect the weights of the bar and the cable. *Ans.* $S_{\max} = 1.93Q$.

6. A pair of adjustable pliers are used for turning a piece of $\frac{3}{4}$-in. pipe as shown in Fig. F. For the dimensions shown, what compressive forces Q are applied to the sides of the pipe when the hand grip is represented by applied collinear forces P as shown? *Ans.* $Q = 6P$.

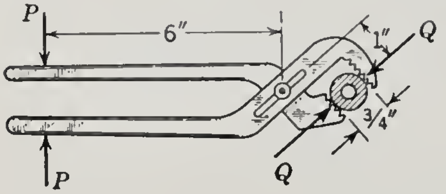


FIG. F

7. A vertical load P is supported by a triangular bracket as shown in Fig. G. Find the forces transmitted to the bolts A and B. Assume that the bolt B fits loosely in a vertical slot in the plate. *Ans.* $R_a = 1.25P$; $R_b = 0.75P$.

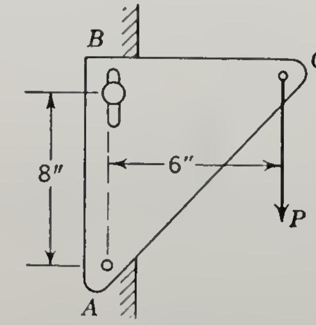


FIG. G

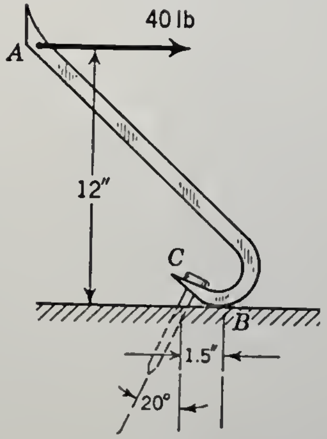


FIG. H

8. Find the magnitude of the pull P exerted on the nail C in Fig. H if a horizontal force of 40 lb is applied to the handle of the wrecking bar as shown. *Ans.* $P = 340$ lb.

9. Determine the forces exerted on the cylinder at B and C by the spanner wrench shown in Fig. I due to a vertical force of 50 lb applied to the handle as shown. Neglect friction at B. *Ans.* $R_b = 240$ lb; $R_c = 245$ lb.

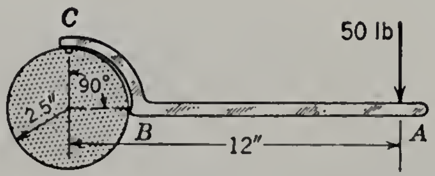


FIG. I

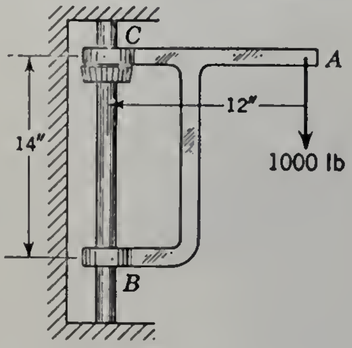


FIG. J

10. A bracket ACB can slide freely on the vertical shaft BC but is held by a small collar attached to the shaft as shown in Fig. J. Neglecting all friction, find the reactions at B and C for the vertical load shown. *Ans.* $R_b = 857$ lb; $R_c = 1{,}317$ lb.

11. Two beams AB and DE are arranged and supported as shown in Fig K. Find the magnitude of the reaction R_e at E due to the force $P = 200$ lb applied at B as shown. *Ans.* $R_e = 100$ lb.

12. A smooth right circular cylinder of radius r rests on a horizontal plane and is kept from rolling by an inclined string AC of length $2r$ (Fig. L). A prismatic bar AB of length $3r$ and weight Q is hinged at point A and leans against the roller as shown. Find the tension S that will be induced in the string AC. *Ans.* $S = 0.433Q$.

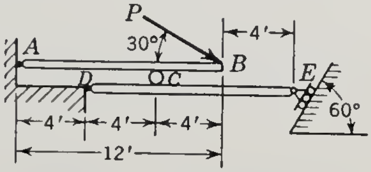


FIG. K

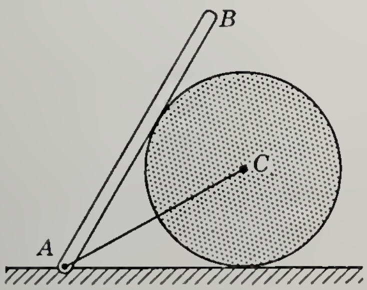


FIG. L

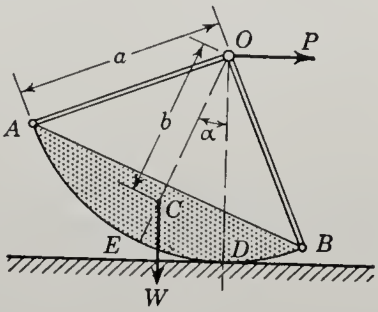


FIG. M

13. A rocker of weight W having a circular shoe AB of radius a and with center at O rests on a horizontal surface and is pulled by a horizontal force P applied at O, as shown in Fig. M. Find the position of equilibrium, as defined by the angle α, which the rocker will assume if its center of gravity is at C, distance b from O along the bisecting radius OE. *Ans.* $\sin \alpha = Pa/Wb$.

1.7. Friction. In all preceding examples where the reaction exerted by a supporting surface was involved, we have assumed the surface to be perfectly smooth so that it could exert only a normal reactive force. While such an assumption of ideal conditions will be entirely justifiable in some cases, this will not always be true. Sometimes the resistance to sliding between two contiguous surfaces will be the dominant factor in determining equilibrium and in such cases, of course, it cannot be neglected. Whenever the surfaces of two bodies are in contact there will be a limited amount of resistance to sliding between them, which is called *friction.* Consider, for example, two plates to be pressed together by normal forces N (Fig. 42). To overcome friction and cause sliding between the contiguous surfaces, certain forces F, acting in the plane of contact, will be required.

The question of friction between clean dry surfaces was first investigated in a complete manner by Coulomb, who published in 1781 the results of a large number of experiments.[1] For given dry surfaces in contact, the results of these experiments may be summarized briefly by the following laws of friction.

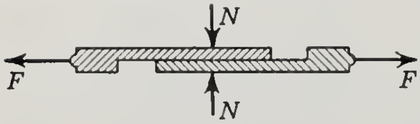


FIG. 42

1. *The total friction that can be developed is independent of the magnitude of the area of contact.*

2. *The total friction that can be developed is proportional to the normal force.*

3. *For low velocities of sliding the total friction that can be developed is practically independent of the velocity,* although the experiments show that the force F necessary to start sliding is greater than that necessary to maintain sliding.

These laws of friction may be expressed by the simple formula

$$F = \mu N \tag{7}$$

where μ is called the *coefficient of friction.* If F is taken as the force necessary to start sliding, μ is called the coefficient of *static friction.* If F is taken as the somewhat smaller force necessary to maintain sliding, once it has been started, μ is called the coefficient of *kinetic friction.* The coefficients of static and kinetic friction vary greatly for different materials and for different conditions of their surfaces. Table 1 lists approximate values of coefficients of friction for various materials.

Table 1. Coefficients of Friction

Materials	Static friction		Kinetic friction	
	μ	φ°	μ	φ°
Leather on wood	0.5 –0.6	27–31	0.3–0.5	17–27
Leather on metal	0.3 –0.5	17–27	About 0.3	About 17
Masonry on dry clay	About 0.5	About 27		
Metal on metal	0.15–0.25	8–14	About 0.1	About 6
Metal on wood	0.4 –0.6	22–31	0.3–0.5	17–27
Rope on wood	0.5 –0.8	27–39	About 0.5	About 27
Stone on stone	0.6 –0.7	31–35		
Stone on wood	About 0.4	About 22		
Wood on wood	0.4 –0.7	22–35	About 0.3	About 17
Steel on ice	About 0.03	About 2	0.015	About 1

---

[1] See C. A. Coulomb, "Théorie des machines simples," Paris, 1821.

To see how friction affects the reactions exerted by supporting surfaces, let us consider the simple case of a small block resting upon a horizontal plane surface and acted upon by a force P making the angle α with the vertical (Fig. 43a). We shall assume, for simplicity, that the force P is large in comparison with the weight of the block so that the gravity force can be neglected or, if preferred, P may be considered as the resultant of the gravity force and some other force not shown. The actual distribution of pressure over the area of contact between the block and the plane will depend upon the point of application of the force P and also upon the angle α, but so long as equilibrium exists, it is evident that this distributed pressure must be sensibly equivalent to a reactive force R which will be equal, opposite, and collinear with the applied force P. If we replace this reaction R by its two components F and N, acting tangentially and normally, respectively, to the surface of contact as shown in the figure, the component F will represent the friction between the surfaces and the component N, the normal force. Thus the condition of equilibrium requires that the relation between the components F and N must be

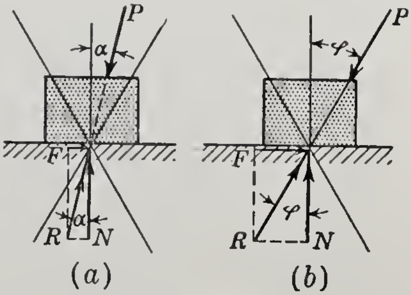


FIG. 43

$$\frac{F}{N} = \tan \alpha \tag{a}$$

When the applied force P makes a certain limiting angle φ with the vertical (Fig. 43b) such that sliding of the block impends, we have, as before, as a condition of equilibrium, that

$$\frac{F}{N} = \tan \varphi \tag{b}$$

and also as a condition of impending sliding (see Eq. 7) that

$$\frac{F}{N} = \mu \tag{c}$$

From Eqs. (b) and (c) we obtain the important relation

$$\tan \varphi = \mu \tag{8}$$

This limiting angle φ, equal to the arc tangent of the coefficient of friction, is called the *angle of friction*.

In the preceding discussion the force P was taken as acting in the plane of the figure. However, we can generalize this discussion and conclude that, so long as the line of action of the applied force P is completely within a certain cone, generated by a line making the angle of static friction φ with the normal to the surface of contact and having its vertex in this surface, the block will remain in equilibrium regardless of the magnitude of the force. This cone is called the *cone of static friction.*

To summarize: Whenever motion is impending, the total reaction R, furnished by the supporting surface, is inclined to the normal by the angle of static friction φ and acts so as to oppose the impending motion (Fig. 43*b*). When motion is not impending, the total reaction R will be inclined to the normal by only so much as is necessary to maintain equilibrium (Fig. 43*a*). If we assume an ideal smooth surface for which the coefficient of friction is zero, the angle of friction is zero and the total reaction R is normal to the surface as assumed heretofore.

In the solution of problems where friction is involved, we may deal either with the total reaction R or with its rectangular components F and N. Usually the components F and N will be more convenient when we are using the algebraic method of projections and the single force R when we are working graphically. Various applications of the foregoing laws of friction will now be considered in the following examples.

EXAMPLES

1. A ladder AB of length l is supported by a horizontal floor at A and by a vertical wall at B and makes an angle α with the horizontal (Fig. 44). Find the maximum distance x up the ladder at which a man of weight W can stand without causing slipping to occur, if the angle of friction between floor and ladder and between wall and ladder is φ. Neglect the weight of the ladder itself.

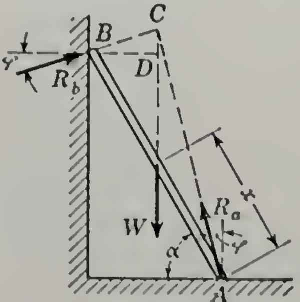


FIG. 44

Solution. When slipping impends, the reactions at A and B will each be inclined to the normal by the angle of friction φ as shown. Prolonging these directions to the intersection C (Fig. 44), we conclude that the vertical gravity force W must also pass through this point, since it is the third of three forces in equilibrium. If the drawing has been constructed to scale, the distance x may be measured directly. Otherwise it may be computed by observing that the distance $DB = l \cos(\alpha + \varphi) \cos \varphi$

and that $x = l - BD \sec \alpha$, from which we finally obtain

$$x = l[1 - \cos (\alpha + \varphi) \cos \varphi \sec \alpha] \tag{d}$$

We see that for the case of perfectly smooth surfaces ($\varphi = 0$) the value of x given by Eq. (d) is zero and some sort of stop at A will be necessary to prevent slipping of the ladder. Again, from Eq. (d), we see that, when $\alpha \geqq 90° - \varphi$, the man may stand safely at the top of the ladder. That is, so long as the ladder makes an angle with the vertical not greater than the angle of friction, there will be no danger of slipping.

2. A rectangular block of width $2b$ and height h rests on a horizontal plane surface (Fig. 45). Find the magnitude P and the height c of the line of action of a horizontal force that will cause tipping and sliding of the block to impend simultaneously. The weight of the block is W, and the coefficient of static friction between the block and the plane is μ.

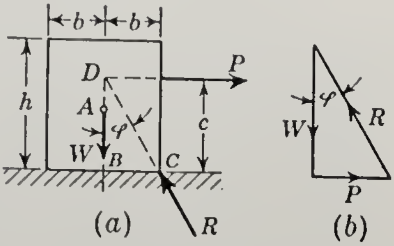


FIG. 45

Solution. When motion of the block impends, it will be in equilibrium under the action of three forces: the gravity force W, which may be considered as applied at the center A of the block; the applied force P; and the reaction R exerted by the floor. As previously pointed out, the actual distribution of pressure over the bottom face of the block, and consequently the point of application of the concentrated reaction R that will produce sensibly the same effect, depends upon the point of application of the applied force P. Now, it is quite evident that when the pressure between the bottom of the block and the plane is concentrated at C the block will be just on the verge of overturning. Thus, taking point C as the point of application of the reaction R and knowing that, to have also impending sliding, it must make the angle of static friction φ with the vertical, its line of action is completely determined, the prolongation of which intersects the line of action of the gravity force W at point D. Since three forces in equilibrium must meet in one point, this determines the point of application of the horizontal force P that will cause tipping and sliding of the block to impend simultaneously. From $\triangle DBC$ (Fig. 45a) the critical height c is found to be

$$c = \frac{b}{\mu} \tag{e}$$

From Eq. (e) it is seen that if we have a block of such proportions that $b/h > \mu$, the block will slide before overturning, regardless of the point of application of the horizontal force P.

From the triangle of forces (Fig. 45b) the magnitude of the horizontal force P necessary to cause sliding of the block to impend is found to be

$$P = W \tan \varphi = \mu W \tag{f}$$

As would be expected, comparison of Eq. (f) with Eq. (7) shows that this horizontal force is equal to the total friction that can be developed between the contiguous surfaces. It will be noted also that the magnitude of this force is independent of its point of application, provided, of course, that sliding impends before overturning takes place.

3. A man wishes to slide a heavy stone block of weight W over a level concrete floor by pulling on it with a rope AB, as shown in Fig. 46*a*. What angle α should the rope make with the horizontal in order that the pull necessary to cause sliding to impend will be a minimum? What will be the magnitude of this minimum force if the angle of friction is φ?

Solution. When conditions are such that sliding of the block impends, it will be in equilibrium under the action of three forces: the gravity force W, the pull $P_{\min}$, and the reaction R which is the resultant of the distributed pressure exerted on the block by the floor (Fig. 46*a*). These three forces must meet in one point and must build a closed triangle when geometrically added. To construct this triangle of forces, we begin with the known vector W, and from the end of this we draw the line that makes the angle of friction φ with it and that is the known limiting direction of the reaction R when motion impends. It is now evident that the shortest vector $P_{\min}$ which will make the closing side of the triangle is one at right angles to the reaction R. Thus we conclude that the least force that will cause sliding of the block to impend will be one making the angle of friction φ with the plane of the floor. That is, $\alpha = \varphi$.

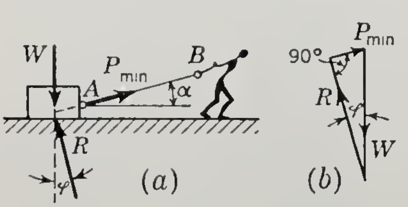


FIG. 46

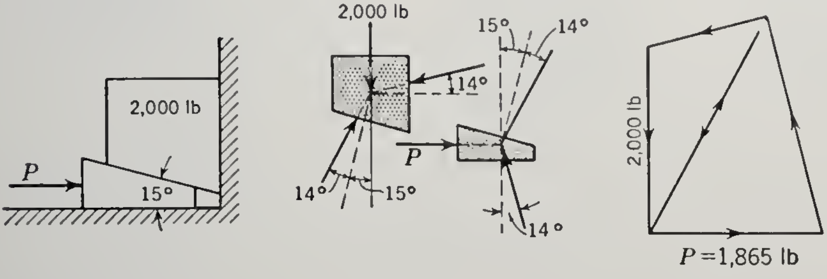
From the triangle of forces (Fig. 46*b*) the magnitude of this least force with which the block can be made to slide is found to be

$$P_{\min} = W \sin \varphi \tag{g}$$

Comparing Eqs. (g) and (f), it is seen that, for cases in which the coefficient of friction is fairly large, considerable effort in sliding a heavy block over a rough surface will be saved by pulling along a line that makes the angle of friction with the plane of sliding. For example, in the case of stone sliding on concrete (assuming $\mu = 0.6$) the least force $P_{\min}$, as given by Eq. (g), will be only 86 per cent of the horizontally applied force P as given by Eq. (f).

4. To raise a heavy stone block weighing 1 ton, the arrangement shown in Fig. 47*a* is used. What horizontal force P will it be necessary to apply to the wedge in order to raise the block if the coefficient of friction for all contiguous surfaces is $\mu = \frac{1}{4}$? Neglect the weight of the wedge.

Solution. For $\mu = \frac{1}{4}$, we have the angle of friction $\varphi \approx 14°$. Then for the condition of impending upward motion of the block, the free-body diagrams for the block and wedge, respectively, will be as shown in Fig. 47*b*. Note that each reactive force inclines to the corresponding normal by the angle of friction and in such a way as to oppose the impending motion. Each body is in equilibrium under the action of three forces that must consequently meet in one point and build a closed triangle. The triangle of forces for the block must be constructed first, after which the one for the wedge may be drawn. These triangles of forces are shown in Fig. 47c, which is made to scale and from which we find $P = 1{,}865$ lb.

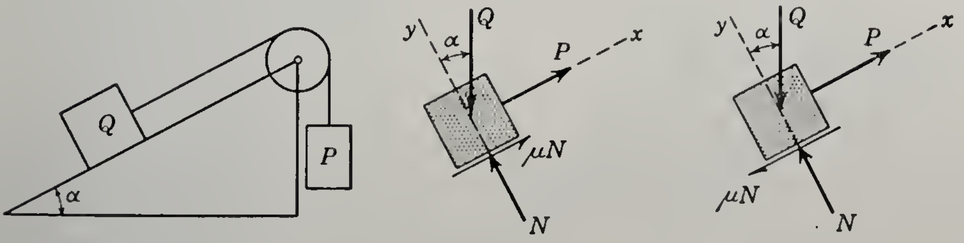


(a) (b) (c)

FIG. 47

5. A block of weight Q rests on an inclined plane and has attached to it a string that overruns a pulley and carries a weight P at its other end as shown in Fig. 48a. If the coefficient of friction between the block Q and the inclined plane is μ, find the limiting values of the ratio P/Q consistent with equilibrium. Neglect friction in the pulley, and assume that the angle of inclination α of the plane is greater than the angle of friction $\varphi = \arctan \mu$.

Solution. If the ratio P/Q is too small, the block Q will slide down the plane pulling the weight P behind it. On the other hand, if the ratio P/Q is too large, the weight P will fall, pulling the block Q up the plane. Hence we conclude that there is a minimum value of P/Q for which slipping of the block Q down the plane impends and a maximum value of P/Q for which slipping of the block Q up the plane impends. The free-body diagram of this block for impending motion down the plane is shown in Fig. 43b; that for impending motion up the plane, in Fig. 48c.



(a) (b) (c)

FIG. 48

In the first case (Fig. 48b), the equations of equilibrium [Eqs. (3)] become

$$P - Q \sin \alpha + \mu N = 0 \qquad -Q \cos \alpha + N = 0$$

and we obtain

$$\left(\frac{P}{Q}\right)_{\min} = \sin \alpha - \mu \cos \alpha \tag{h}$$

In the second case (Fig. 48c), the equations of equilibrium become

$$P - Q \sin \alpha - \mu N = 0 \qquad -Q \cos \alpha + N = 0$$

and we obtain

$$\left(\frac{P}{Q}\right)_{\max} = \sin \alpha + \mu \cos \alpha \tag{i}$$

Equations (h) and (i) can be used for any given numerical data so long as $\alpha > \arctan \mu$.

PROBLEM SET 1.7

1. What must be the angle α between the plane faces of a steel wedge used for splitting logs if there is to be no danger of the wedge slipping out after each blow of the sledge? *Ans.* $\alpha \leqq 2\varphi$.

2. A flat stone slab rests on an inclined skidway that makes an angle α with the horizontal. What is the condition of equilibrium if the angle of friction is φ? *Ans.* $\alpha \leqq \varphi$.

3. What is the necessary coefficient of friction between tires and roadway to enable the four-wheel-drive automobile in Fig. A to climb a 30 per cent grade? *Ans.* $\mu \geqq 0.3$.

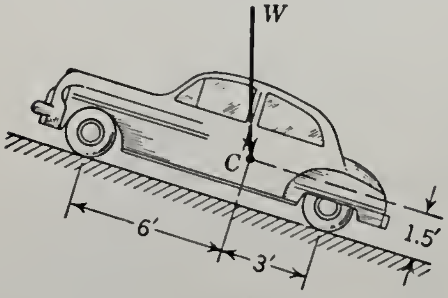


FIG. A

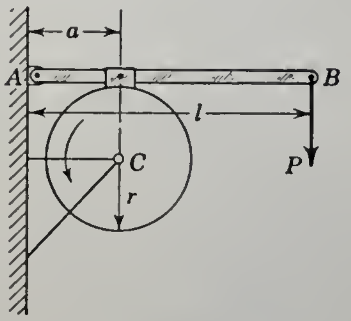


FIG. B

4. A heavy rotating drum of radius r is supported in bearings at C and is braked by the device shown in Fig. B. Calculate the braking moment M_c with respect to point C if the coefficient of kinetic friction between drum and brake shoe is μ. *Ans.* $M_c = \mu Plr/a$.

5. To determine experimentally the coefficient of friction for steel on steel, flat plates, of negligible weight compared with the large top weight W, are stacked on a horizontal plane as shown in Fig. C. Alternate plates are held together by loose-fitting vertical pins A and B. The pin A is anchored to a steel slab, and a horizontal pull applied to the pin B as shown. If there are five movable plates and slipping occurs when the horizontal pull has the magnitude P, what is the coefficient of friction μ? *Ans.* $\mu = P/10W$.

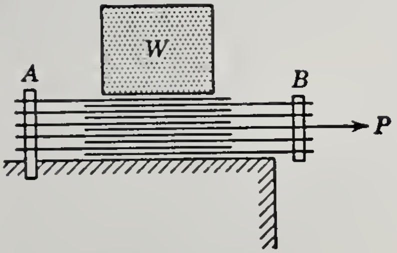
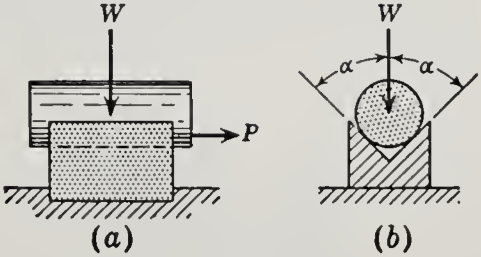


FIG. C



(a)

(b)

FIG. D

6. A short right circular cylinder of weight W rests in a horizontal V notch having the angle 2α as shown in Fig. D. If the coefficient of friction is μ, find the horizontal force P necessary to cause slipping to impend. *Ans.* $P = \mu W/\sin \alpha$.

*7. The ends of a heavy prismatic bar AB are supported by a circular ring in a vertical plane as shown in Fig. E. If the length of the bar is such that it subtends an angle of 90° in the ring and the angles of friction at A and B are each φ, what is the greatest angle of inclination θ that the bar can make with the horizontal in a condition of equilibrium? *Ans.* $\theta \leqq 2\varphi$.

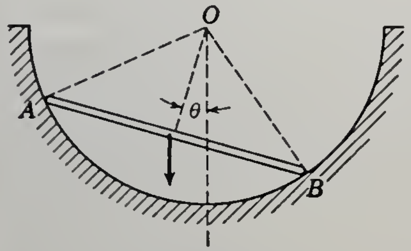


FIG. E

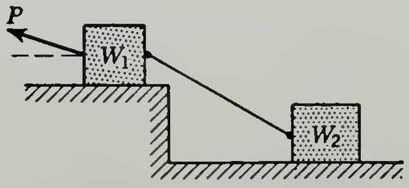


FIG. F

8. Two blocks having weights W_1 and W_2 are connected by a string and rest on horizontal planes as shown in Fig. F. If the angle of friction for each block is φ, find the magnitude and direction of the least force P applied to the upper block that will induce sliding. *Ans.* $P_{\min} = (W_1 + W_2) \sin \varphi$.

9. Two blocks connected by a horizontal link AB are supported on two rough planes as shown in Fig. G. The coefficient of friction for block A on the horizontal plane is $\mu = 0.4$. The angle of friction for block B on the inclined plane is $\varphi = 15°$. What is the smallest weight W of block A for which equilibrium of the system can exist? *Ans.* $W \geqq 1{,}000$ lb.

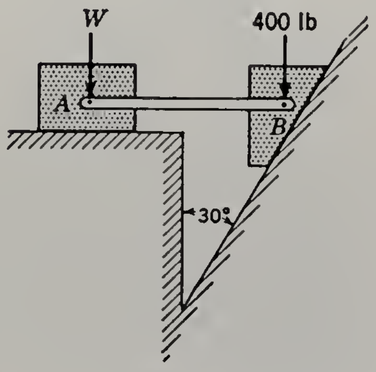


FIG. G

10. Two blocks of weights W_1 and W_2 rest on a rough inclined plane and are connected by a short piece of string as shown in Fig. H. If the coefficients of friction are $\mu_1 = 0.2$ and $\mu_2 = 0.3$, respectively, find the angle of inclination of the plane for which sliding will impend. Assume $W_1 = W_2 = 5$ lb. *Ans.* $\tan \alpha = 0.25$.

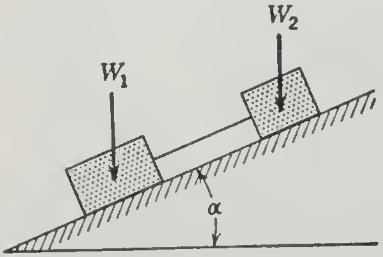


FIG. H

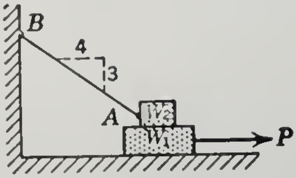


FIG. I

11. A block of weight $W_1 = 200$ lb rests on a horizontal surface and supports on top of it another block of weight $W_2 = 50$ lb (Fig. I). The block W_2 is attached to a vertical wall by the inclined string AB. Find the magnitude of the horizontal force P, applied to the lower block as shown, that will be necessary to cause slipping to impend. The coefficient of static friction for all contiguous surfaces is $\mu = 0.3$. *Ans.* $P = 84.5$ lb.

12. Two rectangular blocks of weights W_1 and W_2 are connected by a flexible cord and rest upon a horizontal and an inclined plane, respectively, with the cord passing over a pulley as shown in Fig. J. In the particular case where $W_1 = W_2$ and the coefficient of static friction μ is the same for all contiguous surfaces, find the angle α of inclination of the inclined plane at which motion of the system will impend. Neglect friction in the pulley. *Ans.* $\tan (\alpha/2) = \mu$.

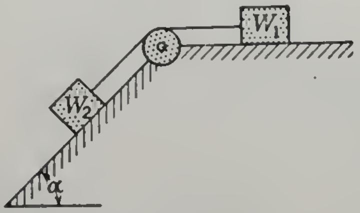


FIG. J

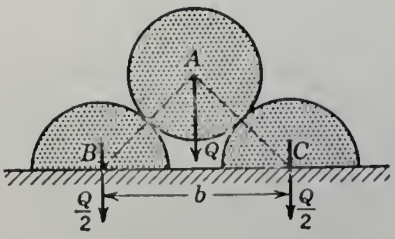


FIG. K

13. A smooth circular cylinder of weight Q and radius r is supported by two semicircular cylinders each of the same radius r and weight $Q/2$, as shown in Fig. K. If the coefficient of static friction between the flat faces of the semicircular cylinders and the horizontal plane on which they rest is $\mu = \frac{1}{2}$ and friction between the cylinders themselves is neglected, determine the maximum distance b between the centers B and C for which equilibrium will be possible without the middle cylinder touching the horizontal plane. *Ans.* $b_{\max} = 2.83r$.

14. Referring to Fig. L, the coefficients of friction are as follows: 0.25 at the floor, 0.30 at the wall, and 0.20 between blocks. Find the minimum value of a horizontal force P applied to the lower block that will hold the system in equilibrium. *Ans.* $P_{\min} = 81.2$ lb.

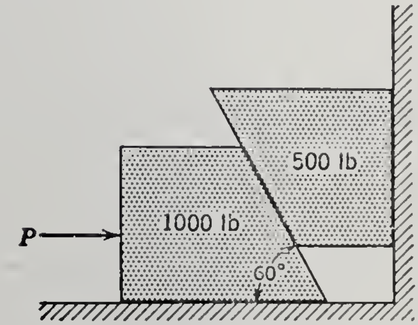


FIG. L

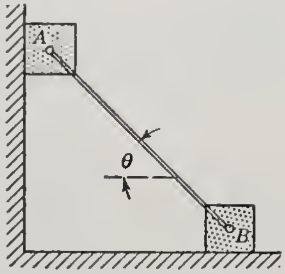


FIG. M

15. Two identical blocks A and B are connected by a rod and rest against vertical and horizontal planes, respectively, as shown in Fig. M. If sliding impends when $\theta = 45°$, determine the coefficient of friction μ, assuming it to be the same at both floor and wall. *Ans.* $\mu = 0.414$.

***16.** Two heavy right circular rollers of diameters D and d, respectively, rest on a rough horizontal plane as shown in Fig. N. The larger roller has a string wound around it to which a horizontal force P can be applied as shown. Assuming that the coefficient of friction μ has the same value for all surfaces of contact, determine the necessary condition under which the large roller can be pulled over the small one. *Ans.* $\mu \geqq \sqrt{d/D}$.

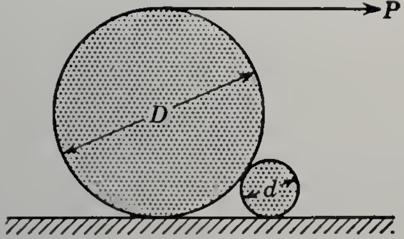


FIG. N

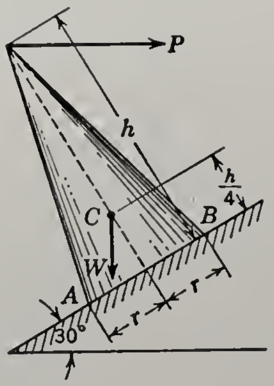


FIG. O

***17.** A solid right circular cone of altitude $h = 12$ in. and radius of base $r = 3$ in. has its center of gravity C on its geometric axis at the distance $h/4 = 3$ in. above the base. This cone rests on an inclined plane AB, which makes an angle of 30° with the horizontal and for which the coefficient of friction is $\mu = 0.5$ (Fig. O). A horizontal force P is applied to the vertex O

of the cone and acts in the vertical plane of the figure as shown. Find the maximum and minimum values of P consistent with equilibrium of the cone if the weight $W = 10$ lb. *Ans.* $P_{\max} = 4.61$ lb; $P_{\min} = 0.598$ lb.

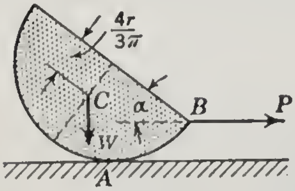


FIG. P

***18.** A short semicircular right cylinder of radius r and weight W rests on a horizontal surface and is pulled at right angles to its geometric axis by a horizontal force P applied at the middle B of the front edge (Fig. P). Find the angle α that the flat face will make with the horizontal plane just before sliding begins if the coefficient of friction at the line of contact A is μ. The gravity force W must be considered as acting at the center of gravity C as shown in the figure. *Ans.* $\sin \alpha = 3\mu\pi/(4 + 3\mu\pi)$.

2

PARALLEL FORCES IN A PLANE

2.1. Two parallel forces. Since two parallel forces do not intersect, we find that the parallelogram law cannot be directly applied to determine their resultant. Hence we must devise new rules to fit this situation. There are three possible cases: (1) two given parallel forces act in the same direction, (2) they act in opposite directions and are unequal in magnitude, and (3) they act in opposite directions and are equal in magnitude. Each case requires somewhat different consideration from the other two. We begin with two forces that act in the same direction. In this case, the same reasoning holds whether the forces are equal or unequal in magnitude.

Two Parallel Forces Acting in the Same Direction. Consider the case of two parallel forces P and Q applied at points A and B of a body, as shown in Fig. 49. It follows from the principle of superposition (see page 8) that the action of these forces will not be changed if we add two equal and opposite forces S acting along the line AB as shown in the figure. Combining the forces S and P, applied at A, we obtain their resultant P_1. In the same manner the forces S and Q, acting at B, are replaced by their resultant Q_1. Thus, instead of two parallel forces P and Q, we have now two nonparallel forces P_1 and Q_1, the resultant of which is the same as the resultant of the given forces P and Q. Next, let us transmit the points of application of the forces P_1 and Q_1 to the point of intersection D of their lines of action and resolve them there into components S, P and S, Q, respectively, as shown in the figure. The

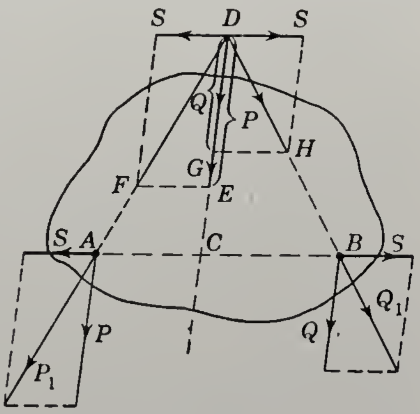


FIG. 49

two forces S, balancing each other, can now be removed, and we obtain only two forces P and Q acting along the same line DC. The resultant of these collinear forces, the same as the resultant of the given forces applied at A and B, is obviously equal to their sum

$$R = P + Q \tag{a}$$

It acts along the line DC parallel to the lines of action of the given forces P and Q and dividing the distance AB between their points of application in the ratio inversely proportional to their magnitudes. This latter statement can be proved as follows: From the similarity of $\triangle ACD$ and $\triangle FED$, we obtain

$$\frac{AC}{CD} = \frac{S}{P} \tag{b}$$

and from the similarity of $\triangle BCD$ and $\triangle HGD$, we obtain

$$\frac{BC}{CD} = \frac{S}{Q} \tag{c}$$

From Eqs. (b) and (c) comes the relationship

$$\frac{AC}{BC} = \frac{Q}{P} \tag{d}$$

as stated above. Thus, Eqs. (a) and (d) together completely determine the magnitude and the position of the line of action of the resultant R.

In determining the line of action of the resultant of two parallel forces acting in the same direction, the method of moments can be used to advantage. Referring again to Fig. 49, it has already been shown that by adding to the parallel forces P and Q two equal and opposite forces S, we obtain the forces P_1 and Q_1 intersecting at a point D. For such intersecting forces, we have Varignon's theorem (see page 41), stating that the sum of the moments of the component forces P_1 and Q_1 is equal to the moment of their resultant R. Observing now that for any moment center the sum of the moments of the two added forces S is always zero, we conclude that the sum of the moments of the given parallel forces P and Q is equal to the sum of the moments of the intersecting forces P_1 and Q_1, which, in turn, is equal to the moment of the resultant R. From this it follows that Varignon's theorem holds also for parallel forces; i.e., *the algebraic sum of the moments of two parallel forces with respect to any moment center in their plane of action is equal to the moment of their resultant with respect to the same center.*

Having the method of finding the resultant of two parallel forces, we may now solve also the inverse problem of resolution of a given force R into two parallel components P and Q acting in the same direction. The case most commonly encountered is that in which the lines of action of the components (on two sides of the given force R) are specified and the magnitudes of P and Q are required (Fig. 50). To find those components, we draw the line AB perpendicular to the given lines of action, as shown. Then using the theorem of moments with point B as a center, we have, since the moment of Q about this point is zero,

$$P \cdot BA = R \cdot BC$$

from which

$$P = R \frac{BC}{BA} \tag{e}$$

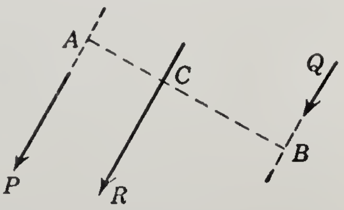


FIG. 50

Similarly, with A as a moment center, we obtain

$$Q = R \frac{AC}{AB} \tag{f}$$

Adding expressions (e) and (f), we find $P + Q = R$, as it should be. The resolution of a given force into more than two parallel components in one plane is an indeterminate problem.

Two Unequal Parallel Forces Acting in Opposite Directions. In this case (Fig. 51) we proceed as before and introduce two equal, opposite, collinear forces S along the line AB joining the points of application of the given parallel forces P and Q. Then, as before, the equivalent forces P_1 and Q_1 are transmitted to their point of intersection C and there resolved into components P, S and Q, S, as shown. Removing, finally, the balanced forces S at C, we are left with two collinear but oppositely directed forces P and Q at C. The resultant of these collinear forces is

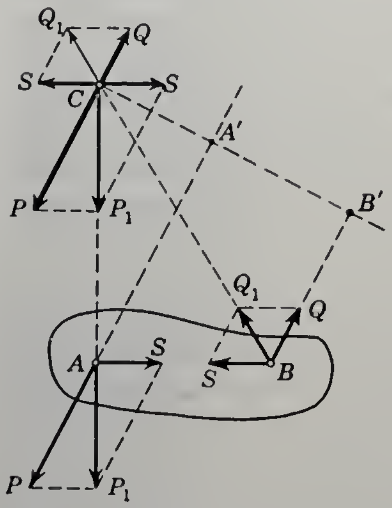


FIG. 51

$$R = P - Q \tag{g}$$

acting at C in the direction of the larger forces P. Proceeding as before, it may be shown that the theorem of moments holds also in this case. Thus, for locating point C, we have, with C as a moment center,

$$Q \cdot CB' - P \cdot CA' = 0$$

from which

$$\frac{CB'}{CA'} = \frac{P}{Q} \tag{h}$$

Thus again the distances of the components from the line of action of the resultant are inversely proportional to their magnitudes, but the line of action of the resultant lies outside the space between the components on the side of the larger force.

Two Equal Parallel Forces Acting in Opposite Directions. A system of two equal parallel forces acting in opposite directions cannot be reduced to one resultant force. This can readily be seen from Fig. 51. If the forces P and Q are equal in magnitude, the two parallelograms APP_1S and BQQ_1S will be equal and corresponding sides will be parallel. Hence P_1 and Q_1 will still be equal and oppositely directed parallel forces, and we never obtain a point of intersection C. Thus, we cannot reduce two equal and opposite but noncollinear forces to any simpler system. Two such forces are called a *couple*, the plane in which they act is called *the plane of the couple*, and the distance between their lines of action is called the *arm of the couple.*

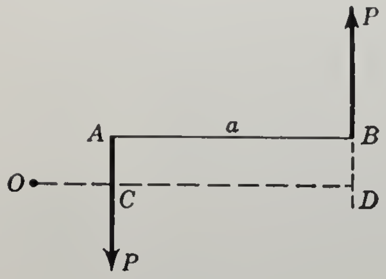


FIG. 52

The algebraic sum of the moments of the two forces of a couple is independent of the position, in the plane of the couple, of the moment center and is always equal to the product of the magnitude of either force and the arm of the couple. Consider, for example, the couple PP with arm AB of length a in Fig. 52, and let O be any arbitrary moment center. Then the algebraic sum of moments is

$$P \cdot OD - P \cdot OC = P(OD - OC) = Pa$$

The same result will be obtained if O lies between the lines of action of the forces. The moment Pa is called the *moment of the couple.* It is positive when the couple tends to produce counterclockwise rotation as shown, and it is negative for clockwise rotation.

The action of a couple on a rigid body will not be changed if its arm is turned in the plane of the couple through any angle α about one of its ends. To prove this statement, let us consider a couple PP with the arm AB, as shown in Fig. 53. We take under any desired angle α with the arm AB a straight-line segment AC equal in length to the arm AB of the given couple. At each of the ends A and C of this line we apply two equal and opposite forces Q and Q' perpendicular to AC and equal in magnitude to P. From the principle of superposition it follows that the addition of these forces which are in equilibrium does not change the action of the given couple. The forces Q together with the forces P give the resultants R, as shown in the figure. These resultants, as diagonals of two rhombuses, divide the angles at A and D in halves; hence they act along the same line AD and as two equal, opposite, and collinear forces can be removed from the system. Then only the forces Q', equal in magnitude to P, remain, and these forces form a couple with the arm AC equal, by construction, to AB and making with it the angle α. This proves that the arm of a couple can be turned around one of its ends, as stated above, without changing the action of the couple on a body to which it is applied.

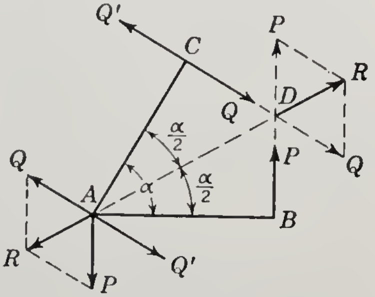


FIG. 53

By successive rotations of the arm of a couple in its plane, first about one end and then the other, the couple can be put in any desired position in its plane. Hence, we can transpose a couple in its plane without changing its action on a body.

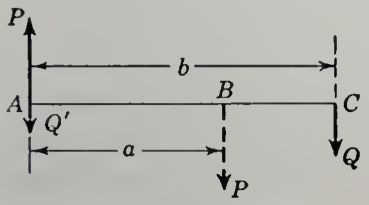


FIG. 54

The action of a couple on a body does not change if we change both the magnitudes of the forces and the arm of the couple in such a way that the moment of the couple remains unchanged. To prove this statement, we shall show how a given couple PP with arm AB (Fig. 54) can be transformed into a couple QQ with arm AC. For this purpose we resolve the force P applied at B into two parallel components Q and Q' applied at points C and A, respectively. From the previously given rules for the resolution of a force into two parallel components it follows that

$$Q = P\frac{a}{b} \qquad Q' = P - Q \tag{i}$$

Now, subtracting the force Q' from the force P at A, we obtain a force Q acting upward. This force Q, together with the force Q applied at C, forms a couple with the arm AC. The moment of this new couple is Qb, which, by using the first of Eqs. (i), becomes equal to Pa. Thus, without changing the action on a body, a given couple can be replaced by another one consisting of different forces with a different arm, provided the moments of the two couples are equal.

Summing up the properties of a couple which have been considered in the foregoing discussion, it can be concluded that two couples acting in the same plane are equivalent if they have equal moments. In short, a couple is completely defined by its plane of action and the magnitude and sign of its moment.

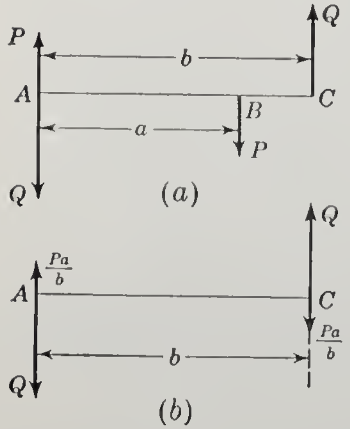


FIG. 55

For the addition of two couples, we have to take only the algebraic sum of their moments and we obtain the moment of the resultant couple. To prove this, let us consider two couples PP and QQ with arms a and b, respectively, as shown in Fig. 55a. By transposition, we can always bring the two arms into coincidence as shown if they are not already so positioned. The couple PP can now be transformed into another couple having an arm of length b in which case the magnitudes of the forces must become equal to Pa/b, as shown in Fig. 55b. Adding, algebraically, the forces at C and A (Fig. 55b), we obtain a couple with the arm equal to b and forces equal to $Q - Pa/b$. The moment of this couple is

$$\left(Q - \frac{Pa}{b}\right)b = Qb - Pa \tag{j}$$

Thus the moment of the resultant couple is equal to the algebraic sum of the moments of the two given couples. The same conclusion can be reached for the case where the moments of the two given couples are of the same sign.

By successive applications of the above procedure, we can replace several couples in one plane by a single resultant couple acting in the same plane, the moment of which is equal to the algebraic sum of the moments of the given couples.

Two couples acting in one plane will be in equilibrium if they have moments of equal magnitude and opposite sign, since, in such a case, proceeding as illustrated in Fig. 55, we obtain each force of the resultant couple equal to zero. We shall generalize this conclusion by the statement that a couple can be balanced only by another couple which is equal in moment, opposite in sign, and coplanar in action with the given couple.

By successive applications of the method of addition of two couples, it can be proved that a system of couples acting in one plane is in equilibrium if the algebraic sum of their moments is equal to zero.

It is possible to resolve a couple into several component couples by choosing the component couples in such a manner that the algebraic sum of their moments is equal to the moment of the given couple.

Resolution of a Force into a Force and a Couple. A given force P applied to a body at any point A can always be replaced by an equal force applied at another point B together with a couple which will be statically equivalent to the original force. To prove this, let the given force P act at A as shown in Fig. 56*a*. Then at B (Fig. 56*b*) we introduce two oppositely directed collinear forces each of magnitude P and parallel to the line of action of the give force P at A. It follows from the law of superposition that the system in Fig. 56*b* is statically equivalent to that in Fig. 56*a*. However, we may now regard the original force P at A and the oppositely directed force P at B as a couple of moment $M = Pa$. Since this couple may now be transformed in any manner in its plane of action so long as its moment remains unchanged, we may finally represent the system as shown in Fig. 56*c*, where the couple is simply indicated by a curved arrow and the magnitude of its moment.

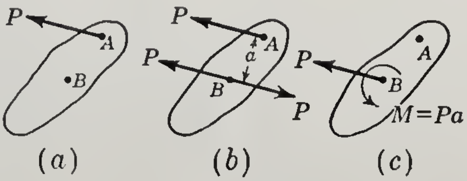


FIG. 56

It will be noted that the moment of the couple introduced in the above manner will always be equal to the product of the original force P and the arbitrary distance a that we decide to move its line of action. This resolution of a force into a force and couple is very useful in many problems of statics.

EXAMPLES

1. A beam AB of length l is supported as shown in Fig. 57 and subjected to equal but opposite vertical forces P at its two ends. Find the reactions at the supports C and D.

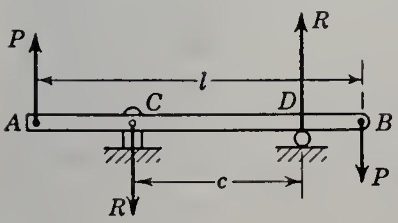


FIG. 57

Solution. Since the active forces P constitute a couple of moment Pl in the clockwise direction, the reactions at C and D must represent a counterclockwise couple of the same moment. Further, since the physical nature of the constraint at D is such that it can only produce a vertical reaction, the reaction at C must also be vertical and the distance c between supports is the arm of the reactive couple. Hence equating moments, we have

$$Rc = Pl$$

from which $R = Pl/c$. These reactions must be directed as shown in the figure.

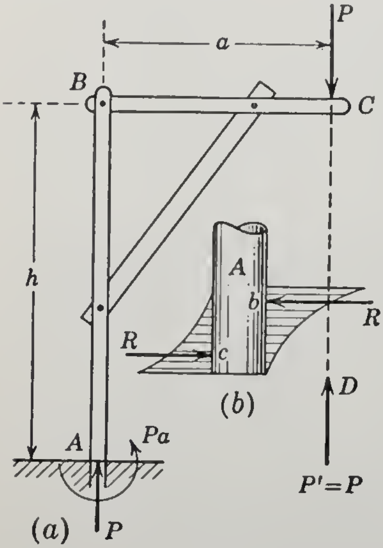


FIG. 58

2. A crane ABC anchored in a cement foundation at A carries a vertical load P at C, as shown in Fig. 58*a*. Neglecting the weight of the crane proper, find the reaction at A.

Solution. Clearly, the reaction at A must be statically equivalent to the equilibrant P' of the active force P, that is, a vertical upward force acting along the line DC. To represent this force as a reaction at A, we resolve it into an equal parallel force P at A and a couple of moment Pa acting as shown in the figure. To understand physically just how the foundation exerts this couple on the bottom of the mast, we must consider the detail of the bottom of the mast as shown in Fig. 58*b*. In trying to tip over to the right, the mast presses against the walls of the hole in which it is seated and, of course, equal and opposite reactive pressure will be exerted on the mast. This distribution of pressure will be something like that indicated in the figure and is sensibly equivalent to a couple RR acting as shown. We see that this kind of constraint, a so-called *built-in end*, is considerably more complex than any we have previously discussed. This example illustrates one of the many useful applications of the resolution of a force into a force and a couple.

PROBLEM SET 2.1

1. (*a*) Resolve the 800-lb force shown in Fig. A into two parallel components P and Q acting, respectively, along aa and bb.

(*b*) Resolve the same force into parallel components P and Q acting, respectively, along bb and cc.

(*c*) Resolve the same force into a force P at B and a couple. Represent the couple by forces F acting along bb and cc.

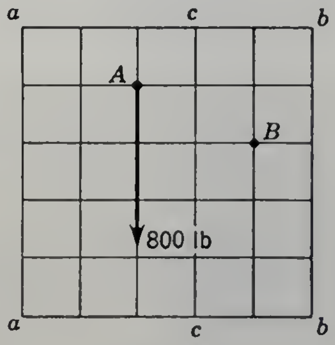


FIG. A

2. A rigid bar $CABD$ supported as shown in Fig. B is acted upon by two equal horizontal forces P applied at C and D. Calculate the reactions that

will be induced at the points of support. Assume $l = 4$ ft, $a = 3$ ft, $b = 2$ ft. *Ans.* $R_b = -R_a = 0.25P$.

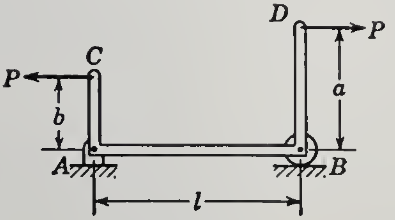


FIG. B

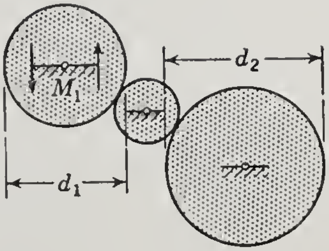


FIG. C

3. Two gears having pitch diameters $d_1 = 6$ in. and $d_2 = 8$ in. are connected by an idling gear as shown in Fig. C. If a couple of moment M_1 is applied to the upper gear as shown, what is the moment M_2 of the couple that must be applied to the lower gear to maintain equilibrium? *Ans.* $M_2 = -1.33\,M_1$.

4. A vertical mast of weight Q guided at A and B is kept in equilibrium by the support C, as shown in Fig. D. Neglecting friction at the guides, determine the vertical reaction R_c and the horizontal reactions R_a and R_b if $a = 1$ ft, $l = 2$ ft. *Ans.* $R_c = Q$; $R_a = -R_b = \frac{1}{2}Q$.

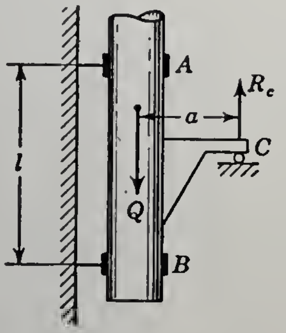


FIG. D

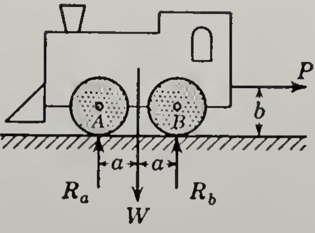


FIG. E

5. Owing to the weight W of the locomotive shown in Fig. E, the reactions at the two points of support A and B will each be equal to $W/2$. When the locomotive is pulling a train and the drawbar pull P is just equal to the total friction at the points of contact A and B, determine the magnitudes of the vertical reactions R_a and R_b. *Ans.* $R_a = \frac{W}{2} - \frac{Pb}{2a}$; $R_b = \frac{W}{2} + \frac{Pb}{2a}$.

2.2. General case of parallel forces in a plane. Let us consider now the case of several parallel forces in one plane (Fig. 59). In such a case we may find the resultant R_1 of the forces F_1, F_2, F_3 acting in one direction and the resultant R_2 of the forces F_4, F_5 acting in the opposite direction by successive applications of the method described in Art. 2.1. As a result of this operation three different cases will be possible.

In the first case R_1 is not equal to R_2 and we obtain two unequal parallel forces acting in opposite directions. The resultant R of these two forces (see Art. 2.1) is obviously the resultant of the given system of forces $F_1, F_2, \ldots, F_5$.

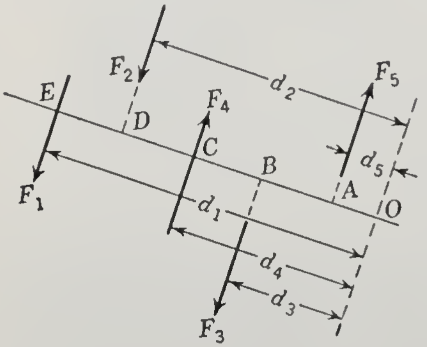


FIG. 59

In the second case the forces R_1 and R_2 will be equal in magnitude, opposite in direction, and acting along two parallel lines. Then the given system of forces reduces to a couple (see Art. 2.1) that we shall call the *resultant couple.*

In the third case the forces R_1 and R_2 will be equal in magnitude, opposite in direction, and collinear in action and the given system of forces is in equilibrium.

The three cases just considered may be distinguished in another way as follows: If the algebraic sum of the given forces is different from zero, the system reduces to a resultant force. In such case the line of action of this resultant force is determined by the condition that its moment with respect to any point in the plane is equal to the algebraic sum of the corresponding moments of the individual forces. When the algebraic sum of the forces is equal to zero, the possibility of a resultant force vanishes and there remain two possibilities: (1) The system reduces to a couple, or (2) the system is in equilibrium. To distinguish between these two cases, the algebraic sum of moments of the forces with respect to any center O in their plane must be calculated. If this sum is different from zero, the system reduces to a resultant couple and the calculated moment gives the moment of this couple. If this sum is zero, the possibility of a resultant couple also vanishes and the system of forces is in equilibrium.

To classify the above three cases analytically, let us consider any system of coplanar parallel forces $Y_1, Y_2, \ldots, Y_n$, referred to coordinate axes x and y as shown in Fig. 60. Choosing the origin of coordinates O as a moment center, we denote by x_i the arm of any force Y_i and by $\bar{x}$ the arm of the resultant force Y. Then this resultant force, if there is one, is defined analytically by the equations

$$Y = \Sigma Y_i \qquad \bar{x} = \frac{\Sigma(Y_i x_i)}{\Sigma Y_i} \tag{9}$$

where the summations are understood to include all forces in the system. The first equation gives the magnitude of the resultant, and the second determines its arm with respect to the chosen moment center. In making the summation of forces, the force Y_i must be taken with positive sign if it acts in the positive direction of the y axis, otherwise negative, while in making the summation of moments, the term $Y_i x_i$ should be taken positive when the moment is counterclockwise, otherwise negative.

When the resultant is a couple, which we shall denote by M, we have

$$\Sigma Y_i = 0 \qquad M = \Sigma(Y_i x_i) \qquad (10)$$

When the given system of forces is in equilibrium,

$$\Sigma Y_i = 0 \qquad \Sigma(Y_i x_i) = 0 \qquad (11)$$

These are the *equations of equilibrium* for a system of parallel forces in one plane.

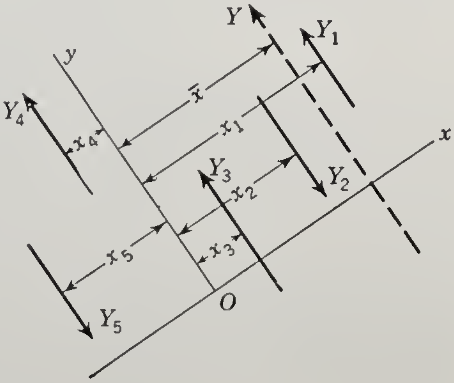


FIG. 60

We observe that, when the first of Eqs. (11) is satisfied, the possibility of a resultant force vanishes and that, when the second is satisfied, the possibility of a resultant couple vanishes. Hence the conclusion that the system is in equilibrium when they are simultaneously satisfied.

The same conditions of equilibrium can also be expressed by two moment equations. If the algebraic sum of moments with respect to a center A is zero, we conclude that the only possibility of a resultant is that of a force which passes through the moment center A. If the algebraic sum of moments about a second center B is also zero, again the system either is in equilibrium or reduces to a resultant force through B. Hence, if the line AB is not parallel to the lines of action of the given forces, all possibility of a resultant vanishes, since it cannot be a force with two different lines of action and we must have equilibrium. Expressed analytically, these two conditions of equilibrium become

$$\Sigma(M_A)_i = 0 \qquad \Sigma(M_B)_i = 0 \qquad (12)$$

where $(M_A)_i$ and $(M_B)_i$ denote the moments of any force Y_i with respect to A and B and the summations are understood to include all forces in the system.

EXAMPLES

1. The forces applied to the roof truss shown in Fig. 61a represent the effect of a wind pressure against the side of the roof. Find the reactions that will be produced at the points of support A and B due to this loading.

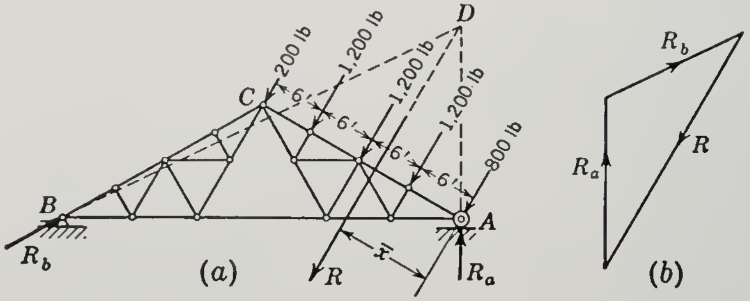


FIG. 61

Solution. So far as the reactions at A and B are concerned, the five parallel forces acting normal to AC can be replaced by their resultant R. From the first of Eqs. (9) the magnitude of this resultant will be

$$R = 800 + 3 \times 1{,}200 + 200 = 4{,}600 \text{ lb}$$

To locate its line of action, we take point A as a moment center and the second of Eqs. (9) gives

$$\bar{x} = \frac{1{,}200(6 + 12 + 18) + 200 \times 24}{4{,}600} = \frac{48{,}000}{4{,}600} = 10.43 \text{ ft}$$

Replacing the active forces by their resultant R, as shown in the figure, we see that the truss is in equilibrium under the action of three forces and the remainder of the solution may be made by the method of Art. 1.5 and the magnitudes of the reactions R_a and R_b scaled from the triangle of forces, shown in Fig. 61b.

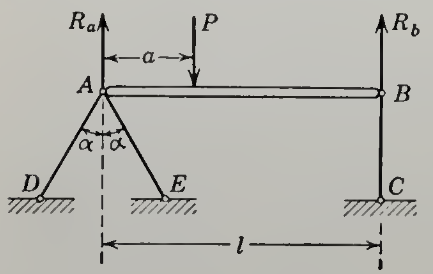


FIG. 62

2. Determine the forces in the hinged bars AD, AE, and BC, supporting the horizontal beam AB on which the vertical load P is acting (Fig. 62). Neglect the weight of the beam.

Solution. The load P is balanced by the reactions R_a and R_b acting at the points of support A and B. Since the vertical bar BC is hinged at its ends, R_b is a vertical force. Hence, R_a must be vertical also, since, if it is not, it gives, together with the force P, a nonvertical resultant which cannot be balanced by the vertical reaction at B. Thus, acting on the beam, we have the three vertical forces R_a, P, and R_b in equilibrium. The simplest way of calculating the reactions R_a and R_b is by using

Eqs. (12). Taking moments, first with respect to point A and then with respect to point B, we obtain the equations

$$R_b l - Pa = 0 \qquad P(l - a) - R_a l = 0$$

from which

$$R_b = \frac{Pa}{l} \qquad \text{and} \qquad R_a = \frac{P(l-a)}{l}$$

As a check on these values, the first of Eqs. (11) may be used, if desired, and we obtain

$$R_a + R_b = P$$

The action of the beam on the bar BC is equal and opposite to the reaction R_b and produces a compressive force in the bar equal to Pa/l. The action of the beam on the hinge at A is equal and opposite to the reaction R_a. Resolving this force into two components acting along the axes of the bars AD and AE, we find the compressive forces in these bars to be $P(l - a)/(2l \cos \alpha)$ each.

3. Two rollers C and D produce vertical forces P and Q on the horizontal beam AB, as shown in Fig. 63. Determine the distance x of the load P from the support A if the reaction R_a is twice as great as the reaction R_b. The weight of the beam is to be neglected, and the following numerical data are given: $P = 2$ tons, $Q = 1$ ton, $l = 12$ ft, $c = 3$ ft.

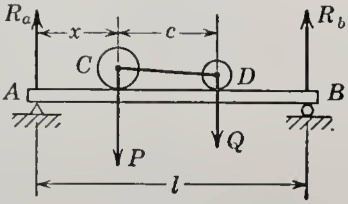


FIG. 63

Solution. Considering the beam as a free body and equating to zero the sum of the moments of all forces with respect to points A and B, we obtain

$$R_b l - Q(x + c) - Px = 0$$
$$Q(l - c - x) + P(l - x) - R_a l = 0$$

The third necessary equation is

$$R_a = 2R_b$$

Eliminating R_a and R_b from these three equations, we obtain

$$x = \frac{l}{3} - \frac{Qc}{P + Q}$$

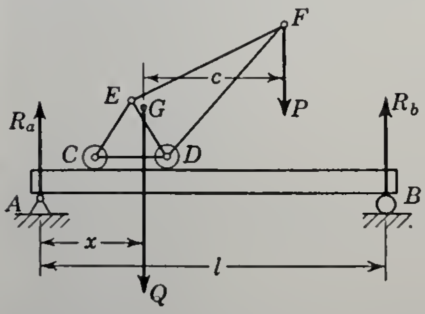


FIG. 64

Substituting the given numerical data, we find $x = 3$ ft.

4. A crane $CDEF$ (Fig. 64) of weight $Q = 5$ tons applied at G supports at F a load $P = 1$ ton and can move along the horizontal girder AB. For the position of the crane shown in the figure, determine the reactions produced at the supports A and B of the girder. Assume that all forces act in one plane, and neglect the weight of the girder. Numerical data are given as follows: $l = 30$ ft, $x = 9$ ft, $c = 12$ ft.

Solution. The forces produced by the wheels of the crane on the girder at C and D represent a system of forces equivalent to P and Q, since the equal and opposite reactions exerted by the girder on the wheels of the crane balance the forces P and Q. Hence we can assume that P and Q act directly on the girder AB. In other words, we shall consider the girder and crane together as a free body, in which case actions and reactions at the points C and D balance each other and do not enter into our consideration. Taking moments of all forces with respect to points A and B, Eqs. (12) become

$$R_b l - P(x + c) - Qx = 0$$
$$P(l - x - c) + Q(l - x) - R_a l = 0$$

Substituting the given numerical data into these equations, we obtain $R_a = 3.8$ tons and $R_b = 2.2$ tons. It will be noted that the values of these reactions are independent of the distance between the wheels of the crane.

5. Two prismatic bars AB and BC of lengths l and $2l$, respectively, are rigidly joined at B and suspended by a string AD, as shown in Fig. 65. Determine the position of equilibrium, as defined by the angle α, that the bars will assume under the action of their weights Q and $2Q$.

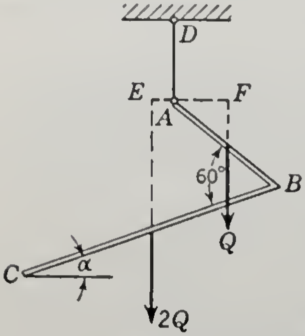


FIG. 65

Solution. Considering both bars together as a free body, we see that they are in equilibrium under the action of three parallel forces representing the weights Q and $2Q$ and the vertical reaction exerted by the string AD at A. Equating to zero the sum of the moments of all forces with respect to point A, we obtain

$$AF = 2AE$$

or, from the geometry of the figure,

$$\frac{l}{2} \cos (60° - \alpha) = 2[l \cos \alpha - l \cos (60° - \alpha)]$$

from which

$$\tan \alpha = \frac{\sqrt{3}}{5}$$

and $\alpha = 19°06'$.

6. The traveling crane shown in Fig. 66 has a total weight $Q = 20$ tons which may be considered as acting at point E. To prevent the crane from tipping to the right under the action of its own weight together with a load P applied at C, a counterweight W is carried at D. This counterweight must be so chosen and placed that, while it prevents such tipping to the right when

the crane supports its maximum load P, there will also be no danger of tipping to the left when the load P is removed. Find the limiting values of the magnitude of the counterweight W and the distance x. The following numerical data are given: $Q = 20$ tons, $P = 20$ tons, $b = 6$ ft, $e = 2$ ft, $l = 12$ ft.

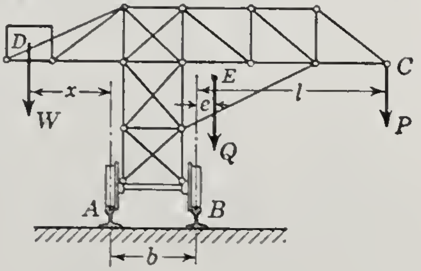


FIG. 66

Solution. Two extreme conditions must be considered: (1) when the load P is acting at C and there is the possibility of overturning of the crane about the rail B, (2) when the load P is removed and, owing to the action of the counterweight W, there is the possibility of overturning about the rail A. In the first case, if overturning about point B impends, there will be no pressure between the rail and wheel at A and the structure will be in equilibrium under the action of the three vertical loads W, Q, and P, together with a vertical reaction at B. Equating to zero the sum of the moments of all forces with respect to point B, we obtain

$$W(x + b) - Qe - Pl = 0 \tag{a}$$

In the second case, when the load P is removed and overturning about point A impends, there will be no pressure between the rail and wheel at B. Again taking moments of all forces, this time with respect to point A, we obtain

$$Wx - Q(b + e) = 0 \tag{b}$$

Solving Eqs. (a) and (b) for W and x, respectively, gives for the desired limiting values of these two quantities

$$W = \frac{Pl}{b} - Q \qquad \text{and} \qquad x = \frac{Q(b + e)b}{Pl - Qb}$$

Using the given numerical data, these values become

$$W = 20 \text{ tons} \qquad \text{and} \qquad x = 8 \text{ ft}$$

It is left as an exercise for the student to prove that the above limiting values of W and x are, respectively, the minimum value of W and the maximum value of x.

PROBLEM SET 2.2

1. A prismatic bar AB of weight $Q = 10$ lb is supported by two vertical wires at its ends and carries at D a load $P = 20$ lb (Fig. A). Determine the forces S_a and S_b in the two wires. *Ans.* $S_a = 20$ lb, tension; $S_b = 10$ lb, tension.

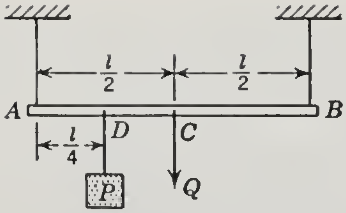


FIG. A

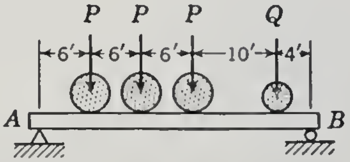


FIG. B

2. The four wheels of a locomotive produce vertical forces on the horizontal girder AB as indicated in Fig. B. Determine the reactions R_a and R_b at the supports if the loads $P = 10$ tons each and $Q = 8$ tons. *Ans.* $R_a = 19\frac{3}{4}$ tons; $R_b = 18\frac{1}{4}$ tons.

3. The beam AB in Fig. C is hinged at A and supported at B by a vertical cord which passes over a frictionless pulley at C and carries at its end a load P as shown. Determine the distance x from A at which a load Q must be placed on the beam if it is to remain in equilibrium in a horizontal position. Neglect the weight of the beam. *Ans.* $x = Pl/Q$.

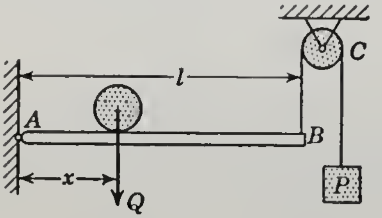


FIG. C

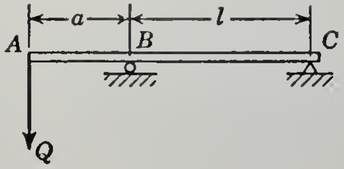


FIG. D

4. A beam AC is supported as shown in Fig. D and subjected to the action of a load Q applied at the free end A. Determine the magnitudes and directions of the reactions at B and C if $a = l/4$ and $Q = 1$ ton. *Ans.* $R_b = 1\frac{1}{4}$ tons, up; $R_c = \frac{1}{4}$ ton, down.

5. Under the action of a load Q a cantilever beam AB presses at points C and B where it is built into a wall, as shown in Fig. E. Neglecting the effect of friction, determine the magnitudes of the reactions at B and C if $l = 10$ ft, $d = 2$ ft, and $Q = 1$ ton. *Ans.* $R_b = 5$ tons; $R_c = 6$ tons.

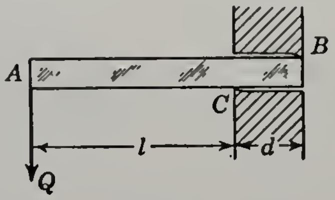


FIG. E

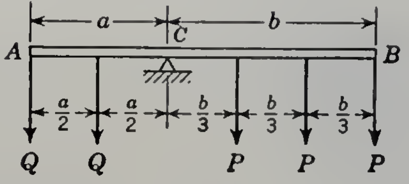


FIG. F

6. Along a lever AB loads Q and P are distributed, as shown in Fig. F. If $Q = 2P$ and the weight of the lever is negligible, determine the ratio $a:b$ of the arms of the lever if it is in equilibrium. *Ans.* $a:b = 2:3$.

7. The beam CE in Fig. G is supported on the beam AB by the three bars CF, DG, and CG, as shown. Find the reactions that will be produced at the points of support A and B of the lower beam due to the action of a load P applied at the free end E of the upper beam if the span $l = 12$ ft and $a = 4$ ft. *Ans.* $R_b = 4P/3$, up; $R_a = P/3$, down.

8. Two identical prismatic bars AB and CD are welded together in the form of a rigid T and suspended in a vertical plane as shown in Fig. H. Calculate the angle α that the bar CD will make with the vertical when a vertical load $P = 10$ lb is applied at B. The weight of each bar is $Q = 5$ lb as shown. *Ans.* $\alpha = 15°57'$.

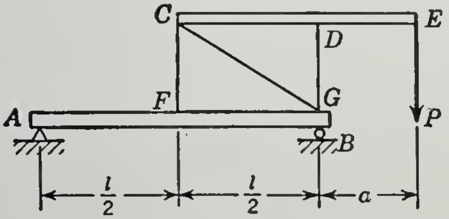


FIG. G

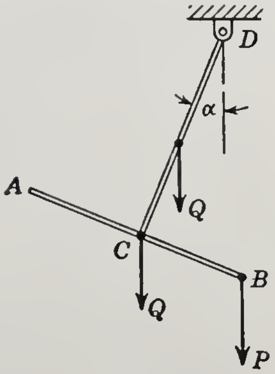


FIG. H

9. Two horizontal beams are arranged as shown in Fig. I. Determine the reaction produced at the support C due to the action of a vertical load P applied to the beam AB as shown. *Ans.* $R_c = Pa/b$.

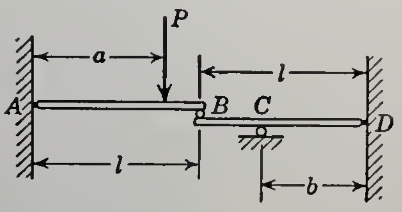


FIG. I

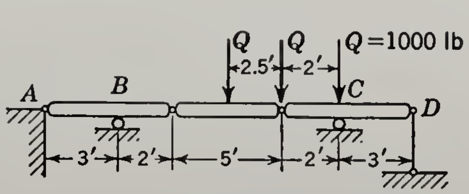


FIG. J

10. Three beams hinged together at their ends are supported and loaded as shown in Fig. J. Determine the reactions at the supports A, B, C, and D. *Ans.* $R_a = 333$ lb, down; $R_b = 833$ lb, up; $R_c = 3{,}500$ lb, up; $R_d = 1{,}000$ lb, down.

11. The system of vertical loads shown in Fig. K represents the pressures transmitted by the wheels of a locomotive to the girder AB. Determine the reactions at A and B if the loads are in tons.

Hint. First, find magnitude and line of action of resultant of the active forces; then use the theorem of three forces to find R_a and R_b. *Ans.* $R_a = 119$ tons; $R_b = 132$ tons.

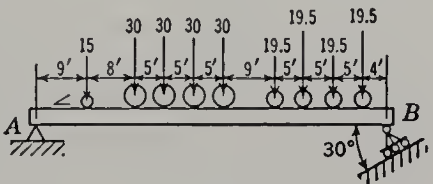


FIG. K

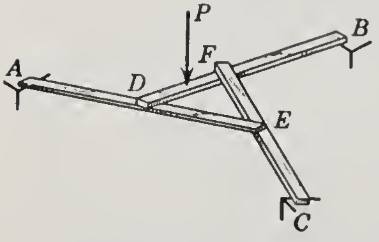


FIG. L

***12.** Three identical bars of lengths l are arranged and supported in a horizontal plane, as shown in Fig. L. Each bar supports the end of another at its mid-point so that DEF is an equilateral triangle with sides of length $l/2$. Find reactions at A, B, C, and interactions at D, E, F, due to a vertical load P applied midway between D and F on the bar DB. *Ans.* $R_a = 3P/7$, $R_b = 5P/14$, $R_c = 3P/14$, $R_d = 6P/7$, $R_e = 3P/7$, $R_f = 3P/14$.

2.3. Center of parallel forces and center of gravity. *Center of Parallel Forces.* Let P and Q (Fig. 67) be two parallel forces applied to a rigid body at points A and B. Then the resultant R is parallel to the given forces, equal to their algebraic sum, and its line of action intersects the line AB at a point C such that

$$BC:AC = P:Q$$

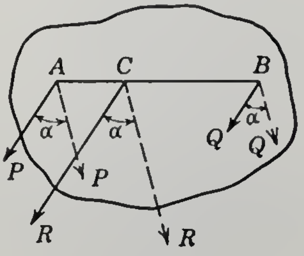


FIG. 67

as shown in Art. 2.1. If the given forces P and Q are rotated about their points of application through any angle α in their plane of action as indicated by the dotted lines in Fig. 67, the resultant R obviously will be rotated through the same angle α and its line of action again passes through point C. Thus we see that point C is the only point through which the resultant of the forces P and Q applied at points A and B acts regardless of the direction of the forces. This point is called the *center of parallel forces* for the case of two parallel forces.

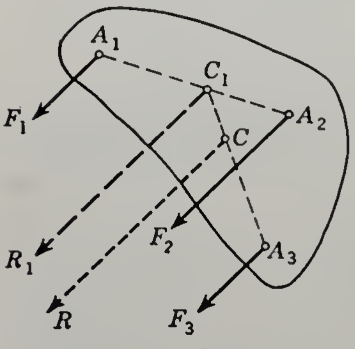


FIG. 68

If we have three parallel forces F_1, F_2, F_3 applied at three given points A_1, A_2, A_3 (Fig. 68), we first locate the center C_1 of F_1 and F_2, as explained above. Then applying the partial resultant R_1 of F_1 and F_2 at C_1, we repeat the procedure and find the center C of two parallel forces R_1 and F_3, as shown. This is the required center of three parallel forces F_1, F_2, F_3 acting at A_1, A_2, A_3, respectively. Obviously, this procedure may be continued for any number of parallel forces $F_1, F_2, \ldots, F_n$, applied at a given points $A_1, A_2, \ldots, A_n$, respectively, and we conclude that there is one and only one point through which the resultant always passes regardless of the direction in which the parallel forces act through their given points of application. This point is called the *center of parallel forces* for the given system of forces applied at the given system of points.

In the particular case of a system of points all lying in one plane of a rigid body, we may locate the center of parallel forces for a system of forces applied at these points by using Eqs. (9). Let $A_1, A_2, \ldots, A_n$ (Fig. 69) be any system of points which all lie in one plane of a rigid body, and let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be their coordinates. To find the center of parallel forces C for a given system of forces $F_1, F_2, \ldots, F_n$ applied at these points and acting in any direction, let us first imagine that the forces act parallel to the y axis. Then, we have a system of forces in one plane and the arm x_c of their resultant may be found by using the second of Eqs. (9) (page 70) from which

$$x_c = \frac{\Sigma(F_i x_i)}{\Sigma F_i} \qquad (13a)$$

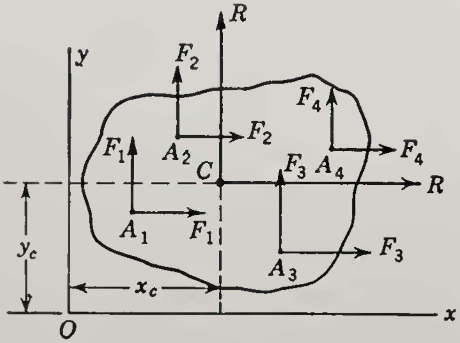


FIG. 69

Now let the forces all be rotated in the plane of the figure until they act parallel to the x axis. In this case the arm y_c of the resultant may be found by again using the second of Eqs. (9), and we obtain

$$y_c = \frac{\Sigma(F_i y_i)}{\Sigma F_i} \qquad (13b)$$

We have already seen that the center of parallel forces of any number of forces applied at a given system of points is independent of the direction in which the forces act. Hence, we conclude that the moment arms x_c and y_c as defined by Eqs. (13) represent the coordinates of this point.

The center of parallel forces for a given system of forces $F_1, \ldots, F_n$ applied at given points $A_1, \ldots, A_n$ in one plane will not be changed if the magnitudes of the forces are all multiplied by the same constant factor. This statement follows at once from the form of Eqs. (13) from which we see that such a factor will appear n times in both the numerator and the denominator of either of these expressions and will therefore cancel out. Thus we conclude that the center of parallel forces for any given system of forces applied at a given system of points in one plane depends only upon the positions of the points and upon the relative magnitudes of the forces.

Center of Gravity. The center of gravity of a body is that point through which the resultant of the distributed gravity force passes regardless of the orientation of the body in space. From this definition it follows that the center of gravity of a rigid body is the center of parallel gravity forces acting on the various particles of the body. Since gravity forces always act vertically downward, it is evident that some rotation of a body through an angle α is equivalent to a corresponding rotation through the same angle of all the gravity forces about their points of application as discussed above.

All physical bodies are, of course, three-dimensional as a consequence of which the gravity forces acting on the various particles of the body represent a system of parallel forces in space. There are, however, certain cases in which we are justified in ignoring one or even two dimensions of a body and assuming that the particles of which it is composed are confined in one plane or even along a line. It is to such special cases that we shall limit our discussion in the remainder of this article.

Consider, for example, the case of a homogeneous thin plate of uniform thickness (Fig. 70*a*). If we imagine such a plate to be subdivided into small prismatic elements as shown and assume that the plate stands vertically on edge, the gravity forces of the above-mentioned elements will represent a system of vertical forces acting in the middle plane of the plate. The center of these parallel gravity forces will evidently lie in the same plane and is the center of gravity of the plate.

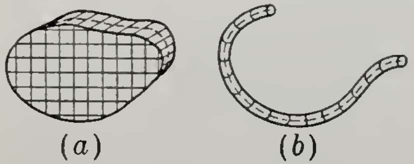

(*a*) (*b*)

FIG. 70

In the case of a homogeneous slender wire of uniform cross section and whose axis is contained in one plane, we proceed in the same manner as above. Imagining the axis of the wire to be divided into small elements as shown in Fig. 70*b* and assuming the plane of this axis to be vertical, the gravity forces acting on the various elements represent a system of parallel forces in a plane, the center of which, in the same plane, represents the center of gravity of the wire.

From the uniform thickness and homogeneity of the plate, it follows that the magnitude of the gravity force of any one of the prismatic elements into which the plate is divided (see Fig. 70*a*) is proportional to the cross-sectional area of that element. In the same way, in the case of the homogeneous slender wire of uniform cross section, we conclude that the gravity force acting on any element is proportional to the length of that element. The importance of these observations lies in the fact, as we have seen already, that, so far as the location of the center of parallel forces is concerned, it makes no difference what this factor of proportionality is since it is the same for each and every element of the plate or wire. Thus we conclude, in the case of the plate, that the position of the center of gravity depends only upon the *shape of figure* of its middle surface and in the case of the slender wire, only upon the *shape of curve* of its axis.

The above discussion brings us to the notion of the center of gravity of the area of a plane figure or of the length of a plane curve. Aside from their identity with the center of gravity of a corresponding thin plate or slender wire, these notions are often of significance in themselves in connection with certain problems of mechanics. However, because geometric figures and curves do not possess the property of weight, the term *center of gravity* as applied to them is something of a misnomer and for this reason the term *centroid* is often used instead. In this book the term center of gravity will be used only with reference to actual physical bodies and the term centroid with reference to geometric figures or curves.

To define analytically the position of the centroid of area of a plane figure, we refer the area to coordinate axes, as shown in Fig. 71*a*. Then denoting by ΔA_i the area of one element and by x_i, y_i the coordinates of its center, Eqs. (13) will take the form

$$x_c = \frac{\Sigma(\Delta A_i x_i)}{\Sigma \Delta A_i} \qquad y_c = \frac{\Sigma(\Delta A_i y_i)}{\Sigma \Delta A_i} \tag{14}$$

where the summations are understood to include all elements of area within the boundary of the figure. In the same manner, we may define the position of the centroid of length of a plane curve (Fig. 71*b*) by the formulas

$$x_c = \frac{\Sigma(\Delta L_i x_i)}{\Sigma \Delta L_i} \qquad y_c = \frac{\Sigma(\Delta L_i y_i)}{\Sigma \Delta L_i} \tag{15}$$

where ΔL_i denotes the length of an element and x_i, y_i the coordinates of its mid-point.

Since we can never perfectly match the boundary of the plane figure in Fig. 71*a* by filling it with small squares ΔA_i, or the true shape of the curve in Fig. 71*b* by straight-line segments ΔL_i, it is evident that expressions (14) and (15) are approximate. However, by making the elements ΔA_i or ΔL_i smaller and smaller, expressions (14) and (15) become more and more accurate. In the limit, when ΔA_i or ΔL

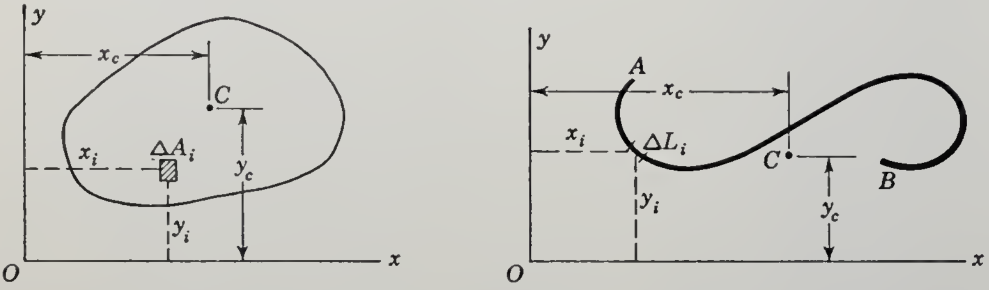
become infinitesimal in size (and infinite in number), expressions (14) and (15) become exact. In such case, we write them in the form

$$x_c = \frac{\int x\,dA}{\int dA} \qquad y_c = \frac{\int y\,dA}{\int dA} \tag{16}$$

for the case of area (Fig. 71*a*) and

$$x_c = \frac{\int x\,dL}{\int dL} \qquad y_c = \frac{\int y\,dL}{\int dL} \tag{17}$$

for the case of length (Fig. 71*b*). The indicated summations can then be made by means of the calculus.



(*a*) (*b*)

FIG. 71

Using formulas (16) or (17), the coordinates of the centroid of any plane figure or curve can be calculated, provided the integrals appearing therein can be evaluated. In general, this is possible, wherever the shape of the figure or curve can be defined by an analytic expression $y = f(x)$. If this is not possible, we divide the area or length into a reasonable number of small but finite elements and then use formulas (14) and (15), making the summations numerically.

Sometimes the position of the centroid of a plane figure or curve can be seen by inspection. For example, if a figure has two axes of symmetry, its centroid lies at their intersection. This statement follows at once from the form of expressions (16) and (17). In Fig. 72, for example, suppose that x and y are axes of symmetry. This means that if the figure is folded about one of these axes every point on one side will coincide with a corresponding point on the other. Thus in the numerators of expressions (16) there will be a minus $x\,dA$ or $y\,dA$ for every plus $x\,dA$ or $y\,dA$ and we obtain $x_c = 0$, $y_c = 0$. Hence the centroid lies at the origin of coordi-

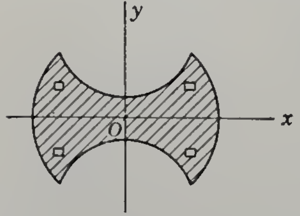


FIG. 72

nates, i.e., at the intersection of axes of symmetry. The same reasoning applies to the centroid of length of the perimeter of such a figure.

Some plane figures while not having any axes of symmetry may be said to be *symmetrical about a point.* That is, there is one point C in the figure which is the mid-point of every conceivable diameter (Fig. 73). In this case also it follows from expressions (16) that this point is the centroid of area of the figure or from expressions (17) that it is the centroid of length of the perimeter.

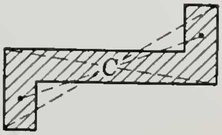


FIG 73

The centroid of the area of any triangle lies at the intersection of its medians. This may be proved as follows: Considering $\triangle ABD$ (Fig. 74), let its area be divided into infinitesimal strips parallel to the side AD as shown. Then it is evident that the centroid of each strip lies at the mid-point of its length. The locus of such points is the median Bb, and we conclude that the centroid of the entire area lies thereon. Repeating the same reasoning for strips parallel to AB or BD, we reach the conclusion stated above. The point of intersection C, of course, lies two-thirds of the length of any median from the corresponding vertex. It should be noted that while point C (Fig. 74) is the centroid of area of the triangle, it is not the centroid of length of the perimeter ABD.

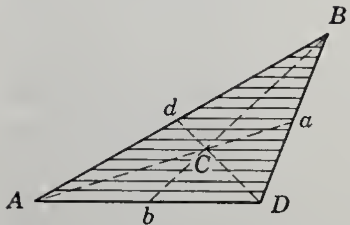
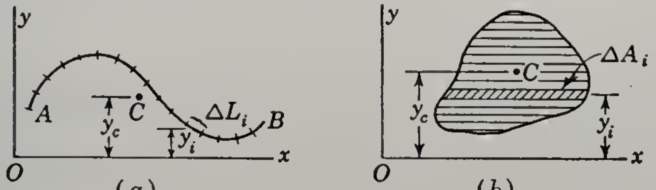


FIG. 74

Theorems of Pappus. 1. *The area of the surface generated by rotating any plane curve about a nonintersecting axis in its plane is equal to the product of the length L of the curve and the distance traveled by its centroid.*



(a) (b)

FIG. 75

2. *The volume of the solid generated by rotating any plane figure about a nonintersecting axis in its plane is equal to the product of the area A of the figure and the distance traveled by its centroid.*

The above statements are called the theorems of Pappus. The first may be proved as follows: Let AB (Fig. 75*a*) be any plane curve of length L that lies in the xy plane and does not intersect the x axis. Let the length of this curve be divided into elements, each so short that it may be considered as a minute portion of a straight line. Let the y coordinate of the mid-point of any element ΔL_i be denoted by y_i, and that of the centroid C of the entire curve AB, by y_c. Now let the plane of the curve be rotated through any angle α (measured in radians) around the x axis. Then, obviously, the distance traveled by the mid-point of an element ΔL_i is αy_i and the area of the curved surface generated by this one element is $\Delta L_i \alpha y_i$. Hence, by virtue of the second of Eqs. (15), the area of the surface generated by the entire curve AB becomes

$$\Sigma\, \Delta L_i \alpha y_i = \alpha \Sigma\, \Delta L_i y_i = \alpha y_c \Sigma\, \Delta L_i = \alpha y_c L \qquad \text{(a)}$$

Since, however, αy_c is the distance traveled by the centroid C of the entire curve, the theorem is proved.

The second theorem may be proved in the same way. Let the area A of any plane figure (Fig. 75*b*) be divided into a large number of very thin strips parallel to the x axis, and let the y coordinate of the middle of any strip of area ΔA_i be denoted by y_i. Now, let the plane of the figure be rotated through any angle α around the x axis. Obviously, any element of area ΔA_i generates a sector of a thin ring, the volume of which can be taken equal to $\Delta A_i \alpha y_i$ since the ring is very thin. Hence, by virtue of the second of Eqs. (14), the total volume generated becomes

$$\Sigma\, \Delta A_i \alpha y_i = \alpha \Sigma\, \Delta A_i y_i = \alpha y_c \Sigma\, \Delta A_i = \alpha y_c A \qquad \text{(b)}$$

which proves the theorem.

The theorems of Pappus are very useful in calculating the surface areas and volumes of various bodies of revolution encountered in engineering, particularly in machine design.

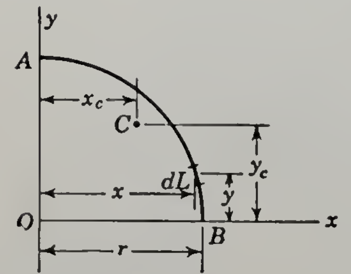


FIG. 76

EXAMPLES

1. Determine, by integration, the coordinates x_c and y_c of the centroid C of the quadrant AB of the arc of a circle of radius r (Fig. 76).

Solution. With the origin of coordinates at the center of the circle and the axes x and y directed as shown, the equation of the curve AB is

$$x^2 + y^2 = r^2$$

Now let the arc AB be divided into elements of infinitesimal length dL like the one shown in the figure. Then the length of any such element evidently is

$$dL = \sqrt{(dx)^2 + (dy)^2} = \frac{r}{y}\, dx$$

Hence, by the second of Eqs. (17), we have

$$y_c = \frac{\int y\,dL}{\int dL} = \frac{\int_0^r y\,\frac{r}{y}\,dx}{\int dL} = \frac{r^2}{\pi r/2} = \frac{2r}{\pi} \tag{c}$$

It follows from symmetry that x_c has the same value.

2. Determine the coordinates x_c and y_c of the centroid C of the area of the spandrel OBD (Fig. 77) if the curve OD is a portion of a parabola with vertical axis Oy.

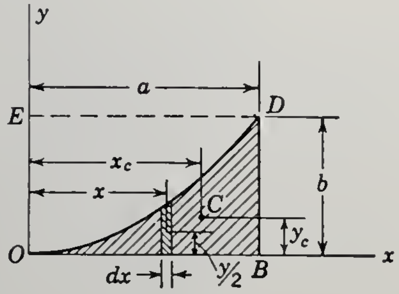


FIG. 77

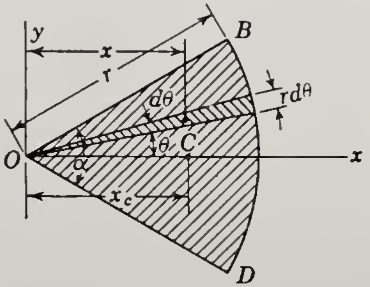


FIG. 78

Solution. The equation of a parabola with vertical axis and vertex at the origin of coordinates is

$$x^2 = 4ky$$

To determine the value of the constant k for the particular case represented in Fig. 77, we observe that when $x = a$, $y = b$. Hence

$$k = \frac{a^2}{4b}$$

and the equation of the curve OD becomes

$$y = \frac{b}{a^2}x^2$$

Now let the shaded area OBD be divided into narrow elements each of height y and width dx like the one shown in the figure. The area dA of any such element obviously is $y\,dx$ and the coordinates of its centroid are x and $y/2$. Hence, by Eqs. (16), we have

$$x_c = \frac{\int x\,dA}{\int dA} = \frac{\int_0^a x\,\frac{b}{a^2}x^2\,dx}{\int_0^a \frac{b}{a^2}x^2\,dx} = \frac{\frac{a^2b}{4}}{\frac{ab}{3}} = \frac{3}{4}a \tag{d}$$

$$y_c = \frac{\int \frac{y}{2}\,dA}{\int dA} = \frac{\int_0^a \frac{b}{2a^2}x^2\,\frac{b}{a^2}x^2\,dx}{\int_0^a \frac{b}{a^2}x^2\,dx} = \frac{\frac{ab^2}{10}}{\frac{ab}{3}} = \frac{3}{10}b \tag{e}$$

3. Determine the coordinates x_c and y_c of the centroid C of the area of the circular sector OBD of radius r and central angle α (Fig. 78).

Solution. In this case let us divide the shaded area of the figure into infinitesimal triangular elements each of altitude r and base $r\,d\theta$ like that shown in the figure. The area dA of any such element will be $r^2\,d\theta/2$ and the x coordinate of its centroid will be $\frac{2}{3}r\cos\theta$. Then by the first of Eqs. (16) we have

$$x_c = \frac{\int x\,dA}{\int dA} = \frac{2\int_0^{\alpha/2}\frac{2}{3}r\cos\theta\,\frac{r^2\,d\theta}{2}}{2\int_0^{\alpha/2}\frac{r^2}{2}\,d\theta} = \frac{\frac{r^3}{3}\sin\frac{\alpha}{2}}{\frac{\alpha r^2}{4}} = \frac{4}{3}\frac{r}{\alpha}\sin\frac{\alpha}{2} \tag{f}$$

From symmetry we conclude that $y_c = 0$. Substituting $\alpha = \pi$ in Eq. (f) gives, for the case of a semicircle, $x_c = 4r/3\pi$.

PROBLEM SET 2.3

1. Determine the coordinates x_c and y_c of the centroid C of the area of one quadrant of an ellipse OAB with major and minor semiaxes a and b, respectively, (Fig. A). *Ans.* $x_c = 4a/3\pi$; $y_c = 4b/3\pi$.

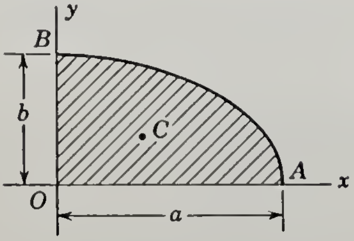


FIG. A

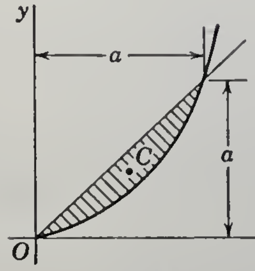


FIG. B

2. Determine the coordinates x_c and y_c of the centroid C of the area between the parabola $y = x^2/a$ and the straight line $y = x$ (Fig. B). *Ans.* $x_c = a/2$; $y_c = 2a/5$.

3. Determine the coordinates x_c and y_c of the centroid C of the length of a circular arc AB of radius r and central angle α (as shown in Fig. C). *Ans.* $x_c = (2r/\alpha)\sin(\alpha/2)$; $y_c = 0$.

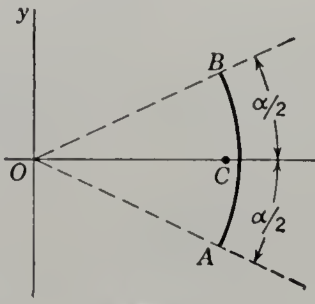


FIG. C

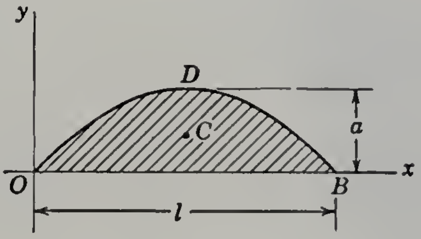


FIG. D

4. Determine the coordinates x_c and y_c of the centroid C of the area between the x axis and the half sine wave ODB (Fig. D).

Hint. The equation of the sine wave is $y = a \sin (\pi x/l)$. *Ans.* $x_c = l/2$; $y_c = \pi a/8$.

5. Using the second theorem of Pappus, calculate the volume of the ring shown in Fig. E if $R = 10$ in., $r = 4$ in. *Ans.* $V = 2{,}502$ in.^{3}

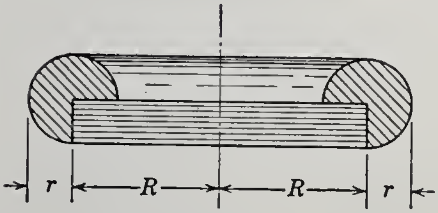


FIG. E

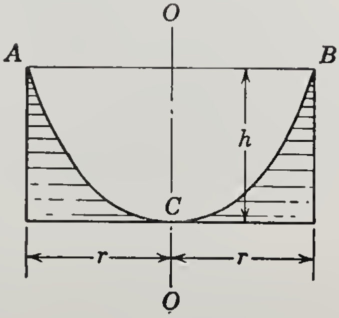


FIG. F

6. A right circular cylindrical tank containing water spins about its vertical geometric axis OO at such speed that the free water surface is a parabaloid ACB (Fig. F). What will be the depth of water in the tank when it comes to rest? *Ans.* $h/2$.

7. Referring to Fig. G, prove that, if the equation of the curve OB, referred to the coordinate axes x and y taken along two adjacent sides of a rectangle $OEBD$, is $y = kx^n$, then the coordinates x_c and y_c of the centroid C of the area of the shaded spandrel OBD are given by the formulas

$$x_c = \frac{n+1}{n+2}a \qquad y_c = \frac{n+1}{4n+2}b$$

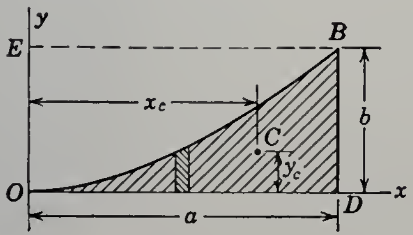


FIG. G

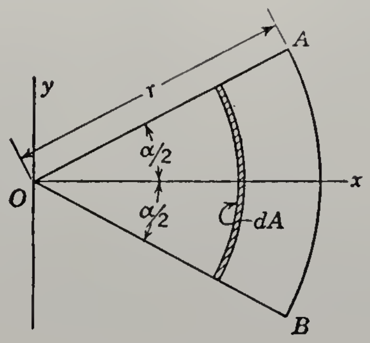


FIG. H

8. Using the result of Prob. 3 and an elemental area dA as shown in Fig. H, locate the centroid C of the area of the circular sector OAB by integration. *Ans.* $x_c = (4r/3\alpha) \sin (\alpha/2)$; $y_c = 0$.

9. Compute, approximately, the coordinate y_c of the centroid of area of the shaded figure in Fig. I. For dimensions, the figure is shown on a $\frac{1}{8}$-in. square grid background and all curves are circular arcs.

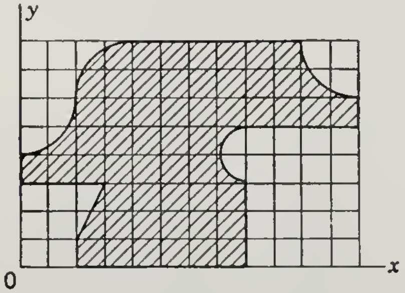


FIG. I

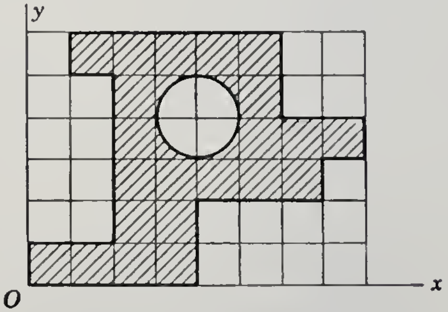


FIG. J

Hint. Subdivide the area into eight horizontal strips and make summations numerically by Eqs. (14). *Ans.* $y_c = 0.543$ in.

10. Calculate, numerically, the coordinates x_c and y_c of the centroid C of the shaded area shown in Fig. J. The superimposed grid has $\frac{1}{8}$-in. squares.

2.4. Centroids of composite plane figures and curves. The positions of the centroids of most common shapes of figures and curves such as triangles, circular or elliptical sectors or quadrants, parabolic spandrels, etc., can readily be found in various handbooks and there is little necessity to recalculate them every time they are needed. The problem most often encountered in engineering practice is that of locating centroids of *composite areas or curves.* A composite area or curve is one which can be broken down into several pieces or components that represent familiar geometric shapes and for which the positions of individual centroids are already known. Two examples of such composite figures are shown in Fig. 79.

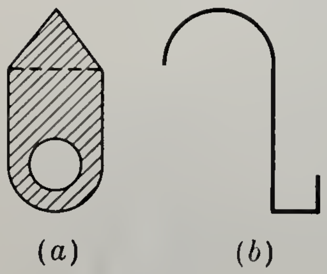


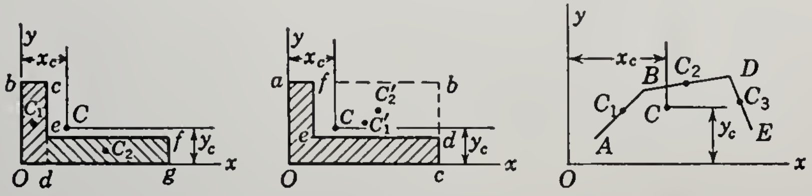
FIG. 79

In locating the centroids of area or length of such figures, it is not necessary to subdivide the figure into infinitesimal elements and then perform involved integrations or cumbersome numerical summations as was done in Art. 2.3. Instead, we simply subdivide the figure into its several component parts, for which the centroids are already known and then use Eqs. (14) or (15), treating each component part as an element. In using Eqs. (14) and (15), it is not necessary that an element ΔA_i or ΔL_i of area or length be infinitesimal or even small so long as we know the position of its centroid.

Consider, for example, the plane figure shown in Fig. 80*a*, consisting of two rectangles *Obcd* and *defg* of areas A_1 and A_2 and with known centroids C_1 and C_2, respectively. Denoting by x_1, y_1 and x_2, y_2 the coordinates of the known centroids C_1 and C_2, respectively, and using Eqs. (14), we obtain for the coordinates x_c and y_c of the centroid C of the composite area the following expressions:

$$x_c = \frac{A_1x_1 + A_2x_2}{A_1 + A_2} \qquad y_c = \frac{A_1y_1 + A_2y_2}{A_1 + A_2} \tag{a}$$

If desired, we can obtain the same coordinates in another way. Referring to Fig. 80*b*, let us consider the shaded area to be obtained by cutting out from the rectangle *Oabc*, of area A_1' and known centroid C_1', the rectangle *efbd* of area A_2' and known centroid C_2'. Obviously the resultant area of the shaded figure is equal to $A_1' - A_2'$, and proceeding as before, it will only be necessary in making the summations indicated in Eqs. (14) to consider the area A_2' as negative. Thus, if we denote by x_1', y_1' and x_2', y_2' the coordinates of the known centroids C_1' and C_2', respectively, we obtain



(*a*) (*b*) (*c*)

FIG. 80

$$x_c = \frac{A_1'x_1' - A_2'x_2'}{A_1' - A_2'} \qquad y_c = \frac{A_1'y_1' - A_2'y_2'}{A_1' - A_2'} \tag{b}$$

The same procedure may be used in determining the coordinates of the centroid of the length of a plane curve composed of several parts for which the individual centroids are known. Consider, for example, the broken line *ABDE* (Fig. 80*c*) composed of the portions *AB*, *BD*, and *DE* of three straight lines of lengths L_1, L_2, and L_3 and known centroids C_1, C_2, and C_3, respectively. Denoting by x_1, y_1, x_2, y_2, and x_3, y_3 the coordinates of the known centroids C_1, C_2, and C_3 respectively, and using Eqs. (15), we obtain

$$x_c = \frac{L_1x_1 + L_2x_2 + L_3x_3}{L_1 + L_2 + L_3} \qquad y_c = \frac{L_1y_1 + L_2y_2 + L_3y_3}{L_1 + L_2 + L_3} \tag{c}$$

The application of Eqs. (14) and (15) to the location of centroids of composite plane figures and curves will now be shown by several practical examples.

EXAMPLES

1. Determine the coordinate y_c of the centroid C of a trapezoid having bases a and b and altitude h as shown in Fig. 81.

Solution. Drawing the line DB, we divide the figure into two triangles ABD and DEB with known centroids C_1 and C_2, respectively, as shown. Then using the second of Eqs. (14), we write

$$y_c = \frac{(ah/2)(h/3) + (bh/2)(2h/3)}{ah/2 + bh/2}$$

which reduces to

$$y_c = \frac{h(a + 2b)}{3(a + b)} \tag{d}$$

It will be noted that when $b = 0$ and the figure is a triangle, expression (d) gives $y_c = h/3$, while for the case of a rectangle ($a = b$), it gives $y_c = h/2$.

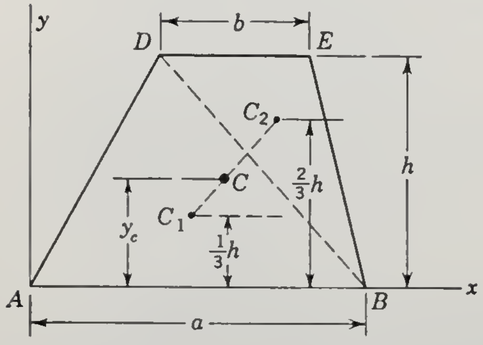


FIG. 81

2. Prove that the centroid C of any trapezoid can be located graphically by the construction illustrated in Fig. 82, i.e., by the intersection of the lines mn and pq.

Solution. Note that the shaded triangles mpC and nqC are geometrically similar. Hence we may write

$$\frac{y_c}{h - y_c} = \frac{a/2 + b}{a + b/2}$$

which is readily brought to the form

$$y_c = \frac{h(a + 2b)}{3(a + b)}$$

as previously obtained in Example 1 above.

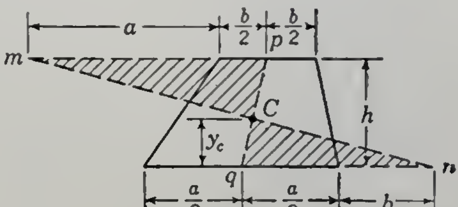


FIG. 82

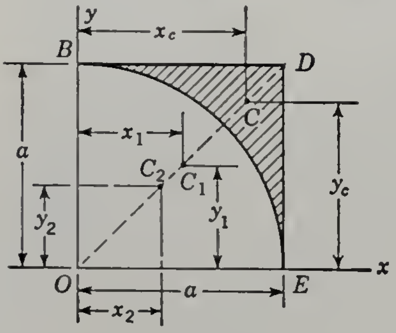


FIG. 83

3. Locate the centroid C of the shaded area of the figure BDE (Fig. 83) which is obtained by cutting the quadrant of a circle of radius a from a square $OBDE$ of the same dimensions.

Solution. From symmetry of the figure it is evident that the desired centroid C lies on the diagonal OD of the square. Hence, choosing coordinate axes x and y as shown in the figure, it will only be necessary to determine one coordinate of the centroid, since $x_c = y_c$. To determine x_c, let us denote by A_1 the area of the square, by x_1 the x coordinate of its centroid C_1, and by A_2, x_2 the corresponding quantities for the quadrant of the circle. Then the first of Eqs. (14) gives

$$x_c = \frac{A_1x_1 - A_2x_2}{A_1 - A_2}$$

For the given dimensions and remembering that $x_2 = y_2 = 4a/3\pi$ (see page 86) this becomes

$$x_c = \frac{a^2(a/2) - (\pi a^2/4)(4a/3\pi)}{a^2 - \pi a^2/4}$$

from which

$$y_c = x_c = \frac{2a}{3(4 - \pi)} = 0.777a \tag{e}$$

4. With reference to the coordinate axes x and y, locate the centroid of the area of the cross section of the stirrup shown in Fig. 84.

Solution. In dealing with this figure it will be simplest to imagine that it is obtained by subtracting the smaller semicircular area A_2 from the larger semicircular area A_1 to give the semicircular shaded ring labeled $A_1 - A_2$ and then to add to this ring the three rectangular shaded areas A_3, A_4, and A_5 as shown in the figure. In cases like this where it is necessary to consider a large number of elements, it is highly desirable to tabulate the data necessary for making the summations indicated in Eqs. (14). Otherwise it is almost impossible to avoid mistakes. A suggested form of tabulation is shown below.

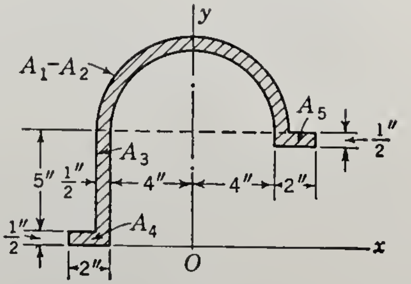


FIG. 84

No.	ΔA_i	x_i	y_i	$\Delta A_i x_i$	$\Delta A_i y_i$
1	31.8	0.00	7.41	0.00	235.60
2	−25.1	0.00	7.20	0.00	−180.70
3	2.5	−4.25	3.00	−10.62	7.50
4	1.0	−5.00	0.25	− 5.00	0.25
5	1.0	+5.00	5.25	+ 5.00	5.25
Σ	11.2	......	....	−10.62	67.90

Using Eqs. (14) and the sums found in the above table, we obtain

$$x_c = \frac{-10.62}{11.2} = -0.95 \text{ in.} \qquad y_c = \frac{67.90}{11.2} = +6.06 \text{ in.}$$

5. The shaded figure shown in Fig. 85 represents the cross section of an 8- by 6-in., 20.2-lb angle iron commonly used in structural engineering. Using the given numerical dimensions of the figure, determine the coordinates x_c and y_c, as shown, of its centroid C.

Solution. Let us first neglect entirely the three rounded corners A_3, A_4, and A_5 and assume that we have simply a figure composed of the two major rectangular portions A_1 and A_2 with known centroids C_1 and C_2, as shown. The centroid C_0 (not shown) for this approximation of the true figure will be very close to the true centroid C, and the determination of its coordinates x_0 and y_0 involves little chance of numerical mistakes. After the centroid C_0 of the approximate figure has been located, the determination of the coordinates x_c and y_c of the true centroid C can be handled as a problem involving only small corrections to the values of x_0 and y_0.

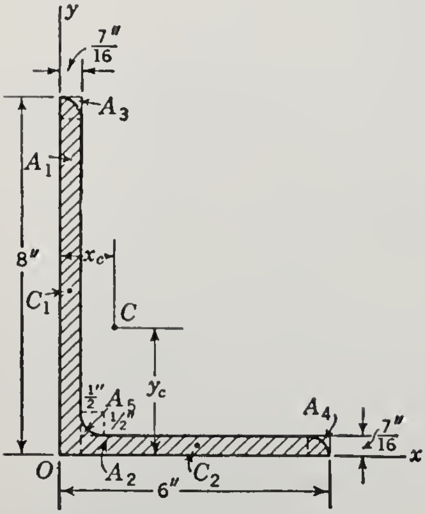


FIG. 85

Beginning, then, with the approximate figure composed of two rectangular portions A_1 and A_2 as shown and using the given dimensions, we may fill in the first two horizontal lines of the table below. This done, Eqs. (14) give

$$x_0 = \frac{8.601}{5.934} = 1.450 \text{ in.} \qquad y_0 = \frac{14.533}{5.934} = 2.449 \text{ in.} \tag{f}$$

No.	ΔA_i	x_i	y_i	$\Delta A_i x_i$	$\Delta A_i y_i$
1	3.500	0.219	4.000	0.766	14.000
2	2.434	3.219	0.219	7.835	0.533
Σ	5.934	1.450	2.449	8.601	14.533
3	−0.041	0.338	7.901	−0.014	−0.325
4	−0.041	5.901	0.338	−0.242	−0.014
5	+0.054	0.549	0.549	0.029	0.029
Σ	5.906			8.374	14.223

To determine now the coordinates x_c and y_c of the centroid C of the true figure, we consider this figure to be obtained by subtracting from the approximate figure, the centroid C_0 of which is defined by Eqs. (f), the small areas A_3 and A_4 and adding to it the small area A_5. Again using the given dimensions and the results of Example 3 [see Eq. (e)], the remaining lines of the table may be filled in. This done, Eqs. (14) give

$$x_c = \frac{8.374}{5.906} = 1.418 \text{ in.} \qquad y_c = \frac{14.223}{5.906} = 2.408 \text{ in.} \tag{g}$$

PROBLEM SET 2.4

1. Determine the coordinate y_c of the shaded area of the figure shown in Fig. A. The following dimensions are given: $a = 6$ in., $b = 1$ in., $c = 2$ in. *Ans.* $y_c = 1.10$ in.

2. If the dimensions a and b of the plane figure shown in Fig. A are fixed, find what the dimension c must be in order that the centroid of the shaded area will lie on the line AB. *Ans.* $c = \sqrt{ab/2}$.

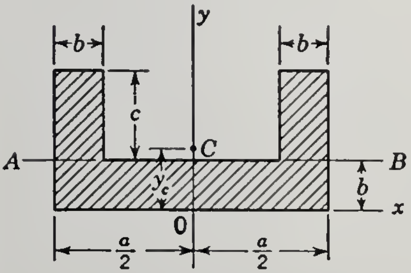


FIG. A

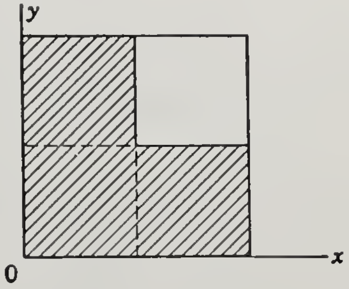


FIG. B

3. Locate the centroid of the shaded three-quarters of the area of a square of dimension a as shown in Fig. B. *Ans.* $x_c = y_c = \frac{5}{12}a$.

4. Locate the centroid of the shaded three-quarters of the area of a square of dimension a as shown in Fig. C. *Ans.* $x_c = a/2$; $y_c = \frac{7}{18}a$.

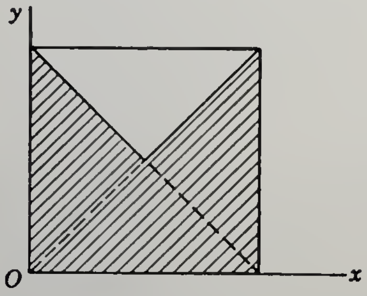


FIG. C

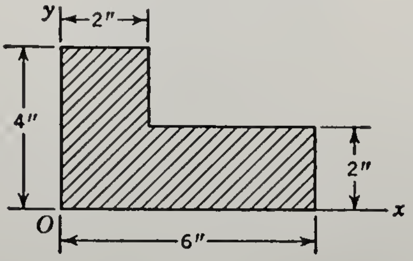


FIG. D

5. Locate the centroid of the shaded area of the figure shown in Fig. D. *Ans.* $x_c = 2.5$ in.; $y_c = 1.5$ in.

6. Referring to Fig. E, locate the centroid of the length of the mean center line of the stirrup with the dimensions shown. *Ans.* $x_c = -0.78$ in. $y_c = 4.78$ in.

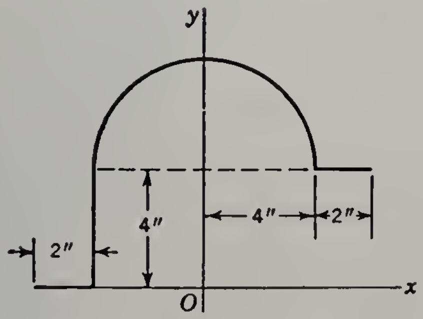


FIG. E

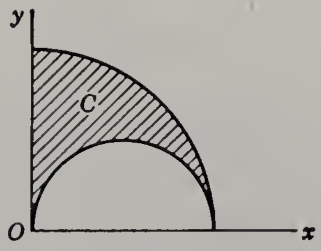


FIG. F

7. Locate the centroid C of the shaded area obtained by cutting a semicircle of diameter a from the quadrant of a circle of radius a as shown in Fig. F. *Ans.* $x_c = 0.349a$; $y_c = 0.636a$.

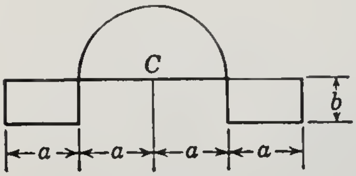


FIG. G

8. A slender homogeneous wire of uniform cross section is bent into the shape shown in Fig. G. If the dimension a is fixed, find the dimension b so that the center of gravity of the wire will coincide with the center C of the semicircular portion. *Ans.* $b = 0.618a$.

9. Locate the centroid C of the shaded area $OABD$ shown in Fig. H. *Ans.* $x_c = 2.63$ in.; $y_c = 2.36$ in.

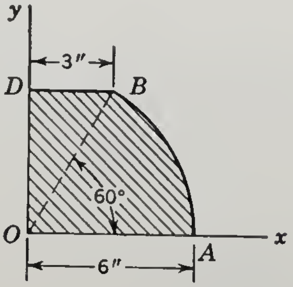


FIG. H

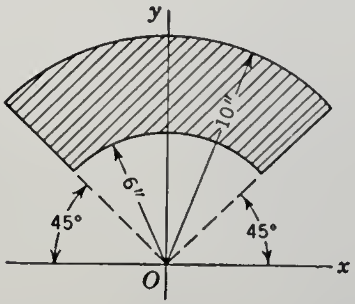


FIG. I

10. Locate the centroid C of the shaded sector of a ring subtending a 90° central angle and symmetrical about the y axis, as shown in Fig. I. *Ans.* $x_c = 0$; $y_c = 7.35$ in.

11. Locate the centroid C of the shaded area of the circular segment BD shown in Fig. J. *Ans.* $x_c = 11.04$ in.; $y_c = 0$.

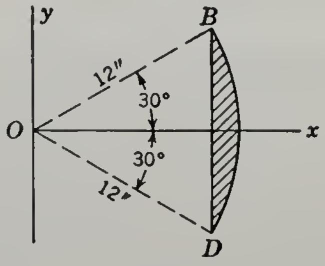


FIG. J

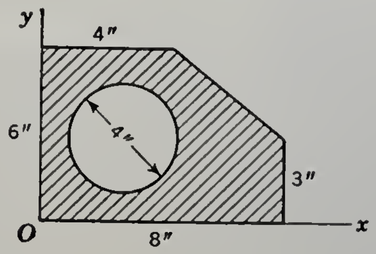


FIG. K

12. Referring to Fig. K, determine the coordinates x_c and y_c of the center of a 4-in.-diameter circular hole cut in a thin plate so that this point will be the centroid of the remaining shaded area. *Ans.* $x_c = 3.62$ in.; $y_c = 2.72$ in.

13. An isosceles triangle ADE is to be cut from a square $ABCD$ of dimension a as shown in Fig. L. Find the altitude y of this triangle so that its vertex E will be the centroid of the remaining shaded area. *Ans.* $y = 0.634a$.

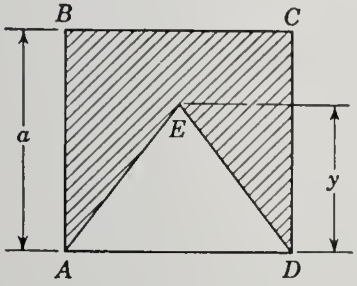


FIG. L

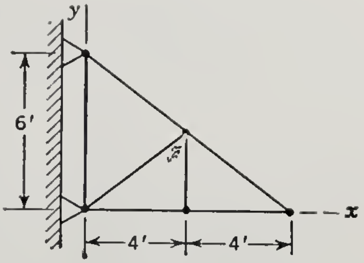


FIG. M

14. Locate the center of gravity of the plane truss shown in Fig. M if all bars have the same weight per unit length. *Ans.* $x_c = 2.94$ ft; $y_c = 1.875$ ft.

15. With respect to coordinate axes x and y, locate the centroid of the shaded area shown in Fig. N. *Ans.* $x_c = 1.9$ in.; $y_c = 0.6$ in.

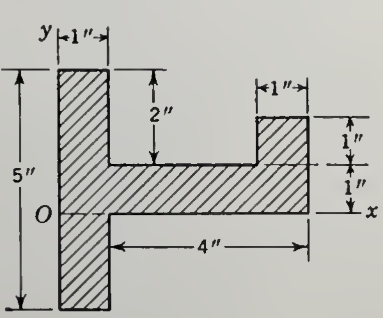


FIG. N

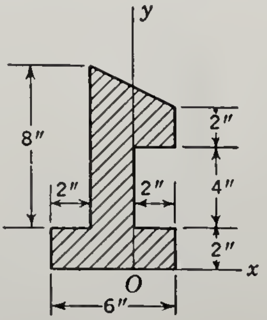


FIG. O

16. With reference to the coordinate axes x and y, locate the centroid of the shaded area of the plane figure shown in Fig. O. *Ans.* $x_c = -0.71$ in.; $y_c = +4.21$ in.

17. With respect to coordinate axes x and y, locate the centroid of the shaded area shown in Fig. P. *Ans.* $x_c = 1.92$ in.; $y_c = 1.91$ in.

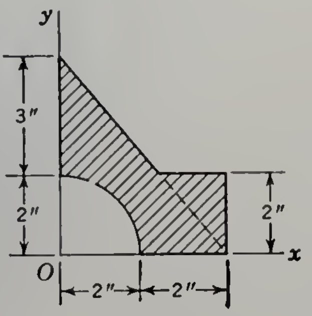


FIG. P

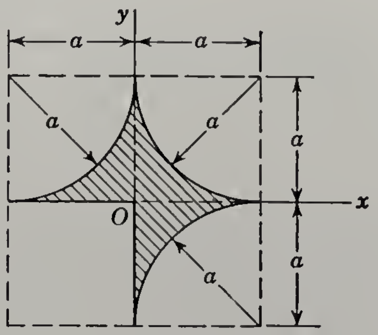


FIG. Q

18. With respect to the coordinate axes shown, locate the centroid of the shaded area in Fig. Q *Ans.* $x_c = y_c = 0.0744a$.

*19. The figure shown in Fig. R represents the cross section of a girder that is built up of plates and angle irons riveted together along their lengths normal to the plane of the figure. Using the given dimensions, determine the coordinate y_c of the centroid C of the section corrected for the reduction of area due to rivet holes. *Ans.* $y_o = 16.12$ in.; $y_c = 15.71$ in.

*20. Determine the coordinate y_c of the centroid C of the cross-sectional area of the built-up girder shown in Fig. S. All rivet holes are $\frac{3}{4}$ in. *Ans.* $y_c = 7.657$ in.

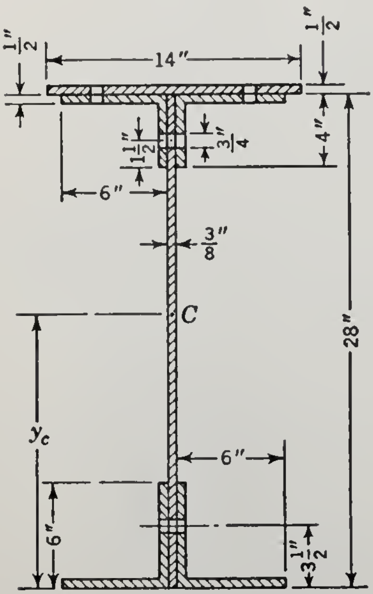


FIG. R

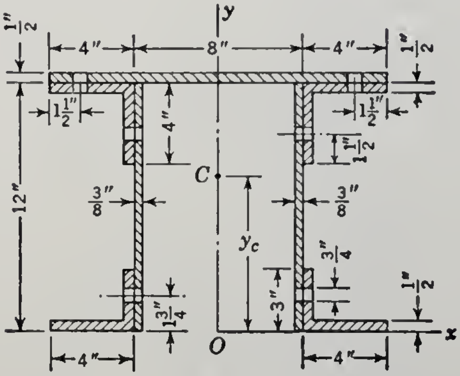
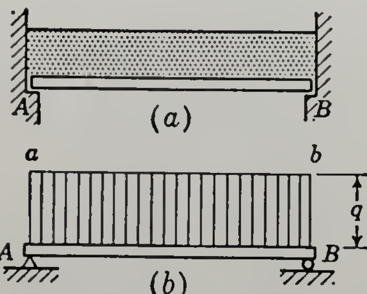


FIG. S

2.5. Distributed force in a plane. In previous problems, we have discussed the equilibrium of bodies under the action of concentrated forces. Sometimes we have to deal with *distributed force* such as water pressure against the face of a dam or ordinary gravity force.

We always can represent such distributed force by a so-called *load diagram.* Consider, for example, a beam AB on which sand is piled to a uniform depth d, as shown in Fig. 86*a*. This loading together with the weight of the beam itself represents a *uniform* distribution of gravity force which can be represented by the rectangular load diagram $AabB$ in Fig. 86*b*. The load per unit length of the *line of application* AB is called the *intensity* of distributed force and is represented in the figure by the uniform height q of the ordinates of the load diagram. If the sand is piled along the beam with variable depth as shown in Fig. 87*a*, we have a nonuniform distribution of gravity force as represented by the load diagram $AabB$ in Fig. 87*b*.



(*a*)
(*b*)

FIG. 86

In this case, any ordinate q of the diagram represents the intensity of load at the corresponding point on the line of application AB.

Let us consider now any case of distributed force in a plane as represented by the load diagram $AabB$ (Fig. 88). We imagine the line of application AB to be divided into an infinite number of elements, each of length dx and at the center of each of which the corresponding element of force dQ acts. In this way we have to deal with a system of parallel forces of various infinitesimal magnitudes, and the principles of Art. 2.2 for determining the magnitude and line of action of their resultant Q must apply. Denoting by q the intensity of force at any point, we conclude that

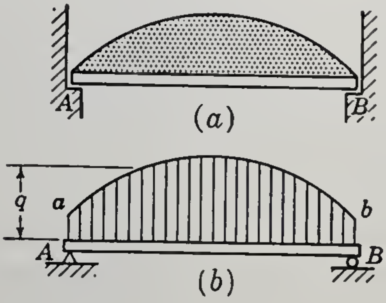


FIG. 87

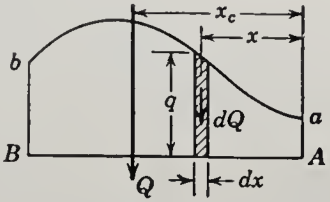


FIG. 88

$$dQ = q\,dx \qquad \text{(a)}$$

Referring to the load diagram (Fig. 88), it is evident that expression (a) also represents the area of a narrow vertical strip of width dx and height q of this diagram. Thus, an element dQ of the distributed force is represented graphically in the load diagram by the corresponding element of area $q\,dx$. Summing up all such elements of force and the corresponding elements of area of the load diagram, we conclude that the resultant force Q is represented graphically by the area $AabB$ of the load diagram.

We consider now the problem of determining the line of action of this resultant. Referring to Fig. 88, the moment with respect to point A of any element of force dQ, such as the one represented by the shaded strip in the diagram, is $dQ\,x$ or, using expression (a), $qx\,dx$, where x is the distance from point A to the element in question. The sum of the moments of all such elements of force with respect to point A may be expressed as follows:

$$\int_A^B qx\,dx \qquad \text{(b)}$$

Denoting by x_c the arm of the resultant Q with respect to point A and equating the moment of this resultant to the sum of the moments of all the elements of force [expression (b)], we obtain

$$Qx_c = \int_A^B qx\,dx \tag{c}$$

from which

$$x_c = \frac{\int_A^B qx\,dx}{Q} \tag{d}$$

Comparing expression (d) with the first of Eqs. (16), we conclude that the resultant force Q acts through the centroid of the area of the load diagram. Thus the load diagram determines both the magnitude and line of action of the resultant force.

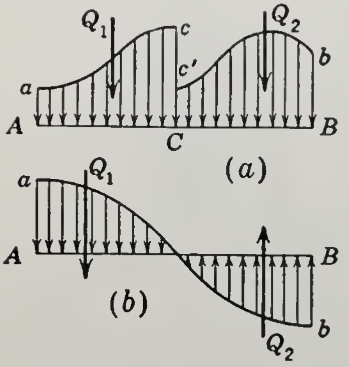


FIG. 89

If, as represented in Fig. 89*a*, the distribution of force in a plane is such that the bounding curve of the load diagram is discontinuous, each portion of the load between points of discontinuity can be treated separately and the resultants Q_1 and Q_2 handled as individual forces. The same treatment can be used also in the case where the sign of the distributed force is different for two portions of the line of application as represented in Fig. 89*b*. In such a case it may be found that the resultants Q_1 and Q_2 are equal in magnitude, in which case the distributed force is equivalent to a couple.

In dealing with bodies subjected to the action of distributed force, we shall always replace such distributed force by its resultant in the form of a single concentrated force or a couple, as the case may be, before attempting to consider the conditions of equilibrium of the body.

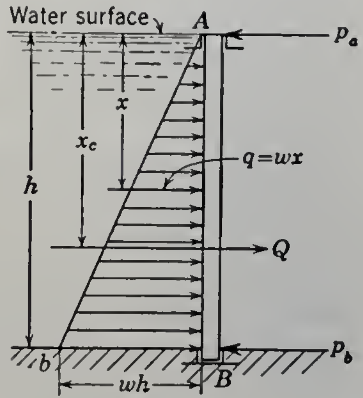


FIG. 90

EXAMPLES

1. A vertical sluice gate when closed is supported along the top and bottom edges AA and BB as shown in Fig. 90 and is subjected to water pressure from one side as indicated. The weight per unit volume of the water is w. Determine the intensities of the reactions furnished by the sup-

ports, assuming that they are uniformly distributed along the lines of support AA and BB perpendicular to the plane of the figure.

Solution. Let us limit our attention to a vertical strip of the gate which is of unit width in the direction perpendicular to the plane of the figure. Since this strip is symmetrically loaded with respect to its middle vertical transverse plane ABb, we may assume the action of all forces acting on it to be concentrated in this plane. Thus we arrive at the problem of the equilibrium of a beam AB under the action of distributed force in one plane together with concentrated reactions p_a and p_b as indicated in the figure and representing the desired intensities of the distributed reactions along AA and BB. Since the intensity of water pressure at any point is proportional to its weight per unit volume w and to the depth x from the surface, we conclude that the load diagram is a triangle AbB as shown.

The intensity of force at the depth h is wh and as the total resultant force Q is given by the area of the load diagram, we conclude that

$$Q = \frac{wh}{2}h = \frac{wh^2}{2} \tag{e}$$

Further, since this force acts through the centroid of the triangular load diagram, we have

$$x_c = \tfrac{2}{3}h$$

and the resultant force Q is completely determined.

Replacing the distributed force by its resultant Q applied as shown in the figure, we can find the reactions p_a and p_b per unit width of gate by using Eqs. (12) of Art. 2.2. Taking moments with respect to points A and B, these equations become

$$-p_bh + Q\frac{2}{3}h = 0 \qquad p_ah - Q\frac{h}{3} = 0$$

from which we find $p_a = Q/3$ and $p_b = \frac{2}{3}Q$. Substituting for Q its value from Eq. (e), we find

$$p_a = \frac{wh^2}{6} \quad \text{and} \quad p_b = \frac{wh^2}{3} \tag{f}$$

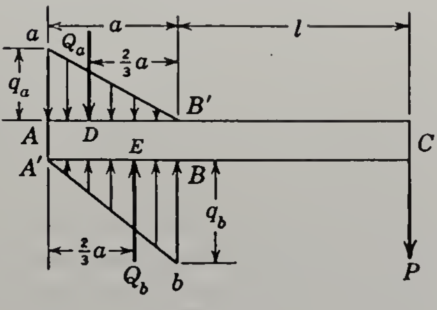


FIG. 91

2. One end of a cantilever beam AC is built into a wall of thickness a as shown in Fig. 91. Owing to the action of a load P applied at the free end C, distributed reactions, as represented by the load diagrams AaB' and $A'Bb$, are produced. Find the maximum intensities q_a and q_b of these reactions. Neglect the weight of the beam, and assume all forces to act in its vertical axial plane of symmetry.

Solution. Since we assume a linear variation in the intensities of pressure along the lines AB' and $A'B$, it follows that the lines of action of the resultants

Q_a and Q_b are as shown in the figure. Replacing the distributed reactions by these resultants, we obtain a system of three parallel forces in a plane which are in equilibrium. Hence, taking moments of the three forces P, Q_a, and Q_b with respect to points D and E and using Eqs. (12), we obtain

$$Q_b\frac{a}{3} - P\left(l + \frac{2}{3}a\right) = 0 \qquad Q_a\frac{a}{3} - P\left(l + \frac{1}{3}a\right) = 0$$

from which

$$Q_a = \frac{3P}{a}\left(l + \frac{1}{3}a\right) \quad \text{and} \quad Q_b = \frac{3P}{a}\left(l + \frac{2}{3}a\right)$$

From the fact that the resultant of a distributed force in a plane is equal to the area of its load diagram, we obtain

$$Q_a = \frac{q_a}{2}a \quad \text{and} \quad Q_b = \frac{q_b}{2}a$$

from which

$$q_a = \frac{2Q_a}{a} \quad \text{and} \quad q_b = \frac{2Q_b}{a}$$

By using the values of Q_a and Q_b given above, we find for the desired maximum intensities

$$q_a = \frac{6P}{a^2}\left(l + \frac{1}{3}a\right) \quad \text{and} \quad q_b = \frac{6P}{a^2}\left(l + \frac{2}{3}a\right) \tag{g}$$

Owing to the appearance of a^2 in the denominators of these expressions, it is seen that the intensities q_a and q_b increase very rapidly with a decrease of the thickness a of the wall. Thus, if we attempt to build the end of the cantilever beam into a very thin wall, it is likely that the intensities of pressure at points A and B will exceed the crushing strength of the masonry of the wall.

3. A gravity dam having a rectangular cross section of height h and width c rests on a bedrock foundation, as shown in Fig. 92*a*, and is submitted to water pressure on one side as shown. Neglecting friction and assuming that the vertical pressure exerted by the foundation is distributed as represented by the trapezoidal load diagram $AabB$, find the intensities q_a and q_b of this reactive pressure. The weight of water per unit volume is w and the weight per unit volume of the masonry in the dam is w_1.

Solution. Again we shall confine our attention to a section of the dam of unit length in the direction normal to the plane of the figure and assume that all forces acting on this section are concentrated in its middle transverse plane. From the principle of superposition, which states that the action of a given system of forces will not be changed if we add to or subtract from that system any other system of forces in equilibrium, it follows that we can think of the section of dam under consideration as being in equilibrium under the action of two superimposed systems of forces each of which alone is in equilibrium. The first system is represented by the weight $W = w_1ch$ together with a uniformly distributed reactive pressure over the base AB, as shown in Fig. 92*b*. The intensity of this distributed reactive force obviously is

$$q_1 = w_1 h \tag{h}$$

The second system consists of the resultant water pressure $P = wh^2/2$ acting at the height $h/3$ above the base, a horizontal reaction of the same magnitude acting at point B, and the reactive force distributed along the line AB, as shown in Fig. 92c. We see at once that the forces P constitute a couple that can be balanced only by another couple of equal magnitude and opposite sign (see Art. 2.1). The distribution of force along the line AB must be taken linear, since the resultant distribution (Fig. 92a) is assumed linear; it is represented by the load diagram shown in Fig. 92c. This distributed force can have for its resultant a couple only if the two triangular portions of the diagram are equal in area. Equating the moment of the couple Q_2Q_2 to the moment of the couple PP, we obtain

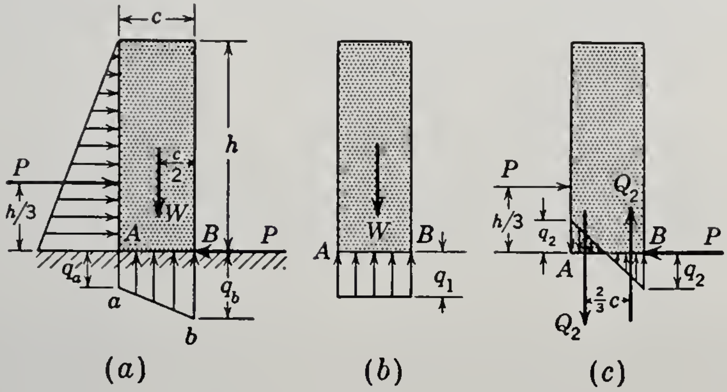


FIG. 92

$$Q_2 \frac{2}{3} c = P \frac{h}{3}$$

Substituting $Q_2 = q_2 c/4$ and $P = wh^2/2$, this becomes

$$\frac{q_2 c}{4} \frac{2}{3} c = \frac{wh^2}{2} \frac{h}{3}$$

from which

$$q_2 = \frac{wh^3}{c^2} \tag{i}$$

The superposition of the two systems of forces represented by Figs. 92b and 92c gives the complete system of forces under which the dam is in equilibrium as represented in Fig. 92a. Hence we conclude that the trapezoidal load diagram $AabB$ (Fig. 92a) is the result of the superposition of the diagrams in Figs. 92b and 92c, and the desired intensities of pressure q_a and q_b are [see Eqs. (h) and (i)]

$$q_a = w_1 h - \frac{wh^3}{c^2} \qquad q_b = w_1 h + \frac{wh^3}{c^2} \tag{j}$$

It is interesting to note from the first of Eqs. (j) that, when

$$w_1 h = \frac{wh^3}{c^2}$$

from which

$$\frac{c}{h} = \sqrt{\frac{w}{w_1}} \tag{k}$$

the intensity of pressure q_a becomes zero. If the ratio of the width of the dam to its height is made less than the value (k), there will be a portion of the line AB that will not press against the foundation. Under such circumstances, water can seep under the dam with the possibility of doing serious damage.

PROBLEM SET 2.5

1. Compute the reactions at the supports B and C of the beam AC loaded as shown in Fig. A. Neglect the weight of the beam itself. *Ans.* $R_b = 733$ lb, up; $R_c = 133$ lb, down.

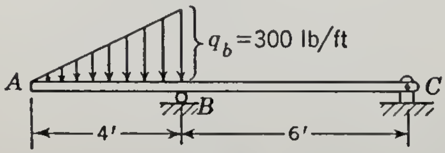


FIG. A

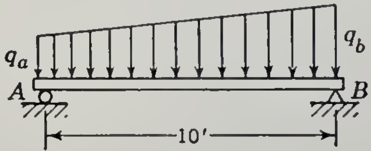


FIG. B

2. Compute the reactions at the supports A and B of the beam loaded as shown in Fig. B if $q_a = 100$ lb/ft, and $q_b = 200$ lb/ft. *Ans.* $R_a = 666.7$ lb; $R_b = 833.3$ lb.

3. An opening 1 ft wide and 4 ft high in a vertical partition between two tanks is closed by a gate $AABB$ hinged along AA and resting against a stop along BB, as shown in Fig. C. If the water levels on the two sides of the partition are as shown, find the total reaction exerted by the stop at B. Assume the specific weight of water to be $w = 62.4$ lb/ft³. *Ans.* $R_b = 499$ lb.

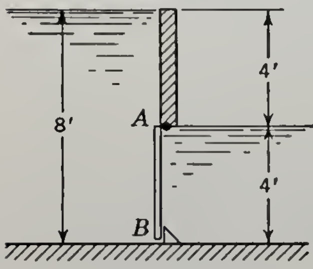


FIG. C

4. Two tanks are separated by an inclined partition the plane of which makes 60 deg. with the horizontal as shown in Fig. D. In this partition there is a rectangular opening 1 ft. wide and 4 ft. long which is closed by a gate hinged along AA as shown. If water ($w = 62.4$ lb/ft³) stands at the level shown on the left of the gate, what vertical pull S applied at B will be required to open the gate against the water pressure? *Ans.* $S = 577$ lb.

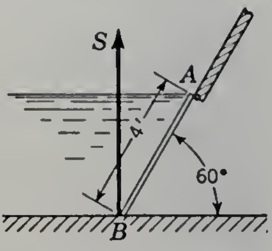


FIG. D

5. Calculate the vertical pull S necessary to open the gate in Prob. 4 if water stands at the depth $h = 6$ ft on the left side of the gate. *Ans.* $S = 1{,}210$ lb.

6. A gravity dam having a trapezoidal cross section is submitted to water pressure against its vertical face, as shown in Fig. E. If the specific weight of water is w and that of masonry $2.5w$, find the minimum width b at the base for which the dam will have a factor of safety of 2 against overturning about B. *Ans.* $b = 15.86$ ft.

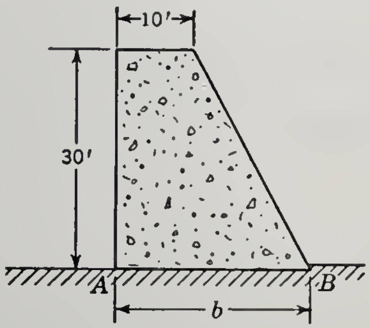


FIG. E

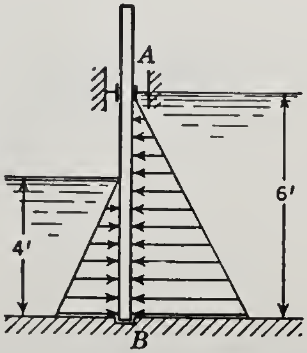


FIG. F

7. When closed, a vertical sluice gate (Fig. F) is supported along the lines AA and BB normal to the plane of the figure and is submitted to the action of water pressure from both sides, as indicated in the figure. If the weight of water per unit volume is $w = 62.4$ lb/ft^3, find the intensities p_a and p_b of the reactions developed by the supports, assuming them to be uniformly distributed along AA and BB. *Ans.* $p_b = 360$ lb/ft; $p_a = 263$ lb/ft.

8. Calculate the reactions R_a and R_b for the beam loaded as shown in Fig. G. Neglect weight of the beam. *Ans.* $R_a = 19$ lb, down; $R_b = 1{,}119$ lb, up.

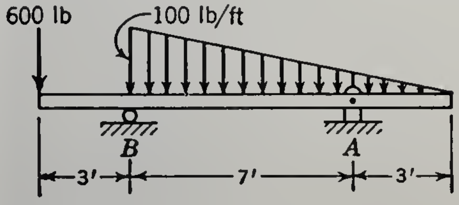


FIG. G

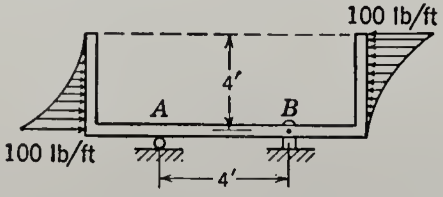


FIG. H

9. If the distributions of load on the end portions of the bar in Fig. H are parabolic, find the reactions at the supports A and B. *Ans.* $R_a = 66.7$ lb, up; $R_b = 66.7$ lb, down.

***10.** A water channel 1 ft wide is closed by a gate AB, hinged at D, as shown in Fig. I. Find the minimum and maximum depths h of water for which the gate does not open automatically because of its own weight. Neglect friction and assume the specific weight of water to be $w = 62.4$ lb/ft^3. *Ans.* $h_{\text{min}} = 0.925$ ft; $h_{\text{max}} = 2.333$ ft

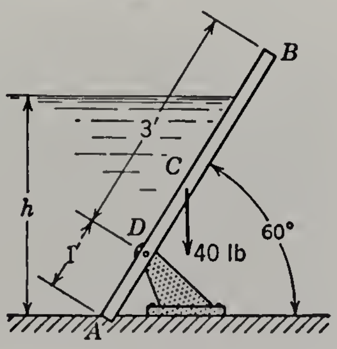
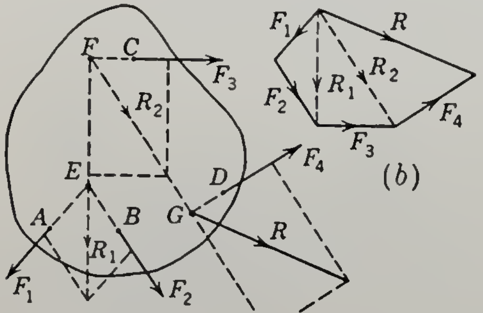


FIG. I

3

GENERAL CASE OF FORCES IN A PLANE

3.1. Composition of forces in a plane. If several coplanar forces applied to a body are not parallel and do not intersect in one point, we have the general case of forces in a plane. Let us consider such a system as represented by the forces $F_1, \ldots, F_4$, applied, respectively, at points A, B, C, and D of the body shown in Fig. 93*a*. To find the resultant of these forces, we begin with any two forces, say, F_1 and F_2, and determine their resultant R_1 by using the parallelogram law as indicated in the figure. Treating next the forces R_1 and F_3 in the same manner, we find their resultant R_2 which evidently is the resultant of the forces F_1, F_2, and F_3. In the same way again, the resultant R of the forces R_2 and F_4, and consequently of the given system of forces $F_1, \ldots, F_4$, is found applied at point G as shown. The point of application of this resultant R may be transmitted to any other point along its line of action if desired. From this discussion, we see that the magnitude and direction of the resultant R are determined by the closing side of the polygon of forces (Fig. 93*b*) and are independent of the points of application of the given forces.



(*a*) (*b*)

FIG. 93

If, in a more general case of n forces $F_1, \ldots, F_n$ in a plane, we find, by successive applications of the parallelogram law that the partial resultant of the first k forces $F_1, \ldots, F_k$ is parallel to the remaining

$n - k$ forces $F_{k+1}, \ldots, F_n$, we use the method of addition of parallel forces, as discussed in Art. 2.2. On the basis of that discussion we conclude that, in general, there are three possibilities: (1) The system of n forces in a plane reduces to a *resultant force*, (2) the system reduces to a *resultant couple*, or (3) the system is in *equilibrium*.

To distinguish between these three cases, we begin with the construction of the polygon of forces. If this polygon is not closed (Fig. 93*b*), the system reduces to a resultant force, the magnitude and direction of which are given by the closing side of the polygon and the line of action of which may be located by using the construction indicated in Fig. 93*a* or the method of adding parallel forces. If the polygon of forces is closed (Fig. 94*b*), we may always divide the forces into two arbitrary groups having resultants R_1 and R_2 which are obviously equal in magnitude and opposite in direction. Then, proceeding as above, the positions of the lines of action of these equal and oppositely directed forces R_1 and R_2 may be found as shown in Fig. 94*a*. If their lines of action do not coincide, they represent the resultant couple of the original system of forces. If their lines of action do coincide, the system is in equilibrium.

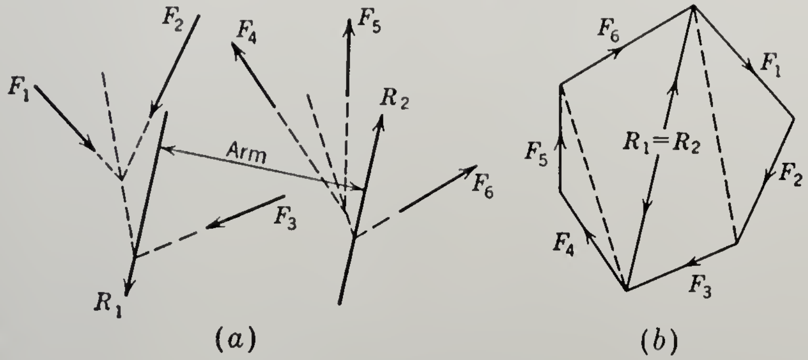


FIG. 94

Composition of a system of forces in a plane by the method illustrated in Figs. 93 and 94 becomes very laborious where many forces are involved and is not generally practicable in the solution of problems. The algebraic equations that have already been used in the case of concurrent and parallel forces in a plane can be applied to better advantage. We have noted above that the magnitude and direction of the resultant force (when one exists) in this general case are obtained in exactly the same manner as in the case of concurrent forces in a plane. Hence, denoting by $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ the projections on rectangular coordinate axes x and y, respectively, of the given forces $F_1, F_2, \ldots, F_n$ and by X and Y the corresponding projections of their resultant R, we may write at once [see Eqs. (1)]

$$X = \Sigma X_i \qquad Y = \Sigma Y_i \tag{a}$$

where the summations are understood to include all forces in the system. The magnitude and direction of the resultant are then obtained from the following equations [see Eqs. (2)]:

$$R = \sqrt{X^2 + Y^2} \qquad \tan \alpha = \frac{Y}{X} \tag{b}$$

where α is the angle that the line of action of R makes with the positive end of the chosen x axis.

The position of the line of action of the resultant may be found by using the theorem of moments. This theorem has already been proved for the cases of concurrent forces and parallel forces in a plane. We now have the general case. From the preceding discussion, it may be seen that in this general case the resultant is obtained by means of successive applications of the principle of the parallelogram of forces or the method of addition of parallel forces. Hence, by employing Varignon's theorem at each step, we conclude that the moment of the resultant about any center in the plane of the given forces is equal to the algebraic sum of the moments of all these forces with respect to the same center. Denoting by $(M_0)_1, (M_0)_2, \ldots, (M_0)_n$ the moments of the given forces $F_1, F_2, \ldots, F_n$ with respect to the origin O and by M_0, the corresponding moment of the resultant R, this statement may be expressed by the equation

$$M_0 = \Sigma(M_0)_i \tag{c}$$

In calculating the algebraic sum of moments indicated here, the use of Eq. (5), page 43, for calculating the moment of any force with respect to the origin O will be found helpful.

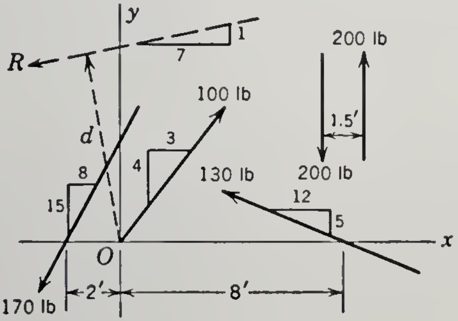
Having the moment of the resultant with respect to the origin of coordinates, its arm d_0 is readily obtained from the equation

$$d_0 = \frac{M_0}{R} \tag{d}$$

This arm, when laid out from the origin O in a direction perpendicular to the known direction of the resultant R [see Eqs. (b)] and so as to give the proper sign of moment, locates a point on the line of action of the resultant. Thus Eqs. (a) to (d) inclusive serve completely to determine the resultant of any system of forces in a plane. If the sums of projections as indicated by Eqs. (a) both become zero, there is obviously no resultant force and the system reduces to a resultant couple as defined by Eq. (c).

EXAMPLES

1. Determine the magnitude, direction, and position of the line of action of the resultant of the coplanar system of forces shown in Fig. 95*a*.



(*a*)

F_i	X_i	Y_i	$(M_o)_i$
170	−80	−150	+300
100	+60	+80	0
130	−120	+50	+400
200	0	−200	+300
200	0	+200	
Σ	−140	−20	+1000

(*b*)

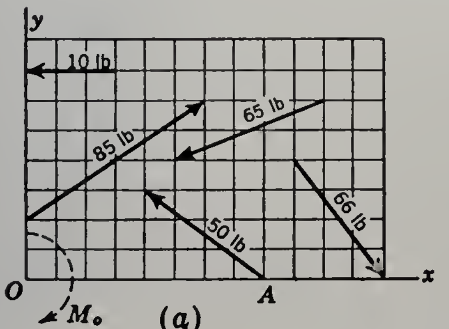
FIG. 95

Solution. Tabulating the forces as shown in Fig. 95*b*, we first compute the projections X_i and Y_i of each force and also its moment $(M_0)_i$ about the origin O, counterclockwise moment being considered as positive. Since the two 200-lb forces form a couple, we record only their combined moment as shown. Making summations as shown at the bottom of each column, and using expressions (a) to (d), we have

$$R = \sqrt{X^2 + Y^2} = \sqrt{(-140)^2 + (-20)^2} = 100\sqrt{2}\text{ lb}$$

$$\tan\alpha = \frac{Y}{X} = \frac{-20}{-140} = \frac{1}{7} \qquad d = \frac{M_o}{R} = \frac{1{,}000}{100\sqrt{2}} = 5\sqrt{2}\text{ ft}$$

Thus the resultant is located as shown in Fig. 95*a*.



(*a*)

F_i	X_i	Y_i	$(M_o)_i$
10	−10.0	0	+70.0
85	+70.7	+47.1	−141.4
65	−60.3	−24.3	+120.6
50	−40.0	+30.0	+240.0
66	+39.6	−52.8	−633.6
Σ	0	0	−344.4

(*b*)

FIG. 96

2. Determine the resultant of the system of coplanar forces shown in Fig. 96*a*. Each division of the superimposed grid is 1 in. square.

Solution. We first compute and record the projections X_i and Y_i of each force as shown in Fig. 96*b*. Summing these projections for all forces, we obtain $X = 0$ and $Y = 0$. Therefore the resultant is not a force.

To see if there is a resultant couple, we compute and record the moment $(M_0)_i$ of each force about the origin O. The calculation of these moments is greatly simplified if we resolve each force into its components X_i and Y_i at the point where its line of action intersects either the x or y axis, so that one component will pass through O. Take, for example, the 50-lb force. Resolving it at A into components X_i, Y_i, we have

$$(M_0)_i = 40 \times 0 + 30 \times 8 = 240 \text{ in.-lb}$$

Moments of the other forces are calculated in a similar manner. Summing moments in accordance with Eq. (c), we get $M_0 = -344.4$ in.-lb. Thus the resultant is a clockwise couple in the plane of action of the forces as shown in Fig. 96*a*.

PROBLEM SET 3.1

1. Determine the resultant of the system of forces in Fig. 95*a* if the directions of the 200-lb forces are reversed so that they represent a clockwise couple. *Ans.* $R = 100\sqrt{2}$ lb; $\tan \alpha = \frac{1}{7}$; $d = 2\sqrt{2}$ ft.

2. What is the moment of a couple that must be added to the system of forces in Fig. 95*a* to make the resultant R pass through the origin O? *Ans.* $M = -1{,}000$ ft-lb.

3. A gravity dam of trapezoidal cross section is acted upon by coplanar forces, as shown in Fig. A. Calculate the magnitude and direction of the resultant R and the distance x to the point where its line of action cuts the base line OA. The magnitudes of the given forces are shown in kip units (1 kip = 1,000 lb). *Ans.* $R = 71.0$ kips; $\theta = 70°50'$; $x = 11.7$ ft.

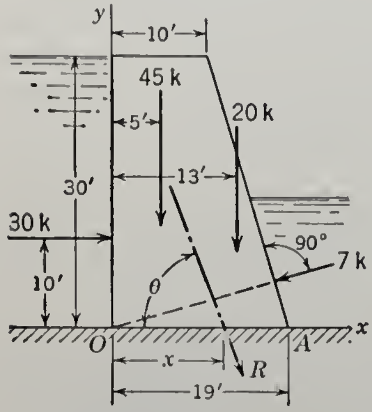


FIG. A

4. Determine and locate the resultant force R on the dam in Fig. A, by the graphical method illustrated in Fig. 93.

5. Loads act on a roof truss as shown in Fig. B. The loads on the bottom chord are vertical; those on the top chord are perpendicular to the chord lines and act midway between chord points. Find magnitude, direction, and position of the resultant force.

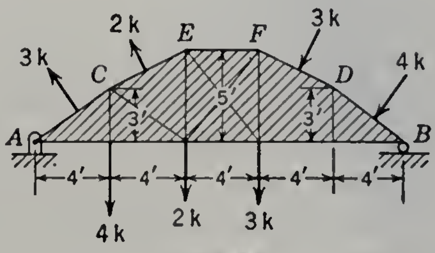


FIG. B

3.2. Equilibrium of forces in a plane. In Art. 3.1, we have seen that a system of forces in one plane can reduce to a resultant force, a resultant couple, or a state of equilibrium. If expressions (a) vanish, there is no resultant force, and if expression (c) vanishes, there is no resultant couple. Thus the system of forces is in equilibrium if

$$\Sigma X_i = 0 \qquad \Sigma Y_i = 0 \qquad \Sigma (M_0)_i = 0 \tag{18}$$

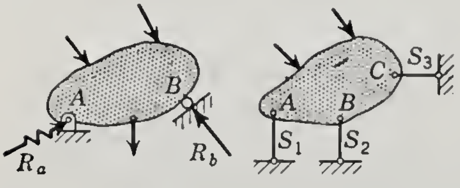
These expressions represent the *equations of equilibrium* for the general case of forces acting in one plane.

Since concurrent coplanar force systems and parallel coplanar force systems are only special cases of forces in a plane, the above equations of equilibrium must also hold for these cases. In the case of concurrent forces in a plane the third of Eqs. (18) will always be satisfied if the first two of the same system are satisfied, and hence, for this case, they reduce to the system of Eqs. (3) (see page 26). In the case of parallel forces in a plane, by taking the y axis parallel to the lines of action of the forces, there will be no projections on the x axis and the first of Eqs. (18) is always satisfied. Thus we are left with a system of two equations, equivalent to Eqs. (11) (see page 71).

It is sometimes advantageous to replace Eqs. (18) by three moment equations, analogous to Eqs. (6) derived for the case of concurrent forces in a plane. In the general case of forces in a plane it can be proved that, if the algebraic sum of the moments of all the forces, with respect to any three centers A, B, and C in their plane and not all on one straight line, is zero, the system of forces is in equilibrium. To prove this statement, we reason as follows: If the algebraic sum of the moments of all forces with respect to any point, say A, is zero, it can be concluded at once that the system is not reducible to a resultant couple. However, we may still have a resultant force passing through the moment center A or a state of equilibrium. Evidently then, if the algebraic sums of the moments of all forces with respect to any two points, say A and B, are simultaneously zero, the only possibility of a resultant is a force acting along the line AB. Finally, if the algebraic sums of the moments of all forces, not only with respect to A and B but also with respect to a third point C, are zero and C is not on the line AB, the possibility of a resultant force falls completely and we must have equilibrium. Expressing these three conditions of equilibrium algebraically, we have

$$\Sigma (M_A)_i = 0 \qquad \Sigma (M_B)_i = 0 \qquad \Sigma (M_C)_i = 0 \tag{19}$$

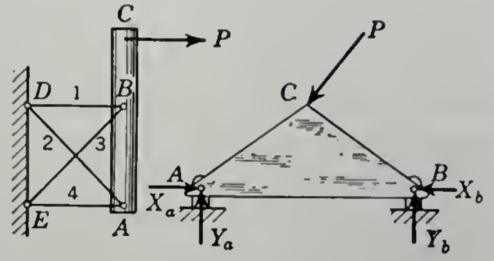
Reconsidering the set of equations (18) or (19), we observe that in both cases, three independent conditions are not only necessary but also sufficient to ensure equilibrium of a system of coplanar forces. Naturally, with three equations we can determine only three unknowns. This means that in dealing with constrained bodies where unknown reactions are to be evaluated, we shall not be able to determine the magnitudes of more than three such forces, or possibly the magnitude and direction of one and the magnitude of another. For this reason a system of physical constraints of a rigid body in a plane which gives rise to just three unknowns is said to be *statically determinate.*



(*a*) (*b*)

FIG. 97

Common systems of constraints which satisfy the above condition are shown in Fig. 97. In Fig. 97*a* we have a body supported by a *hinge* A and a *roller* B, which completely constrain the body in the plane of the figure. The roller at B determines physically the direction of R_b but the direction of R_a remains unknown as indicated by the wavy-line vector. Thus the three unknowns in this case are the magnitude and direction of R_a and the magnitude of R_b. In Fig. 97*b* a body is constrained in one plane by three hinged bars which are not parallel and do not intersect in one point.[1] In this case the lines of action of three reactive forces S_1, S_2, and S_3 (coinciding with the axes of the bars producing them) are known and again we have only three unknown magnitudes to determine.



(*a*) (*b*)

FIG. 98

Any system of supports of a rigid body in one plane which contains more than three degrees of constraint will set up reactive forces involving more than three unknowns. In such case the equations of equilibrium (18) or (19) will be inadequate and the system is said to be *statically indeterminate.* Supports in excess of those just necessary and sufficient to prevent motion are called *redundant supports.* Two examples of systems with redundant constraints are shown in Fig. 98. In each case the corresponding system of reactions will involve four unknowns and the systems are *statically indeterminate.*

---

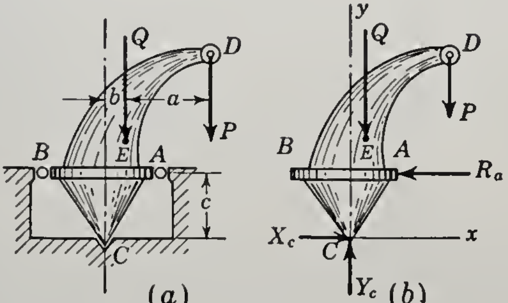
[1] If the three bars in Fig. 97*b* are parallel or intersect in one point, the body will always have some freedom of motion in the plane of the figure and is not adequately constrained, i.e., it will move under the action of applied loads.

Applications of Eqs. (18) and (19) in the calculation of statically determinate reactions will now be illustrated by several examples.

EXAMPLES

1. A revolving crane (Fig. 99*a*) supported by a pivot C and a horizontal ring AB carries, besides its own weight Q applied at E, a vertical load P applied at D. Determine the reactions at the points of support if $P = 4$ tons, $Q = 2$ tons, $a = 15$ ft, $b = 3$ ft, and $c = 6$ ft.

Solution. We begin with a free-body diagram of the entire crane isolated from its supports (Fig. 99*b*). Besides the active forces P and Q, we have to consider reactive forces exerted by the supporting ring AB and the socket at C. Assuming some slight play between the crane and ring, we conclude that for the loading shown, the crane will press on the ring only at one point A. Thus we have a horizontal reaction at A, as shown, and no reaction at B. The reaction on the pivot C can have any direction through this point. Thus, in general it can have two independent rectangular components X_c and Y_c, as shown. The magnitudes of these two components define the reaction at C just as well as the magnitude and direction of a single force and are easier to work with analytically.



(*a*) (*b*)

FIG. 99

Choosing coordinate axes as shown in Fig. 99*b* and taking C as a moment center, Eqs. (18) become

$$\begin{aligned} X_c - R_a &= 0 \\ Y_c - P - Q &= 0 \qquad\qquad \text{(a)} \\ R_a c - P(a + b) - Qb &= 0 \end{aligned}$$

It should be noted that by taking point C as a moment center, we eliminate the unknowns X_c and Y_c from the third equation, which can then be solved directly for R_a. Substituting the given numerical data into Eqs. (a), we obtain $R_a = 13$ tons, $X_c = 13$ tons, $Y_c = 6$ tons.

2. A prismatic bar AB of weight Q and length l is supported by a small roller at C and presses against a smooth vertical wall at A (Fig. 100). If an additional load P is applied at the end B of the bar, find the position of equilibrium of the bar as defined by the angle α that its axis makes with the horizontal.

Solution. Neglecting friction, the reactions R_a and R_c will have directions as shown in the figure. The point of intersection O of their lines of action is at the distance $b = a/\cos^2 \alpha$ from point A. When equilibrium of the bar exists, the algebraic sum of the moments of all forces with respect to point O must be zero. Hence

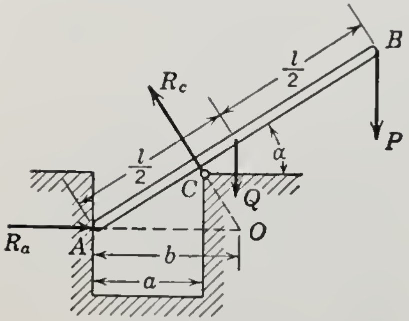


FIG. 100

$$Q\left(\frac{a}{\cos^2 \alpha} - \frac{l}{2}\cos \alpha\right) = P\left(l \cos \alpha - \frac{a}{\cos^2 \alpha}\right)$$

from which

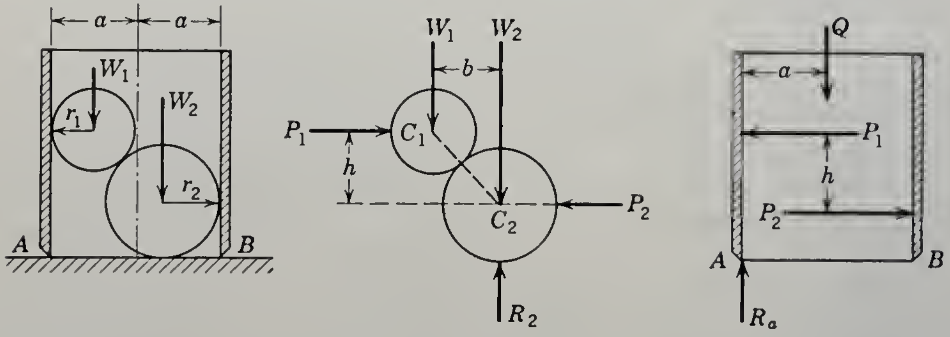
$$\cos \alpha = \sqrt[3]{\frac{2(P + Q)a}{(2P + Q)l}} \tag{b}$$

For the particular case where $P = 0$, Eq. (b) reduces to

$$\cos \alpha = \sqrt[3]{\frac{2a}{l}}$$

It is assumed, of course, that $a < l/2$.

3. A hollow right circular cylinder of radius a is open at both ends and rests on a smooth horizontal plane as shown in Fig. 101*a*. Inside the cylinder are two spheres having weights W_1 and W_2 and radii r_1 and r_2, respectively. The lower sphere also rests on the horizontal plane. Neglecting friction, find the minimum weight Q of the cylinder in order that it will not tip over.



(*a*) (*b*) (*c*)

FIG. 101

Solution. Let us consider first a free-body diagram for the two spheres, assuming that they are joined together at their point of contact as shown in Fig. 101*b*. By virtue of the assumption of smooth surfaces, we conclude that the reactive forces P_1 and P_2 exerted on the spheres by the walls of the cylinder are horizontal forces as shown and likewise that the reaction R_2 on the bottom of the lower sphere is a vertical force. Thus the two spheres are in equilibrium under the action of the five coplanar forces shown in Fig. 101*b*. Using the first and last of Eqs. (18) with C_2 as a moment center, we obtain

$$P_1 - P_2 = 0 \qquad W_1 b - P_1 h = 0$$

from which

$$P_1 = P_2 = \frac{W_1 b}{h} \tag{c}$$

We now turn our attention to a free-body diagram of the cylinder (Fig. 101*c*). The action of the spheres on the inside of the cylinder are represented by forces equal and opposite to the previously considered forces P_1 and P_2 in Fig. 101*b*. Referring to Eq. (c), we see that these forces constitute a couple of moment $P_1 h = W_1 b$ that tends to overturn the cylinder about point A. Resisting this overturning moment is the weight Q of the cylinder acting along its geometric axis. When overturning impends, the reactive force on the bottom of the cylinder is concentrated at A as shown. Equating to zero the algebraic sum of the moments of all forces with respect to this point, we obtain

$$Qa = W_1 b$$

from which

$$Q = W_1 \frac{b}{a} \tag{d}$$

Referring back to Fig. 101*a*, we see that

$$b = 2a - r_1 - r_2$$

and expression (d) becomes

$$Q = W_1 \frac{2a - r_1 - r_2}{a} \tag{e}$$

In the particular cases of two identical spheres each of weight W and radius r, we have

$$b = 2(a - r)$$

and expression (d) becomes

$$Q = 2W \frac{a - r}{a} \tag{f}$$

It must be kept in mind that expressions (e) and (f) represent in each case the minimum value of the weight Q for which equilibrium can exist.

PROBLEM SET 3.2

1. A beam ABC hinged to a fixed support at A and resting on a roller at B is loaded as shown in Fig. A. Determine the reactions at the points of support. *Ans.* $X_a = -400$ lb; $Y_a = 167$ lb; $R_b = 633$ lb.

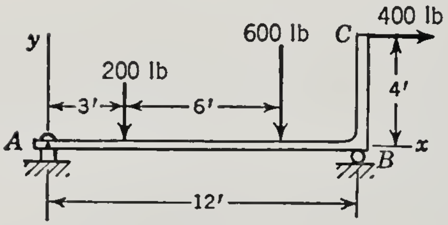


FIG. A

2. A horizontal beam AB hinged to a vertical wall at A and supported by a tie rod CD is subjected to the loading shown in Fig. B. Calculate the horizontal and vertical components of the reaction at A and the tension S in the tie rod. *Ans.* $X_a = 433$ lb; $Y_a = -50$ lb; $S = 500$ lb.

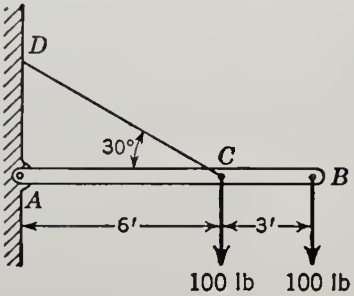


FIG. B

3. The vertical mast AB of a crane (Fig. C) is supported by a pivot at B and a guide at A. Neglecting friction, determine the reactions produced at A and B due to the loads P and Q acting on the crane as shown. The following numerical data are given: $P = 2$ tons, $Q = 1$ ton, $a = 8$ ft, $b = 12$ ft, $c = 14$ ft. *Ans.* $R_a = 2$ tons; $R_b = 3.61$ tons.

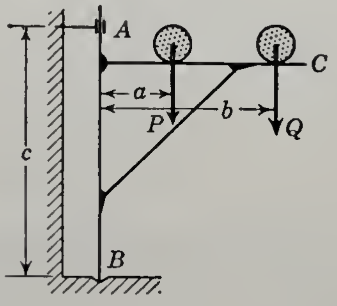


FIG. C

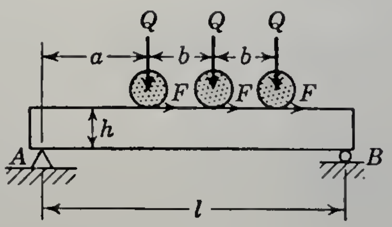


FIG. D

4. Determine the reactions R_a and R_b produced at the supports A and B of a girder (Fig. D) by the vertical loads Q and the tractive forces F exerted by the wheels of a locomotive. Neglect friction in the roller at B. The following numerical data are given: $Q = 20$ tons, $F = 1$ ton, $a = 10$ ft, $b = 6$ ft, $h = 5$ ft, $l = 40$ ft. *Ans.* $R_a = 35\frac{3}{4}$ tons; $R_b = 24\frac{3}{8}$ tons.

5. Determine the reaction at A and the force S in the bar DE due to the action of the loads P and Q applied to the crane, as shown in Fig. E. Neglect the weight of the crane and assume ideal hinges at A, D, and E. Assume that $P = 500$ lb, $Q = 300$ lb, $a = 6$ ft. *Ans.* $R_a = 1{,}140$ lb; $S = 1{,}555$ lb, compression.

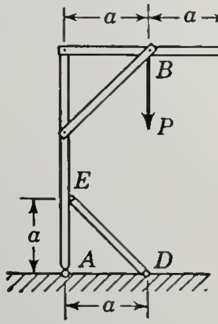


FIG. E

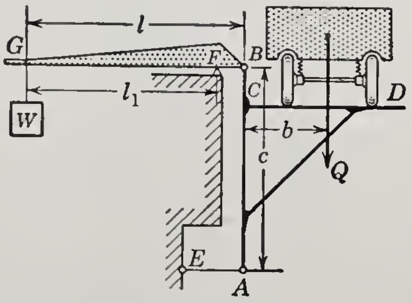


FIG. F

6. A horizontal platform CD carries a truck of weight Q (Fig. F) and is rigidly attached to the vertical bar AB, which is hinged at the bottom to the horizontal bar AE and at the top to the horizontal lever BFG. Determine the weight Q of the truck if a known weight W hanging at G holds the platform and its load in equilibrium. The weight of the empty platform is just balanced by that of the lever. *Ans.* $Q = Wl_1/(l - l_1)$.

7. The system in Fig. G is in equilibrium for the conditions of loading shown. Subsequently the load Q is increased by 200 lb. How far must the load P be moved to the right to preserve equilibrium? *Ans.* 8 in.

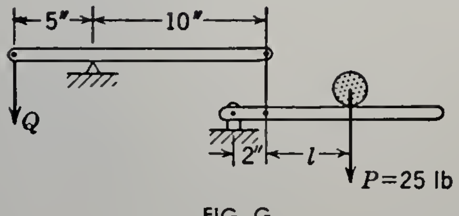


FIG. G

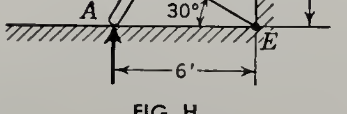


FIG. H

8. A 150-lb man stands on the middle rung of a 50-lb ladder, as shown in Fig. H. Assuming that the floor and wall are perfectly smooth and that slipping is prevented by a string DE, find the tension S in this string and also the reactions R_a and R_b at A and B. *Ans.* $S = 81.2$ lb; $R_a = 240.6$ lb; $R_b = 70.4$ lb.

9. A prismatic bar AB of weight Q and length l rests at A against a smooth horizontal floor, and under the action of its gravity force Q presses against

supports at C and D (Fig. I). Neglecting friction, determine the magnitudes of the reactions R_a, R_d, and R_c. *Ans.* $R_a = Q$; $R_c = R_d = (Ql/2a) \cos \alpha$.

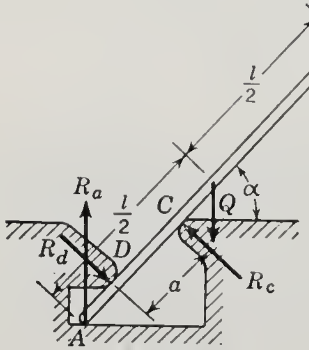


FIG. I

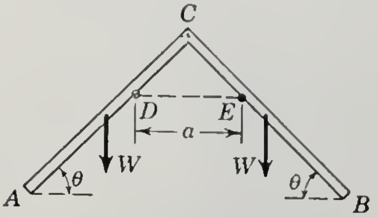


FIG. J

10. Two slender prismatic bars AC and BC, each of length l and weight W, are hinged together at C and supported in a vertical plane by two pegs at D and E as shown in Fig. J. Neglecting all friction, find the angle θ that each bar will make with the horizontal in the condition of equilibrium. *Ans.* $\cos \theta = \sqrt[3]{a/l}$.

11. A heavy prismatic timber of weight W and length l is supported horizontally between two fixed fulcrums A and B, as shown in Fig. K. If the coefficient of friction between the timber and each fulcrum is μ, find the magnitude of a horizontal force P applied as shown that will cause impending sliding of the timber to the right. The following numerical data are given: $W = 100$ lb, $a = 20$ in., $h = 12$ in., $c = 10$ in., $l = 70$ in., $d = 12$ in., and $\mu = \frac{1}{3}$. *Ans.* $P = 50$ lb.

***12.** Find the magnitude and direction of the least force P necessary to cause impending sliding of the timber in Fig. K. Compare this result with the value of the corresponding horizontal force as obtained in the preceding problem. *Ans.* $P_{\min} = 45.2$ lb, inclined downward by 25°17′ to the horizontal.

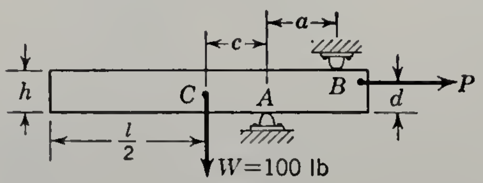


FIG. K

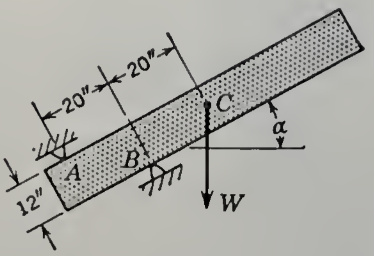


FIG. L

***13.** A heavy prismatic timber of weight W is supported in a vertical plane as shown in Fig. L. If the coefficient of friction between the timber and each of the supports A and B is $\mu = \frac{1}{3}$ and the dimensions are as shown, find the maximum value of the angle α consistent with equilibrium. *Ans.* $\alpha \leqq 43°$.

14. A heavy block of weight W rests on a rough horizontal plane as shown in Fig. M. Hinged to this block is a slender bar AB of length l which leans against a small frictionless roller at D and carries a vertical load P at its free end B. Find the magnitude of P for which sliding of the block will impend if the coefficient of friction on the horizontal plane is μ. The following numerical data are given: $\theta = 30°$, $l = 30$ in., $h = 10$ in., $\mu = \frac{1}{3}$. Neglect the weight of the bar AB completely. *Ans.* $P = 0.482W$.

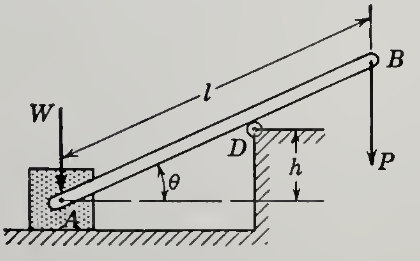


FIG. M

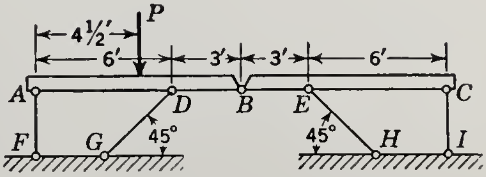


FIG. N

*15. Two beams AB and BC, joined together by a hinge B, are supported by four bars, hinged at their ends (Fig. N). Determine the force produced in each of these bars due to the action of the load $P = 500$ lb. The dimensions of the structure are shown in the figure. *Ans.* $S_a = 188$ lb, compression; $S_d = S_e = 265$ lb, compression; $S_c = 62$ lb, tension.

3.3. Plane trusses: method of joints. The various methods of dealing with the equilibrium of a system of forces confined to one plane find a wide application in the analysis of *plane trusses*. A plane truss is defined as a system of bars, all lying in one plane and joined together at their ends in such a way as to form a rigid framework.[1] A typical example is shown in Fig. 102*a*, which represents a common type of bridge truss. If this truss, supported in a vertical plane as shown, is subjected to the action of a force Q applied at one of the joints and also acting in the plane of the truss, reactions will be produced at the points of support. These reactions may be determined by any of the methods already discussed in Arts. 1.5 or 3.2. Under the action of this system of balanced forces external to the truss as a whole, axial forces will be produced in the various bars. The determination of these internal forces constitutes the *analysis* of the truss.

Sometimes, instead of fastening the bars of a truss together by means of pins at their ends (Fig. 102*a*), they are connected by means of plates at each joint to which the ends of the bars are riveted or welded. The usual type of riveted connection is shown in Fig. 102*b*. In either case it is usual practice in the analysis of trusses to make the following simplifying assumptions: (1) that the bars are connected at their ends by frictionless hinges; (2) that the axes of all bars lie in one plane, called the middle plane of the truss; and (3) that all forces acting on the truss are applied at the hinges only and also lie in the plane of the truss.

---

[1] For various rules of formation of plane trusses, see the authors' "Theory of Structures," pp. 44, 76, McGraw-Hill, New York, 1945.

These three assumptions are of great significance. They amount, in short, to the replacement of the actual *physical truss* of Fig. 102*a* or 102*b* by the *idealized truss* of Fig. 102*c*, consisting of a system of weightless[1] bars all lying in one plane and joined together at their ends by frictionless hinges to which external forces, acting only in the plane of the truss, are applied. For such an idealized case it is evident, as discussed in Art. 1.1, that each bar of the truss is in equilibrium under the action of forces applied at its two ends only and therefore is in either tension or compression. From this it follows that the reactions which each bar of the truss exerts on the hinges at its two ends can be represented by two equal, opposite, and collinear forces which have for their common line of action the axis of the bar. We have then, at each joint of the truss, a system of concurrent forces in one plane that must be in equilibrium (Fig. 102*c*). Thus on the basis of the above assumptions, the various methods, previously studied, of handling the problem of equilibrium of concurrent forces in a plane can now be applied directly to the analysis of trusses. Such procedure is called the method of joints.

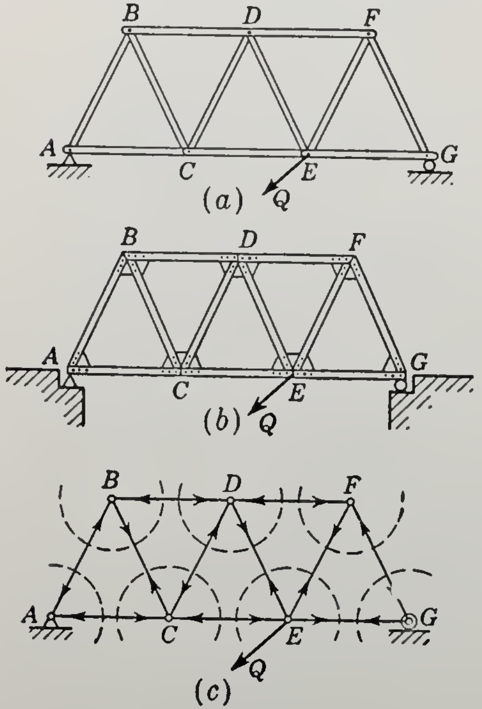


FIG. 102

EXAMPLES

1. A plane truss *ABE* is supported and loaded in a vertical plane as shown in Fig. 103*a*. Determine graphically the reactions at *A* and *E* and the axial force in each bar.

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[1] The analysis of a truss under its own weight is usually treated as a separate problem, so that for all other loading conditions the weights of the bars are entirely neglected.

Solution. Under the action of the applied force Q, reactions R_a and R_e will be produced at the points of support as shown. Considering the entire structure as a free body and applying the method of Art. 1.5, these two reactions can be determined without difficulty. The direction of the force Q, applied at D, being known and the reaction R_e being normal to the surface on which the roller rests, the reaction R_a at A must have the line of action AG, as indicated in the figure. The three forces Q, R_a, and R_e also must build a closed triangle, as shown in Fig. 103*b*, and thus the balanced system of forces external to the structure as a whole is completely determined.

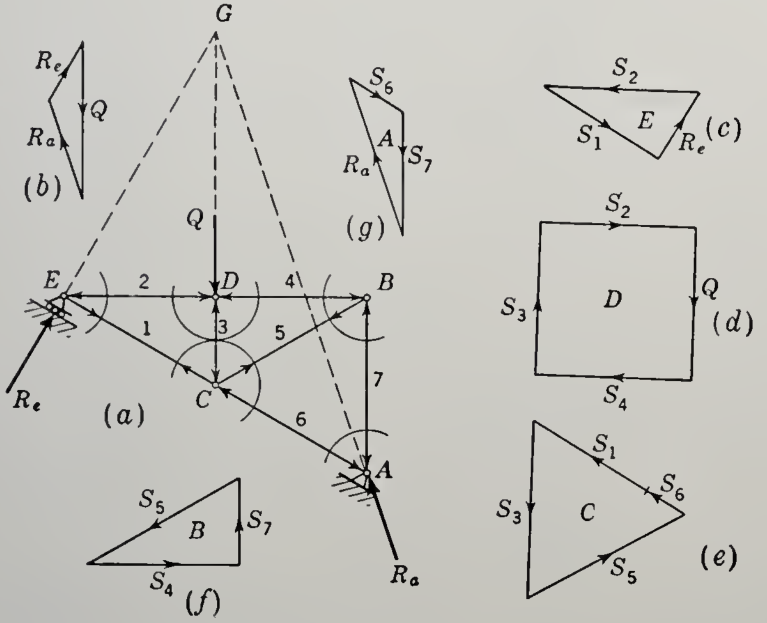


FIG. 103

As already explained above, these external forces produce axial forces in the various bars of the truss and each bar in turn exerts on the hinges at its ends two equal but oppositely directed reactions which have for their common line of action the axis of the bar. The magnitude of either of these reactions obviously represents the force in the bar, and whether it is directed away from the hinge or toward it determines, correspondingly, whether the bar is in tension or compression. For convenience of reference let us number each bar of the truss, as shown in Fig. 103*a*, and denote by S_i the force in any bar i.

We now begin the analysis of the truss by consideration of the equilibrium of the hinge E. Considering this hinge as a free body, as indicated by the circle around joint E, we find acting upon it the previously determined force R_e together with the reactions S_1 and S_2 exerted by the bars 1 and 2. We shall not know at once whether these last two forces should be directed toward the hinge or away from it, but keeping in mind that their lines of action must coincide with the axes of the bars which produce them, the triangle of forces (Fig. 103*c*) for the hinge E can be drawn and the magnitudes of the forces S_1 and S_2 determined by scaling the corresponding sides of the triangle. Now, from the directions of the arrows, which must follow each other tail to head on the sides of the triangle of forces, we see that the bar 1 pulls on the hinge, indicating tension, and the bar 2 presses against the hinge, indicating compression. Arrows indicating such reactions are now placed at E, as shown in Fig. 103*a*.

Let us consider next the equilibrium of the hinge D. We first replace the bar 2 by its reaction S_2 equal but opposite to the previously considered reaction of this bar on the hinge E. There remain then but two unknown forces at D, representing the reactions on this hinge of the bars 3 and 4. Again we do not know whether these last two forces are directed toward the hinge or away from it, but knowing the directions of their lines of action we can construct the polygon of forces shown in Fig. 103*d*, from which the forces S_3 and S_4 are determined as before. In this case we see from the arrows on the sides of the polygon of forces that both bars are pressing on the hinge and hence are in compression. Arrows indicating such reactions can now be placed on the axes of the bars 3 and 4 at D, as shown in Fig. 103*a*.

Proceeding next to a consideration of the hinge C, the reactions exerted on this hinge by the bars 1 and 3 will be known and again only two bars with unknown forces will be encountered. The force polygon for this hinge, from which S_5 and S_6 may be determined, is shown in Fig. 103*e*. Finally the hinge B may be considered, after which the forces in all the bars of the truss will have been determined.

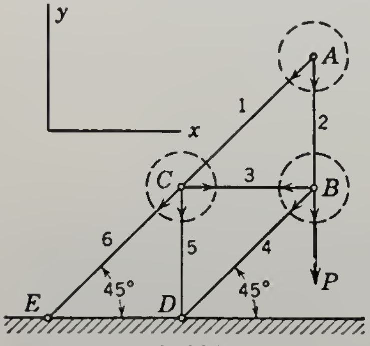


FIG. 104

2. Determine analytically the axial forces in the bars of the plane truss supported and loaded as shown in Fig. 104.

Solution. Owing to the fact that this truss is formed as an integral part of its foundation, it will not be necessary to consider the reactions at D and E. The analysis of internal forces can be commenced at once with a consideration of the conditions of equilibrium of the joint A. Isolating this hinge as a free body, we encounter only two forces representing the reactions of the bars 1 and 2 on this hinge. Since two forces can be in equilibrium only if they are equal, opposite, and collinear, we conclude that the forces in these two bars are zero, since their axes are not collinear. Hence the bars 1 and 2 may be imagined to be completely removed from the truss without altering the forces in any of the remaining bars.

If now we consider the equilibrium of the hinge B, we find acting upon it three forces: the load P and the two forces S_3 and S_4 representing the reactions exerted by the bars 3 and 4. Since we do not know at once whether the reactions S_3 and S_4 are directed toward or away from the hinge but only that their lines of action must coincide with the axes of the bars that produce them, let us assume that they are directed as shown. That is, we assume both bars to be in tension. If later in our calculation we find for either of these forces a negative value, it will simply indicate that the assumed direction for that force is incorrect and hence that the bar exerting it is in compression. Thus, by consistently assuming all bars to be in tension, positive results will automatically indicate tension and negative results, compression. Projecting forces at B onto coordinate axes x and y as shown, Eqs. (3) become

$$-S_3 - 0.707S_4 = 0 \qquad -P - 0.707S_4 = 0$$

from which we find $S_3 = +P$ and $S_4 = -\sqrt{2}\,P$.

We proceed now to a consideration of the equilibrium of the hinge C. Remembering that $S_1 = 0$ and $S_3 = +P$, the equations of equilibrium for this hinge become

$$+S_3 - 0.707S_6 = 0 \qquad -S_5 - 0.707S_6 = 0$$

from which $S_5 = -P$ and $S_6 = +\sqrt{2}\,P$.

3. Determine the force produced in each bar of the tower shown in Fig. 105 due to a horizontal force P applied at the top as shown.

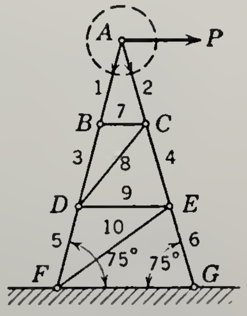


FIG. 105

Solution. Considerable time can be saved by recognizing at once that there is no force produced in any of the bars 7, 8, 9, or 10. This can be proved as follows: Considering the equilibrium of the hinge B and equating to zero the algebraic sum of the projections of all forces acting on this hinge onto an axis at right angles to AD, we conclude that the force in the bar 7 is zero, since the reaction of this bar on the hinge B is the only force that can have any projection on such an axis. Thus the bar 7 can be imagined to be removed, and we are left with a similar situation at the hinge C and conclude accordingly that the force in the bar 8 is zero. Subsequent considerations of the hinges D and E result in the same conclusion regarding the bars 9 and 10. Out of this discussion we can evidently state the following rule: If in any truss there be a joint at which only three bars meet and two of these bars lie along the same straight line, then the force in the third bar is zero, provided that there is no external force applied at the joint in question.

To complete the analysis, we now isolate the hinge A as a free body and imagine that the bars 1 and 2 pull on this hinge with forces S_1 and S_2. Then projecting forces onto a vertical axis, we obtain

$$S_1 \cos 15° + S_2 \cos 15° = 0$$

From this we conclude that $S_2 = -S_1$. Projecting forces at A onto a horizontal axis, we write

$$-S_1 \sin 15° + S_2 \sin 15° + P = 0$$

Substituting $-S_1$ for S_2 in this equation, we obtain

$$-2S_1 \sin 15° + P = 0$$

from which

$$S_1 = \frac{P}{2 \sin 15°} = +1.93P$$

Now since the bars 7, 8, 9, and 10 have been shown to be inactive, there are only collinear forces at each of the joints B, C, D, and E, and we conclude that

$$S_1 = S_3 = S_5 = +1.93P \qquad S_2 = S_4 = S_6 = -1.93P$$

PROBLEM SET 3.3

1. Calculate the axial force S_i in each bar of the simple truss supported and loaded as shown in Fig. A. The triangle ACB is isosceles with 30° angles at A and B and $P = 1{,}000$ lb. *Ans.* $S_1 = -667$ lb; $S_2 = -1{,}333$ lb; $S_3 = +1{,}154$ lb; $S_4 = +577$ lb; $S_5 = +1{,}154$ lb.

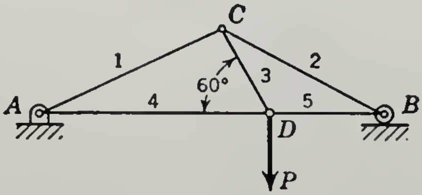


FIG. A

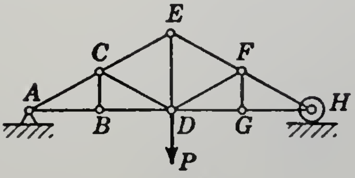


FIG. B

2. Prove that a tensile force equal to the applied load P is produced in the bar DE of the truss shown in Fig. B.

3. Determine the axial forces in the bars 1, 2, 3, 4, and 5 of the plane truss supported and loaded as shown in Fig. C. *Ans.* $S_1 = -P$; $S_2 = +P$; $S_3 = -0.5P$; $S_4 = +0.442P$; $S_5 = -0.333P$.

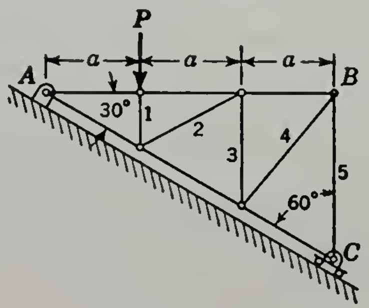


FIG. C

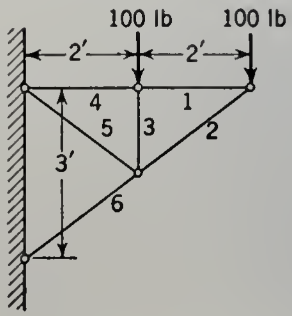


FIG. D

4. Determine the axial force in each bar of the plane truss loaded as shown in Fig. D. *Ans.* $S_1 = +133$ lb; $S_2 = -167$ lb; $S_3 = -100$ lb; $S_4 = +133$ lb; $S_5 = +83$ lb; $S_6 = -250$ lb.

5. Determine the force S in the bar CD of the simple truss supported and loaded as shown in Fig. E; $\triangle ABC$ is equilateral. *Ans.* $S = -0.866P$.

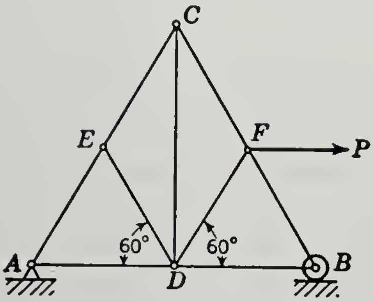

C
E
F
P
60°
60°
A
D
B

FIG. E

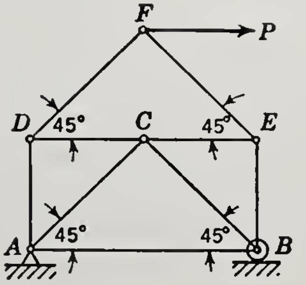

F
P
D
45°
C
45°
E
45°
45°
A
B

FIG. F

6. Determine the force S in the bar AB of the simple truss supported and loaded as shown in Fig. F. *Ans.* $S = +\frac{1}{2}P$.

7. Determine the axial force in each bar of the plane truss loaded as shown in Fig. G. *Ans.* $S_1 = -S_4 = +2P$; $S_2 = -S_6 = -2.236P$; $S_3 = +P$; $S_5 = 0$.

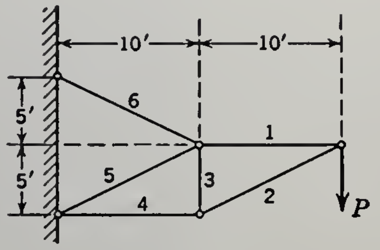

10′
10′
5′
5′
6
1
5
3
4
2
P

FIG. G

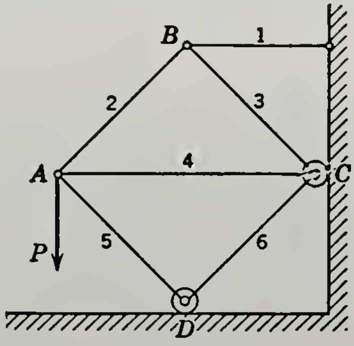

B
1
2
3
4
A
C
P
5
6
D

FIG. H

8. Determine the axial force in each bar of the plane truss supported and loaded as shown in Fig. H. $ABCD$ is a square; AC is horizontal. *Ans.* $S_1 = +P$; $S_2 = -S_3 = -S_5 = -S_6 = +0.707P$; $S_4 = 0$.

9. Determine the axial force S_i in each bar of the plane truss supported and loaded as shown in Fig. I. *Ans.* $S_1 = -1{,}000$ lb; $S_3 = +2{,}000$ lb; $S_5 = -2{,}000$ lb; $S_7 = -3{,}000$ lb; $S_8 = +3{,}000$ lb; $S_9 = -707$ lb.

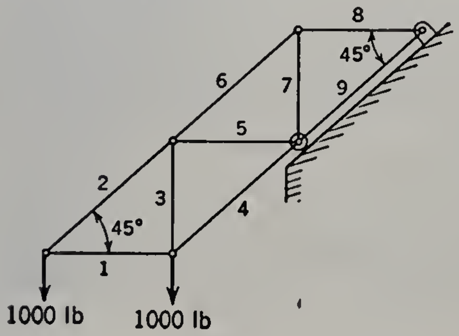

8
45°
6
7
9
5
2
3
45°
4
1
1000 lb
1000 lb

FIG I

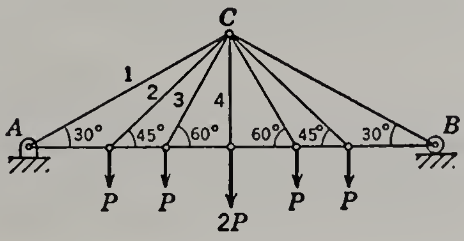
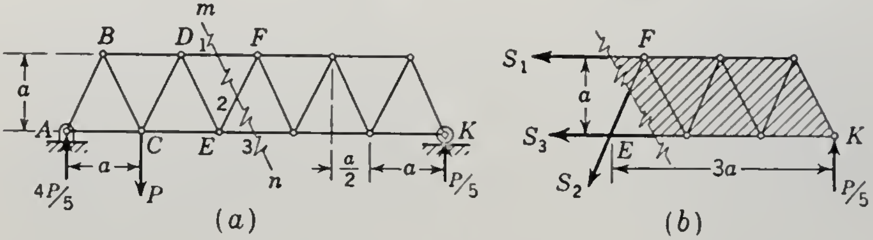

C
1
2
3
4
A
30°
45°
60°
60°
45°
30°
B
P
P
2P
P
P

FIG. J

10. Using the method of joints, calculate the axial force in each of the bars 1, 2, 3, and 4 of the plane truss shown in Fig. J. *Ans.* $S_1 = -6P$; $S_2 = +1.414P$; $S_3 = +1.155P$; $S_4 = +2P$.

3.4. Plane trusses: method of sections. In Art. 3.3 we considered the analysis of plane trusses by the *method of joints.* We shall consider now another method of calculating the axial forces in the bars of a truss, the use of which makes it possible to determine the force in some chosen bar without going through successive considerations of the conditions of equilibrium of all hinges of the truss. Referring to the truss shown in Fig. 106*a*, let us assume that it is required to determine the forces in the bars 1, 2, 3, due to a vertical load P applied as shown. We first consider the entire truss as a free body and find the reactions at A and K to be $4P/5$ and $P/5$, respectively, as shown. Now instead of considering the equilibrium of the hinges A, B, C, D, and E in succession, as would be necessary by the method of joints, let us imagine that by a section mn (Fig. 106*a*) the truss is cut into two parts and then consider the conditions of equilibrium of the part to the right of this section (Fig. 106*b*). Acting upon this free body, we have the vertical reaction at K together with the three forces S_1, S_2, S_3, representing the axial forces in the three bars that were cut by the section mn. The directions of these forces must, of course, coincide with the axes of the bars, so that only their magnitudes remain unknown. Thus we obtain a system of coplanar forces in equilibrium, and either system of equations (18) or (19) may be employed to determine the magnitudes of the three unknown forces. For example, equating to zero the algebraic sum of moments of all forces with respect to point E (thus eliminating from consideration two of the unknown forces), we obtain



(*a*) (*b*)

FIG. 106

$$S_1 a + \frac{P}{5} 3a = 0$$

from which $S_1 = -3P/5$. The negative sign, of course, indicates compression instead of tension, as assumed. In a similar manner, by taking point F as a moment center, we find $S_3 = +P/2$, tension. S_2

can be evaluated most easily by equating to zero the alegbraic sum of projections of all forces on a vertical axis. Thus, we obtain

$$\frac{P}{5} - S_2 \frac{2}{\sqrt{5}} = 0$$

from which $S_2 = +P/2\sqrt{5}$, tension.

The foregoing procedure in the analysis of trusses is called the *method of sections.* It consists essentially in the isolation of a portion of the truss by a section in such a way as to cause those internal forces that we wish to evaluate to become external forces on the isolated free body. By this procedure, we usually arrive at the case of equilibrium of a general system of forces in a plane and the usual conditions of equilibrium [Eqs. (18) or (19)] may be employed to evaluate the unknown forces as was done above. The success or failure of the method rests entirely upon the choice of section. In general, a section should cut only three bars, since only three unknowns can be determined from three equations of equilibrium. However, there are special cases where we may successfully cut more than three bars, and some of these will be illustrated in the following examples. Here we shall always assume tension to be positive and compression, negative.

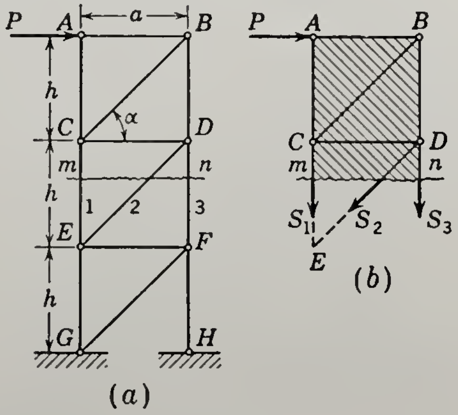


FIG. 107

EXAMPLES

1. Using the method of sections, determine the axial forces in bars 1, 2, and 3 of the tower loaded as shown in Fig. 107*a*.

Solution. We begin by making a section *mn* cutting the three bars for which the axial forces are required and consider the portion of the tower above this section as a free body (Fig. 107*b*). Acting on this free body, we have the applied force *P* and the forces S_1, S_2, and S_3 representing the axial forces in the cut bars 1, 2, and 3.

Equating to zero the algebraic sum of moments of these forces with respect to point *D*, we obtain

$$S_1 a - Ph = 0$$

from which $S_1 = +Ph/a$, tension. Similarly, with *E* as a moment center, we have

$$-S_3 a - P\,2h = 0$$

from which $S_3 = -2Ph/a$, compression. Finally, equating to zero the algebraic sum of the horizontal projections of the forces, we find

$$P - S_2 \cos \alpha = 0$$

from which $S_2 = +P/\cos \alpha$, tension, where, of course, $\cos \alpha = a/\sqrt{h^2 + a^2}$.

2. Determine the axial forces produced in the bars 1 and 4 of the tower shown in Fig. 108*a* due to a horizontal force P applied at the joint C.

Solution. In this case, we make a section mn that cuts four bars but in such a way that the axes of three of them intersect in one point, thus making it possible to determine the axial force in the fourth by using that point of intersection as a moment center.

Considering then the equilibrium of the free body in Fig. 108*b* and using successively as moment centers the points E and D, we find

$$S_1a - P\,2h = 0 \qquad -S_4a - P\,2h = 0$$

from which $S_1 = +2Ph/a$, tension, and $S_4 = -2Ph/a$, compression.

Equating to zero the algebraic sum of the horizontal projections of the forces in Fig. 108*b*, we obtain

$$S_3 - S_2 = P$$

but the individual magnitudes of these two forces cannot be determined from the conditions of equilibrium of the free body shown.

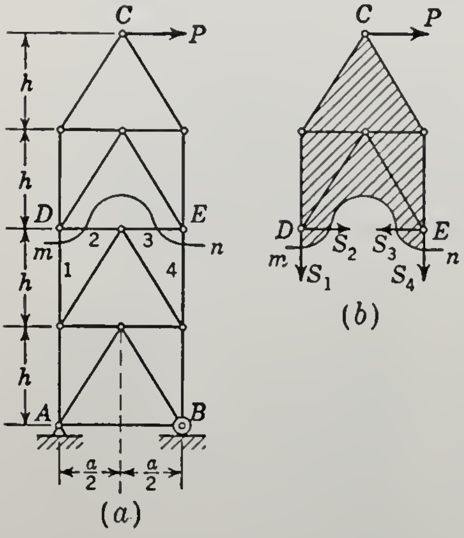


FIG. 108

PROBLEM SET 3.4

1. Using the method of sections, find the axial force in each of the bars 1, 2, 3, of the plane truss in Fig. A. *Ans.* $S_1 = +3P; S_2 = -P; S_3 = -\sqrt{3}P$.

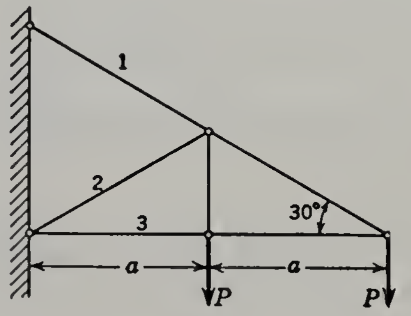


FIG. A

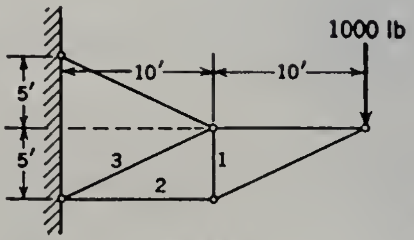


FIG. B

2. Using the method of sections, find the axial force in each of bars 1, 2, 3 of the plane truss in Fig. B. *Ans.* $S_1 = +1{,}000$ lb; $S_2 = -2{,}000$ lb; $S_3 = 0$.

3. Referring to Fig. C, find the axial force in the bar x: (*a*) using the method of joints, (*b*) using the method of sections. ABC is equilateral. *Ans.* $S_x = -\sqrt{3}\,P/2$.

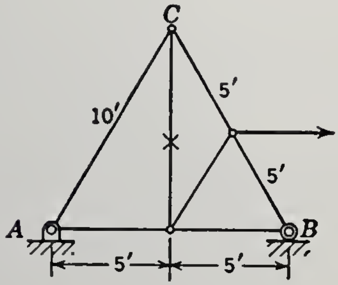

C
10′
5′
x
P
5′
A
B
5′
5′

FIG. C

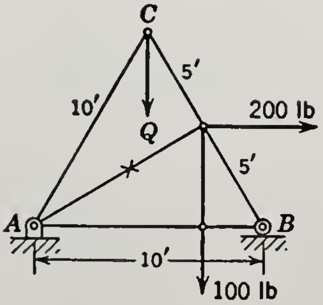

C
5′
10′
Q
200 lb
x
5′
A
B
10′
100 lb

FIG. D

4. Referring to Fig. D, find the axial force in the bar x: (*a*) using the method of joints, (*b*) using the method of sections. ABC is equilateral. *Ans.* $S_x = +123$ lb.

5. Determine, by the method of sections, the axial force in each of the bars 1, 2, and 3 of the plane truss shown in Fig. E. *Ans.* $S_1 = -P$; $S_2 = +5P/4$; $S_3 = -5P/4$.

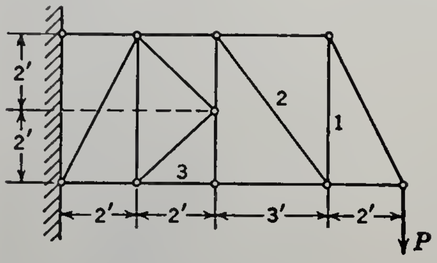

2′
2′
2
1
3
2′
2′
3′
2′
P

FIG. E

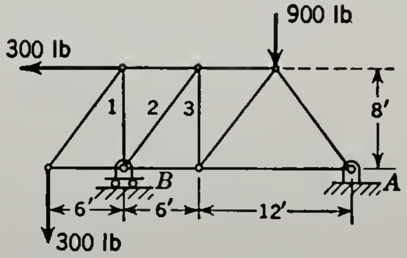

900 lb
300 lb
1
2
3
8′
B
A
6′
6′
12′
300 lb

FIG. F

6. Determine, by the method of sections, the axial force in each of the bars 1, 2, and 3 of the plane truss shown in Fig. F. *Ans.* $S_1 = -300$ lb; $S_2 = -666$ lb; $S_3 = +533$ lb.

7. Using the method of sections, calculate the axial force in each of the bars 1, 2, and 3 of the plane cantilever truss loaded as shown in Fig. G. *Ans.* $S_1 = -5.33P$; $S_2 = +2P$; $S_3 = -1.67P$.

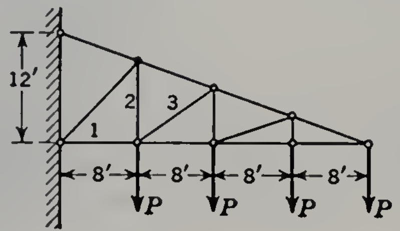

12′
2
3
1
8′
8′
8′
8′
P
P
P
P

FIG. G

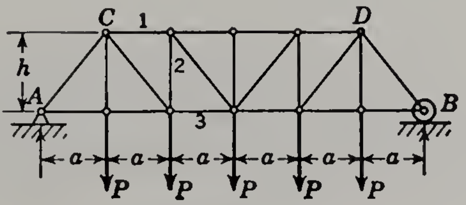

C
1
D
h
A
2
3
B
a
a
a
a
a
a
P
P
P
P
P

FIG. H

8. Determine the forces in bars 1, 2, and 3 of the plane truss loaded and supported as shown in Fig. H. *Ans.* $S_1 = -4Pa/h$; $S_2 = -P/2$; $S_3 = +4Pa/h$.

9. Determine the forces in bars 1, 2, and 3 of the plane truss loaded and supported as shown in Fig. I. *Ans.* $S_1 = S_2 = S_3 = -3Qa/2h$.

10. By using the method of sections, find the forces produced in bars 1, 2, and 3 of the truss shown in Fig. J due to the action of a ball of weight $W = 1{,}000$ lb, resting at the hinge D and supported by a cable EF as shown. The cable EF is parallel to AC. *Ans.* $S_1 = -416.7$ lb; $S_2 = +500.0$ lb; $S_3 = +333.3$ lb.

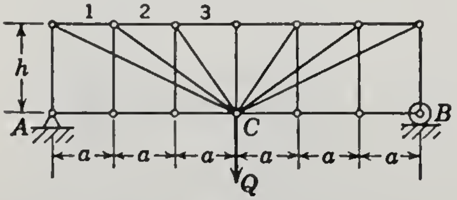

1
2
3
h
A
C
B
a
a
a
a
a
a
Q

FIG. I

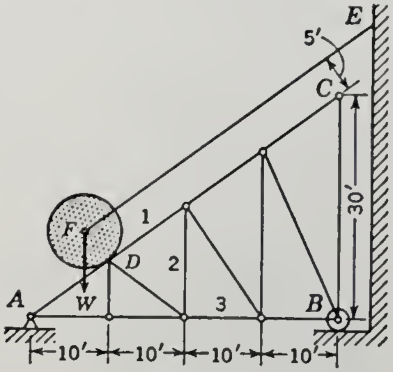

E
5'
C
30'
F
1
D
2
W
3
A
B
10'
10'
10'
10'

FIG. J

11. By using the method of sections, determine the forces in bars 1, 2, and 3 of the truss supported and loaded as shown in Fig. K. *Ans.* $S_1 = -0.825Q$; $S_2 = +0.424Q$; $S_3 = +0.500Q$.

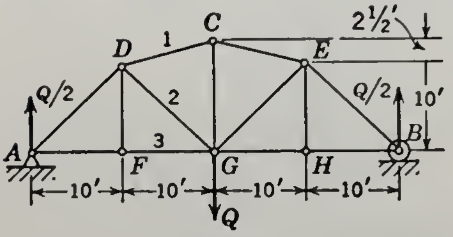

C
D
1
E
2½'
Q/2
2
Q/2
10'
3
B
A
F
G
H
10'
10'
10'
10'
Q

FIG. K

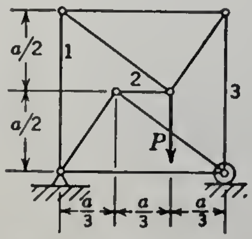

a/2
1
2
3
a/2
P
a/3
a/3
a/3

FIG. L

12. Determine the forces in bars 1, 2, and 3 of the plane truss supported and loaded as shown in Fig. L. *Ans.* $S_1 = -P/3$; $S_2 = 0$; $S_3 = -2P/3$.

13. Using the method of sections, find the axial force in each of the bars 1 and 2 of the plane truss loaded and supported as shown in Fig. M. *Ans.* $S_1 = -\frac{4}{3}P$; $S_2 = +\frac{4}{3}P$.

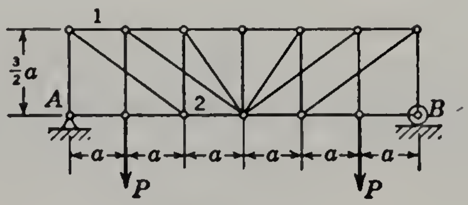

1
3/2 a
A
2
B
a
a
a
a
a
a
P
P

FIG. M

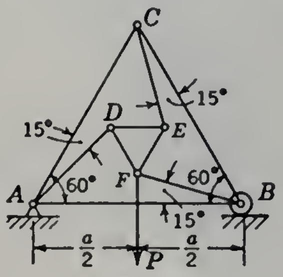

C
D
15°
15°
E
A
60°
F
60°
B
15°
a/2
P
a/2

FIG. N

*14. Determine the force S in the bar AB of the plane truss loaded and supported as shown in Fig. N. *Ans.* $S = +0.43P$.

*15. Determine the forces in bars 1, 2, and 3 of the plane truss supported and loaded as shown in Fig. O; $\angle CAB = \angle DBA = 60°$, $\angle CBA = \angle DAB = 30°$. *Ans.* $S_1 = +0.866\,P$; $S_2 = -0.769\,P$; $S_3 = -0.963\,P$.

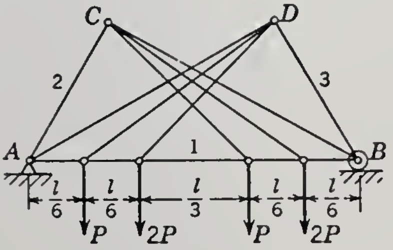


FIG. O

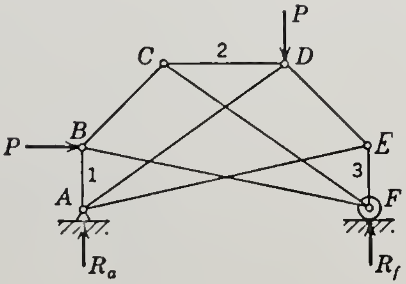
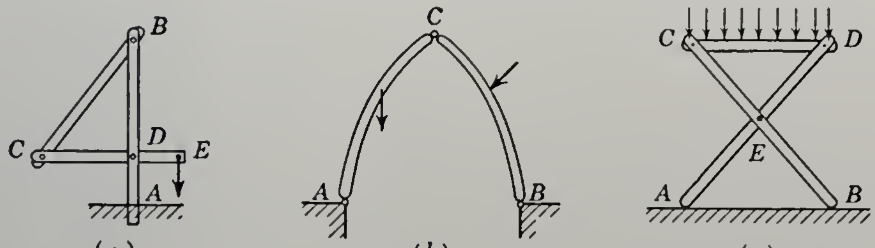


FIG. P

*16. Determine the forces in bars 1, 2, and 3 of the plane truss loaded and supported as shown in Fig. P if $ABCDEF$ is one-half of a regular octagon. *Ans.* $S_1 = -0.293P$; $S_2 = -P$; $S_3 = -1.207P$.

3.5. Plane frames: method of members. In engineering problems of statics we frequently encounter structures consisting of several bars or members in one plane so pinned together as to form a rigid system, but not otherwise satisfying the definition of a plane truss (see page 117). Several examples of such *frame structures* are shown in Fig. 109. They differ from trusses principally in one respect: the various members are not limited to the action of forces applied at their ends only. Thus the bars composing such structures are subjected to *bending* as well as simple tension or compression and are called *flexure members.* In the case of frames containing such flexure members, the actions and reactions between the several parts of the system will not be collinear with the axes of the members as in the case of trusses. This makes it necessary in the analysis of such structures to consider separately the conditions of equilibrium of each member of the frame

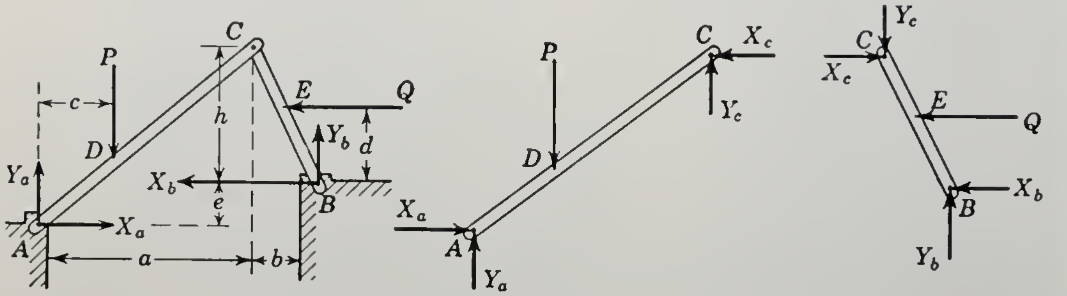


(*a*) (*b*) (*c*)

FIG. 109

rather than of its joints. Thus in contradistinction to the method of joints for analyzing trusses, we use what we shall call the *method of members* for the analysis of frames.

The general problem in the analysis of a frame structure under given loading is to find the magnitudes and directions (or more often the rectangular components) of the forces transmitted from one member to another through the connecting pins. To do this analytically, we isolate each member of the structure as a free body and show all forces acting on it by vectors. Consider, for example, the *three-hinged arch* ACB loaded as shown in Fig. 110*a*. This structure consists of two rigid bars or ribs AC and BC hinged together at C and to the foundation at A and B. Since each member is acted upon by a force at some intermediate point, it is subjected to bending, and the forces at its ends will not be directed along the axis of the member. Accordingly, we do not know the direction of the reaction at either A or B; therefore we represent these reactions by rectangular components X_a, Y_a, and X_b, Y_b, as shown. Considering the equilibrium of the entire structure, we encounter a system of coplanar forces with four unknowns and with only three equations of equilibrium and the problem appears to be indeterminate. To avoid this difficulty, we disconnect the two bars at C and isolate each one as a free body, Figs. 110*b* and 110*c*. Then on each of these free bodies we have, in addition to the forces already considered, a pair of rectangular components X_c, Y_c, representing the force transmitted from one bar to the other through the hinge C. If we assume these forces on the free body AC to be directed as shown in Fig. 110*b*, then it follows from the law of action and reaction that the corresponding forces on BC must be oppositely directed, as shown in Fig. 110*c*. It makes no difference which way we direct the forces X_c, Y_c, as long as they are *opposite* on the two free bodies. If either of



(*a*) (*b*) (*c*)

FIG. 110

them is incorrectly assumed, we shall simply obtain a result with negative sign after calculation.

We now have two simultaneous free-body diagrams (Fig. 110*b* and 110*c*) involving altogether six unknowns. Since for each free body we have three equations of equilibrium, Eqs. (18) or (19), it follows that the problem is statically determinate, i.e., we have six equations and six unknowns. With points A and B as moment centers, Eqs. (18) for the two systems will be as follows:

For the free body in Fig. 110*b*,

$$\begin{aligned} X_a - X_c &= 0 \\ Y_a + Y_c - P &= 0 \\ X_c(h + e) + Y_c a - Pc &= 0 \end{aligned} \tag{a}$$

For the free body in Fig. 110*c*,

$$\begin{aligned} X_c - X_b - Q &= 0 \\ Y_b - Y_c &= 0 \\ Y_c b - X_c h + Qd &= 0 \end{aligned} \tag{b}$$

Having numerical data for the dimensions a, b, c, d, e, h, and the loads P and Q, these six equations are easily solved for the six unknowns X_a, Y_a, X_b, Y_b, and X_c, Y_c. When the rectangular components have all been found, the magnitude and direction of each pin force can be calculated by using Eqs. (2) page 25, if desired.

Before going further, it should be noted that the procedure outlined above brings us in general to a large number of simultaneous equations involving as many unknowns. The solution of such a set of equations can be troublesome if each equation contains many or all of the unknowns. Therefore in writing such equations, we should choose such moment centers and axes of projections that as many of the unknowns as possible are eliminated from each equation. This is usually not difficult to manage. When a system of forces in one plane is in equilibrium, the algebraic sum of their projections on *any axis* must vanish, and the algebraic sum of their moments about *any point* must vanish. There is nothing mandatory about taking x and y axes horizontal and vertical, respectively, or using the origin O as a moment center.

EXAMPLES

1. A folding campstool rests upon a smooth horizontal floor and is loaded as shown in Fig. 111*a*. Determine the magnitude of the shear force on the pin at E and the position of the load P on the bar CD (that is, the value of α) to make this force a maximum.

Solution. We begin with a consideration of the equilibrium of the entire frame. Since the floor is assumed to be smooth, the reactions R_a and R_b will be vertical forces, and we obtain the free-body diagram as shown in Fig. 111*a*. Then using points B and A, successively, as moment centers, we find

$$R_a = (1 - \alpha)P \qquad \text{and} \qquad R_b = \alpha P$$

To proceed further, we make a separate free-body diagram for each leg of the stool, representing the unknown pin reactions at D, C, and E by rectangular components, as shown in Figs. 111*b* and 111*c*. We do not know the directions of these forces but only that the components at E on AD must be equal and opposite to those at E on BC, since they are in the nature of action and reaction. Now writing $\Sigma M_d = 0$ for the free body AD and $\Sigma M_c = 0$ for the free body BC, we obtain

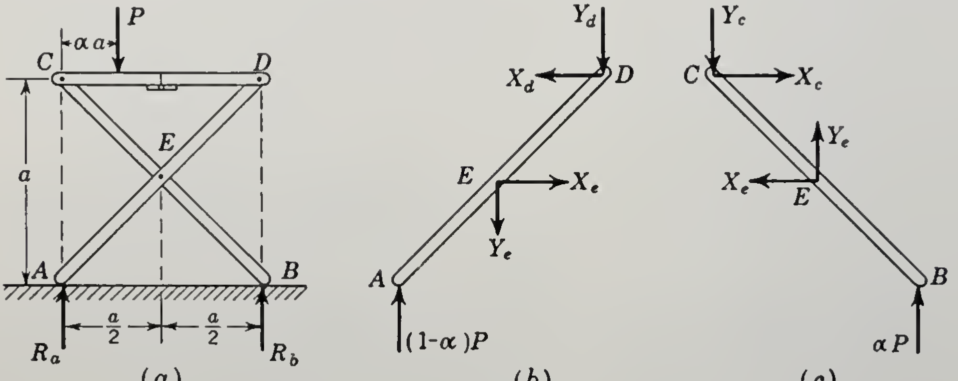


FIG. 111

$$-(1 - \alpha)Pa + Y_e\frac{a}{2} + X_e\frac{a}{2} = 0$$

$$\alpha Pa + Y_e\frac{a}{2} - X_e\frac{a}{2} = 0$$

from which

$$X_e = P \qquad \text{and} \qquad Y_e = (1 - 2\alpha)P$$

The fact that we obtain positive values for these components shows that the assumed directions in Figs. 111*b* and 111*c* were correct.

For the resultant shear force on the pin at E, we have

$$R_e = P\sqrt{1 + (1 - 2\alpha)^2}$$

the maximum value of which is

$$(R_e)_{\max} = \sqrt{2}\,P$$

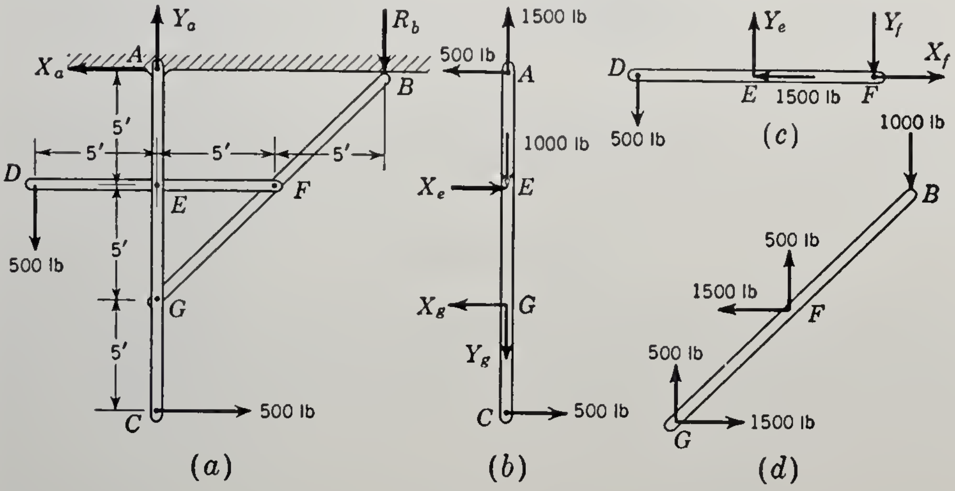
when $\alpha = 0$ or 1, that is, when the load P is at C or D.

2. A frame consisting of three members is supported and loaded as shown in Fig. 112*a*. Find the rectangular components of the forces transmitted from one member to another through the connecting pins *E*, *F*, *G*.

Solution. We begin with a free-body diagram of the entire frame, as shown in Fig. 112*a*. Using *A* as a moment center, and neglecting any friction at *B*, Eqs. (18) may be written as follows:

$$-X_a + 500 = 0$$
$$Y_a - R_b - 500 = 0$$
$$-10R_b + 500 \times 5 + 500 \times 15 = 0$$

from which we obtain $X_a = 500$ lb, $Y_a = 1{,}500$ lb, $R_b = 1{,}000$ lb.



(*a*) (*b*) (*d*)

(*c*)

FIG. 112

We now make a separate free-body diagram for each of the three members, as shown in Fig. 112*b* to 112*d*. On these free-body diagrams, we place the given loads at *D* and *C* and the numerical values of the previously found reactions at *A* and *B*, as shown. At each of the points *E*, *F*, *G*, we indicate two rectangular components of force representing the interactions between members at these points of connection, but we do not immediately attach any arrows or symbols to these vectors.

Now let us consider the equilibrium of the vertical bar *AC* in Fig. 112*b*. At this time none of the four forces acting at *E* and *G* is known, but, nevertheless, we can find X_e and X_g by writing $\Sigma M_g = 0$ and $\Sigma M_e = 0$, which give

$$500 \times 5 + 500 \times 10 - 5X_e = 0$$
$$500 \times 10 + 500 \times 5 - 5X_g = 0$$

From these two equations, we obtain $X_e = 1{,}500$ lb, $X_g = 1{,}500$ lb, directed as assumed. Accordingly, we at once record 1,500 lb to the left at *E* in Fig. 112*c* and 1,500 lb to the right at *G* in Fig. 112*d*.

Next, we consider the equilibrium of the horizontal bar *DF* in Fig. 112*c*. With *F* as a moment center, the equations of equilibrium [Eqs. (18)] for this free body become

$$X_f - 1{,}500 = 0$$
$$Y_e - Y_f - 500 = 0$$
$$5Y_e - 500 \times 10 = 0$$

from which $X_f = 1{,}500$ lb, $Y_e = 1{,}000$ lb, and $Y_f = 500$ lb, all directed as assumed. These numerical values may now be recorded at E in Fig. 112*b* and at F in Fig. 112*d*, as shown.

Returning to the free body in Fig. 112*b* and writing $\Sigma Y_i = 0$, we obtain

$$1{,}500 - 1{,}000 - Y_g = 0$$

from which $Y_g = 500$ lb directed down. We immediately place a corresponding 500-lb force up at G in Fig. 112*d*, and we see that all forces have now been evaluated.

As a check on our arithmetic, we observe that the forces on the inclined bar in Fig. 112*d*, which have all been found from considerations of the other free-body diagrams, satisfy the three conditions of equilibrium for this bar.

PROBLEM SET 3.5

1. The frame structure as shown in Fig. A supports a load $Q = 1{,}000$ lb. Assuming ideal pins at all joints, find the compressive force S in the bar BC and the shear force R_d on the pin at D. The pulley at E has a radius $r = 1$ ft. *Ans.* $S = 1{,}250$ lb; $R_d = 2{,}016$ lb.

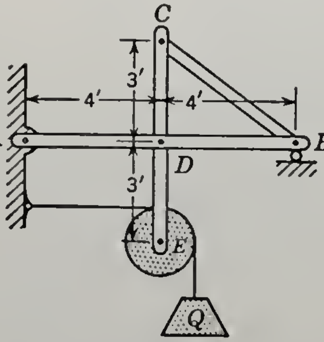


FIG. A

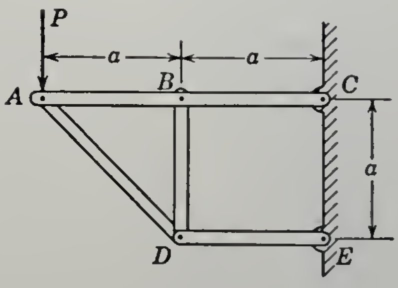


FIG. B

2. Find the axial force S induced in the vertical member BD of the plane frame supported and loaded as shown in Fig. B. *Ans.* $S = +2P$, tension.

3. Referring to the compound beam in Fig. C, determine the shear force on the connecting pin B and the axial force S_i in each of the four supporting bars. *Ans.* $R_b = 225$ lb; $S_1 = -188$ lb; $S_2 = S_3 = -265$ lb; $S_4 = 62$ lb.

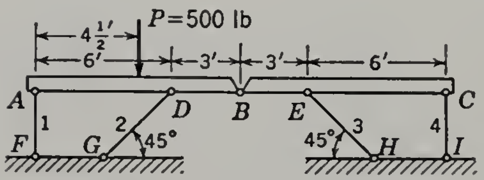


FIG. C

4. Referring to Fig. D, find the tension S induced in the tie rod AB of the frame ABC supported and loaded as shown. *Ans.* $S = 130$ lb.

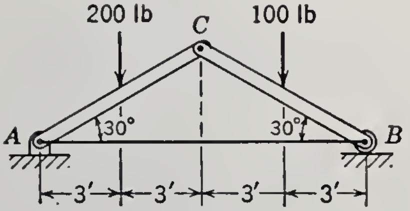


FIG. D

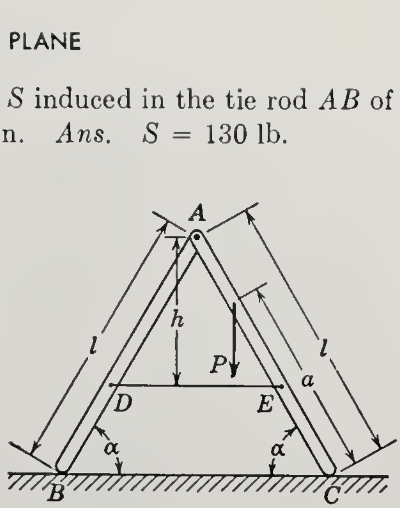


FIG. E

5. A ladder, consisting of two equal parts AB and AC hinged together at A and connected by a horizontal string DE, rests on a smooth horizontal plane (Fig. E). Determine the tensile force S that will be produced in the string by the applied load P. The following numerical data are given: $l = 10$ ft, $a = 7$ ft, $h = 5$ ft, $\alpha = 60°$. *Ans.* $S = 0.35P$.

6. A plane figure-four frame $ABDE$ is supported on an inclined plane and loaded as shown in Fig. F. Calculate the axial force induced in the member BD. *Ans.* $S_{bd} = 106.7$ lb, tension.

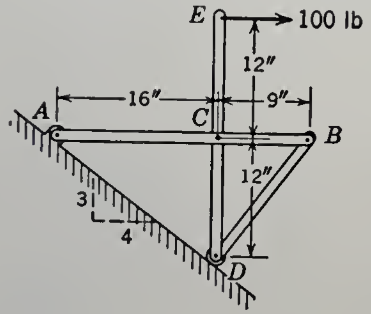


FIG. F

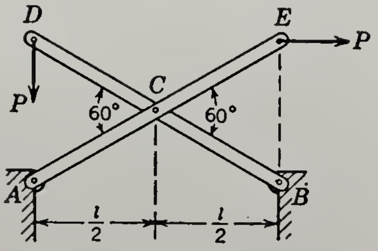


FIG. G

7. Determine the horizontal and vertical components of the reactions at A and B for the X frame supported and loaded in one plane as shown in Fig. G. *Ans.* $X_a = 1.732P$, in; $Y_a = 0.423P$, up; $X_b = 2.732P$, in; $Y_b = 0.577P$, up.

8. A bracket made up of two members BD and AC is loaded as shown in Fig. H. Neglecting the weights of the members and pulleys, calculate the horizontal and vertical components of the reactions at A and B. Each pulley has a radius $r = 1$ ft. *Ans.* $X_a = -X_b = 1{,}167$ lb; $Y_a = +500$ lb; $Y_b = 0$.

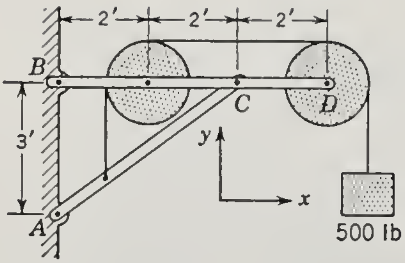


FIG. H

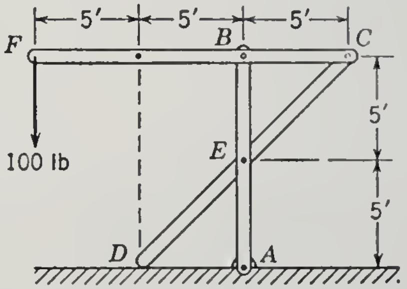


FIG. I

9. Calculate the shear forces on the pins A, B, C, and E of the plane frame supported and loaded as shown in Fig. I. Neglect the weights of the members, and assume that the horizontal floor is smooth. *Ans.* $R_a = 100$ lb; $R_b = 300$ lb; $R_c = 200$ lb; $R_e = 400$ lb.

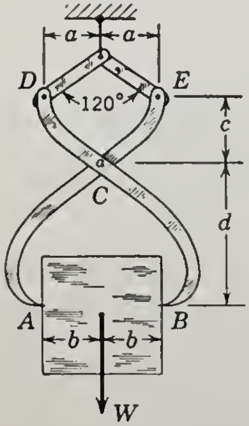


FIG. J

10. Calculate the pressure exerted against the sides of an ice cube of weight $W = 100$ lb by the points A and B of the tongs which support it (Fig. J). The dimensions of the tongs are as follows: $a = 4$ in., $b = 8$ in., $c = 8$ in., $d = 16$ in. *Ans.* $R_a = R_b = 95$ lb.

11. Calculate the shear force R_c on the pin C of the ice tongs in Fig. J. *Ans.* $R_c = 167.4$ lb.

12. Find the reactions at the supports A and B of the semicircular three-hinged arch loaded as shown in Fig. K. *Ans.* $R_a = 4.01$ tons, $R_b = 4.15$ tons.

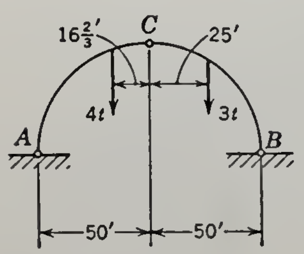


FIG. K

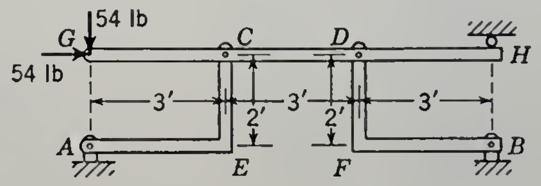


FIG. L

13. Calculate the horizontal and vertical components of the reactions at A and B of the frame structure loaded as shown in Fig. L. *Ans.* $X_a = 63$ lb; $Y_a = 42$ lb; $X_b = 117$ lb; $Y_b = 78$ lb.

14. Calculate the components of reactive forces at A and B of the frame structure in Fig. L if the horizontal bar GH pushes against a smooth vertical wall at H instead of the roller as shown. *Ans.* $X_a = 162$ lb; $Y_a = 108$ lb; $X_b = 81$ lb; $Y_b = 54$ lb.

15. Calculate the total shear force on each of the pins A, B, and F of the frame structure loaded as shown in Fig. M. Each bar is inclined at 45° and

AE and *BD* are connected at their mid-points *F*. *Ans.* $R_a = R_b = 1{,}118$ lb; $R_f = 1{,}414$ lb.

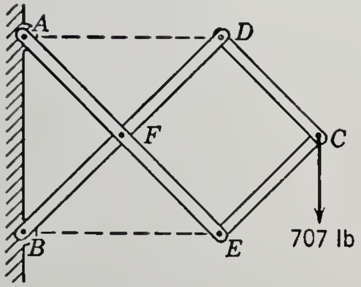


FIG. M

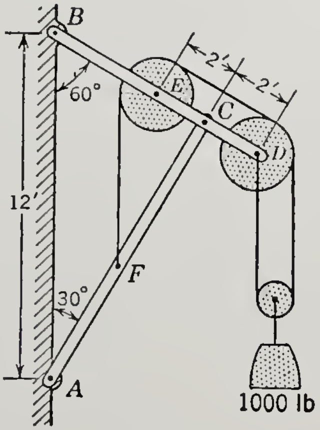


FIG. N

16. Determine the horizontal and vertical components of the reactions at *A* and *B* of the frame shown in Fig. N. Each of the stationary pulleys at *D* and *E* is 32 in. in diameter and the weights of pulleys and members are to be neglected. *Ans.* $X_a = 633$ lb; $Y_a = 802$ lb; $X_b = 633$ lb; $Y_b = 198$ lb.

3.6. The funicular polygon. While the method of projections and moments described in Art. 3.2 can always be used in the solution of problems of statics, it is sometimes advantageous to use graphical methods in working with a system of forces in a plane. The method of composition of forces by successive applications of the parallelogram law, besides being very tedious, is not always satisfactory, since the lines of action of the forces sometimes do not intersect within the limits of the drawing and sometimes we have parallel forces. It is the purpose of this article to present a graphical method of analysis which can be applied to any system of forces in a plane.

Let us consider first the simple case of two forces *P* and *Q* acting on a body, as shown in Fig. 113*a*. We begin with the construction of the polygon of forces *ABC* (Fig. 113*b*), the closing side $\overline{AC}$ of which gives both the magnitude and direction of the resultant *R* of the given forces. There remains, then, only the determination of the position of its line of action. To accomplish this, we take an arbitrary point *O*, called a *pole*, in the plane of the polygon of forces and join it by lines 1, 2, and 3 with the apexes of the polygon. These lines are called *rays* and like the other lines in the figure may be considered as vectors representing forces. Taking, for instance, the arrows as indicated inside $\triangle AOB$, we may consider the force *P* as the resultant of the forces 1 and 2. In the same manner, the force *Q* may be considered as the resultant of the forces 2 and 3, the directions of which are indicated by the arrows inside $\triangle BOC$. Referring, now, to Fig. 113*a*, it is evident that the action of the forces P and Q will not be changed if each of them is replaced by its two components indicated in Fig. 113*b*. These replacements will be made in the following manner: Beginning with any point a in the plane of action of the forces (Fig. 113*a*), we draw the line ab parallel to the ray AO. From the point of intersection b of this line with the line of action of the force P, we draw the line bc parallel to the ray BO, and from the point of intersection c of this line with the line of action of the force Q, we draw the line cd parallel to the ray CO. The polygon $abcd$, obtained in this way, is called a *funicular polygon* for the forces P and Q. The apexes of this polygon are on the lines of action of the given forces, and its sides are parallel to the rays of the polygon of forces. We assume now that, at point b, the forces 1 and 2 replacing the force P and, at point c, the forces 2 and 3 replacing the force Q are applied as shown in Fig. 113*a*. In this way the given system of forces P and Q is replaced by a system of four forces applied at points b and c. Since the pair of forces 2 acting along the line bc are equal and opposite, they are in equilibrium and may be removed from the system and there remain only the forces 1 and 3, which are equivalent in action to the given forces P and Q. The magnitude and direction of the resultant of these forces are given in Fig. 113*b* by the vector $\overline{AC}$ and a point on its line of action is given by the point of intersection e of the forces 1 and 3 acting along the first and last sides of the funicular polygon (Fig. 113*a*).

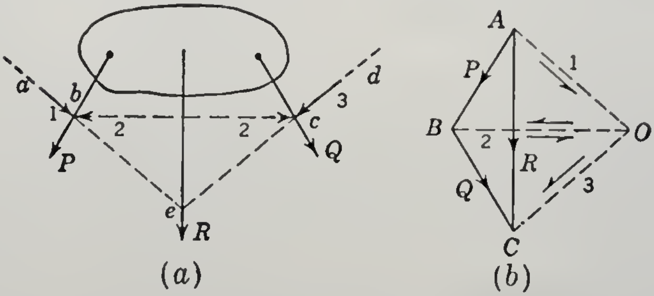


FIG. 113

If we imagine a weightless flexible string going along the sides ab, bc, and cd of the funicular polygon with its ends fixed at a and d, this string will evidently be in equilibrium under the action of the forces P and Q. The tensile forces in the portions ab and bc of the string will be numerically equal to the forces 1 and 2, indicated in the figure at b, and will balance the force P. In the same way, the tensile forces in the portions *bc* and *cd* of the string balance the force Q. This relation between the constructed polygon *abcd* and the configuration of equilibrium of a string submitted to the action of the given forces explains the origin of the name "funicular polygon," i.e., *string polygon*.

The graphical constructions discussed above are perfectly general and can be used also in the case of parallel forces in a plane. We take as a second example the simple case of two parallel forces P and Q applied to a body as shown in Fig. 114*a* and begin with the construction of the polygon of forces which, in this case, is represented by a portion ABC of a straight line Fig. 114*b*) parallel to the lines of action of the forces.[1] By choosing a pole O and drawing the rays 1, 2, and 3, we may construct a funicular polygon *abcd* (Fig. 114*a*) as before with its sides parallel to these rays. Replacing the force P by its components 1 and 2 applied at point b and the force Q by its components 2 and 3 applied at point c, we see again that the forces 2 balance each other and can be removed leaving only two nonparallel forces 1 and 3, which are equivalent to the given forces P and Q. The magnitude and direction of the resultant of these forces are given by the vector $\overline{AC}$ in the polygon of forces (Fig. 114*b*), and a point on its line of action is obtained by the point of intersection e of the first and last sides of the funicular polygon (Fig. 114*a*).

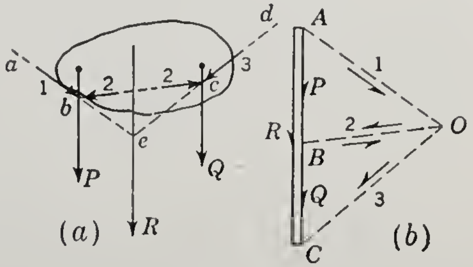


FIG. 114

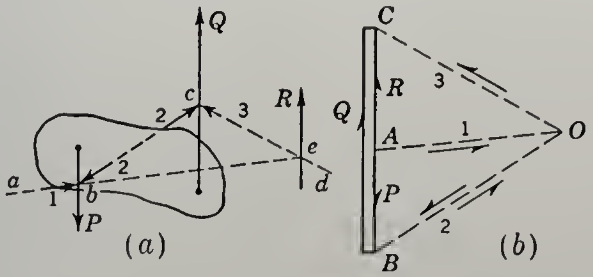


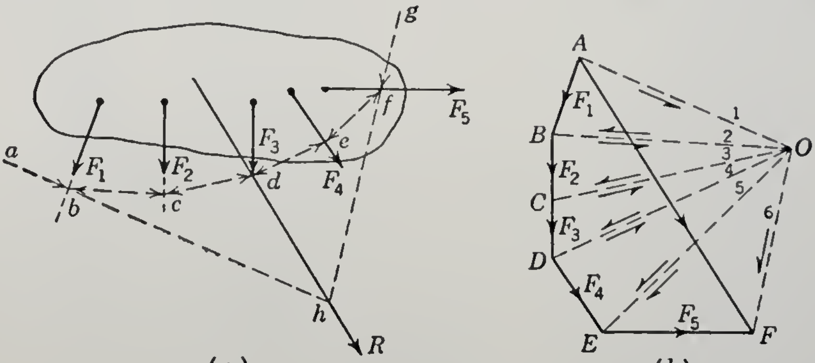
FIG. 115

The same procedure may be used in the case of two unequal parallel forces acting in opposite directions, as shown in Fig. 115*a*. In this case again, we begin with the polygon of forces ABC, select a pole O, and draw rays 1, 2, and 3. Starting with any point a in the plane of action of the given forces, the funicular polygon *abcd* with its sides parallel, respectively, to rays OA, OB, and OC is obtained as shown (Fig. 115*a*). The magnitude and direction of the resultant are given by the closing side $\overline{AC}$ of the polygon of forces (Fig. 115*b*), and a point on its line of action, by the intersection *e* of the first and last sides of the funicular polygon (Fig. 115*a*).

---

[1] The slight separation of the two vertical lines in Fig. 114*b* is made only for more easily distinguishing the several superimposed vectors.

Let us consider now a general case of several coplanar forces $F_1, \ldots, F_5$ applied to a body as shown in Fig. 116*a*. We begin with the construction of the polygon of forces $ABCDEF$. Choosing an arbitrary pole O, drawing the rays 1, 2, 3, 4, 5, 6, and constructing, in the plane of action of the forces, the lines *ab*, *bc*, . . . , *fg*, parallel to these rays, we obtain the funicular polygon *abcdefg* as shown. At the apexes of this polygon, each of the given forces $F_1, \ldots, F_5$ is replaced by its two components as represented by rays in Fig. 116*b*. The forces acting along the sides *bc*, *cd*, *de*, and *ef* are pairs of equal and opposite forces and may be removed from the system. There remain only the forces 1 and 6 acting at points *b* and *f* which are equivalent to the given forces $F_1, \ldots, F_5$. The magnitude and direction of the resultant of these forces are given by the closing side $\overline{AF}$ of the polygon of forces (Fig. 116*b*), and a point on its line of action, by the intersection *h* of the first and last sides of the funicular polygon (Fig. 116*a*).



(*a*) (*b*)

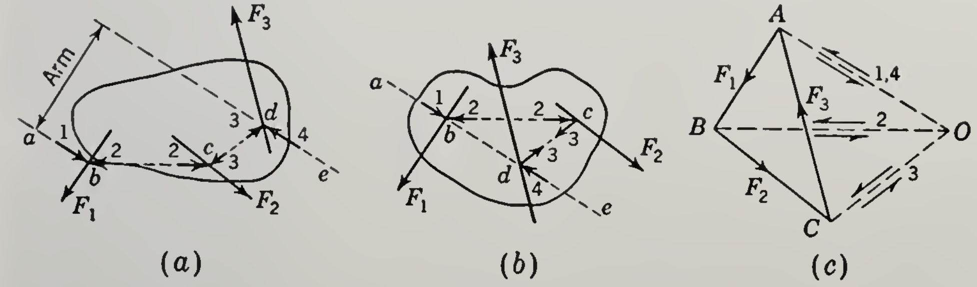
FIG. 116

From the foregoing examples, it is seen that the graphical determination of the resultant of any given system of forces in a plane reduces to the construction of two polygons: the polygon of forces and a funicular polygon. From the closing side of the polygon of forces we obtain the magnitude and direction of the resultant; from the intersection of the first and last sides of the funicular polygon we obtain a point on its line of action.

If the polygon of forces is closed, the first and last rays coincide;

hence the first and last sides of the funicular polygon become parallel or coincide. In the first case, the two equal forces acting along these sides represent a resultant couple. In the second case, they balance each other and the given system of forces is in equilibrium.

The above two cases are illustrated in Fig. 117. Three given forces F_1, F_2, and F_3 are of such magnitude and direction that their vectors build a closed polygon (Fig. 117*c*). Hence the rays 1 and 4, directed to the beginning of the first vector $\overline{AB}$ and to the end of the last, $\overline{CA}$, coincide in one ray OA. If the given forces $F_1, \ldots, F_3$ are applied at three points in a body as shown in Fig. 117*a*, the first side *ab* and the last side *de* of the funicular polygon are parallel but do not coincide.



(*a*) (*b*) (*c*)

FIG. 117

Hence the unbalanced forces 1 and 4 acting along these sides represent a couple that is equivalent to the given system of forces. The moment of this couple is found by scaling the length of its arm from Fig. 117*a* and the magnitude of either force from ray OA (Fig. 117*c*).

If the same forces $F_1, \ldots, F_3$ are applied at three points in a body as shown in Fig. 117*b*, the first and last sides of the funicular polygon coincide, and the equal but opposite forces 1 and 4 acting along these coincident sides balance each other. Hence, in this case, the given system of forces is in equilibrium.

It is seen that by using the polygon of forces together with the funicular polygon, the three possible cases which may occur in the case of a system of forces in a plane can be investigated entirely by graphical methods. If the polygon of forces is not closed, the given system of forces reduces to a resultant force. If the polygon of forces is closed but the first and last sides of the funicular polygon, which are parallel, do not coincide, the system reduces to a resultant couple. If the polygon of forces is closed and the first and last sides of the funicular polygon coincide, i.e., the funicular polygon is closed also, the system of forces is in equilibrium.

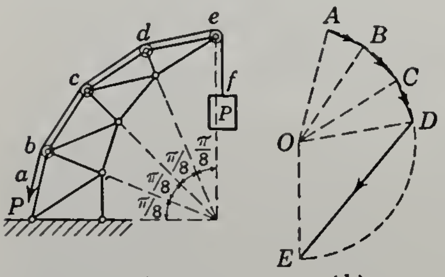
From the foregoing discussion, we conclude that for any system of forces in a plane the graphical conditions of equilibrium will be (1) *a closed force polygon and* (2) *a closed funicular polygon.* These conditions are entirely independent of the order in which the forces are geometrically added in constructing the polygon of forces, of the position in the plane of the polygon of forces of the pole O, and of the point in the plane of action of the forces at which the construction of the funicular polygon is started. They are equivalent to the analytical conditions of equilibrium as represented by Eqs. (18) or (19) of Art. 3.2 and, in general, may be used for the graphical solution of any problem of statics involving a system of coplanar forces in equilibrium.

However, just as the analytical solution of a problem of statics can often be greatly simplified by a proper choice of a moment center or of an axis of projection, so can the graphical solution of such a problem often be greatly simplified by a proper choice of the order in which the forces are added in constructing the polygon of forces, of the position of the pole O, or of a starting point in the plane of action of the forces for the construction of the funicular polygon. The most expedient procedure to be followed in any particular instance depends largely upon the conditions of the problem, and no set of rules can ever take the place of a sound understanding of the fundamental relation between the funicular polygon and the polygon of forces together with its pole and rays.

EXAMPLES

1. The hoisting cable of a crane (Fig. 118*a*) is carried over a series of pulleys *b, c, d,* and *e* which are attached to the joints of the crane as shown. Determine the forces exerted by the axles of the pulleys on the joints of the truss to which they are attached.

Solution. These forces may be obtained in a very simple manner by observing that the flexible cable itself represents a funicular polygon for the required system of forces. Neglecting friction in the pulleys, it is evident that the tensile force in the cable is uniform throughout its length and numerically equal to the load P. Now, with any pole O as a center (Fig. 118*b*), let a circle be drawn, the radius of which represents, to some scale, the magnitude of the force P in the cable, and from this pole let rays OA, OB, OC, OD, and OE, parallel to the portions *ab, bc, cd, de,* and *ef,* respectively, of the cable be drawn. Then, obviously, $ABCDE$ is the polygon of forces for the required



(*a*) (*b*)

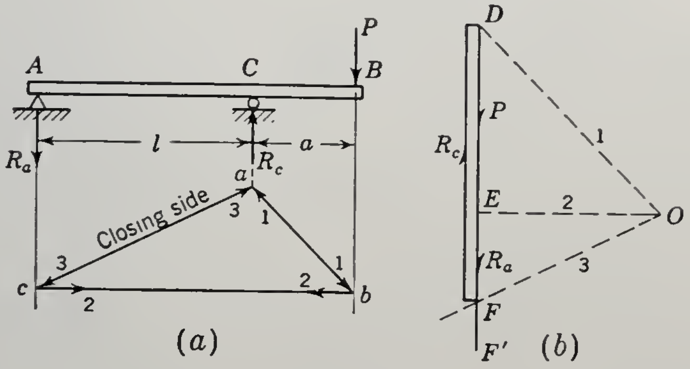
FIG. 118

system and the sides AB, . . . , DE represent the forces acting on the joints of the truss.

2. Determine graphically the reactions R_a and R_c produced at the supports A and C of the horizontal beam AB (Fig. 119*a*) due to the action of a vertical load P applied at the free end B.

Solution. It is evident from the conditions of support that the reactions R_a and R_c are vertical forces, the lines of action of which pass through the points A and C, respectively. The three parallel forces P, R_a, and R_c, being in equilibrium, must build a closed polygon of forces, and their funicular polygon must be closed.

We begin the construction of the polygon of forces (Fig. 119*b*) by laying out to scale the vector $\overline{DE}$ representing the known force P. Following this will come the vector $\overline{EF}$ representing the reaction R_a and then the vector $\overline{FD}$ representing the reaction R_c and closing the polygon of forces. However, since we know the magnitude of neither R_a nor R_c, the polygon of forces cannot yet be completed and we can do no more than draw the line EF' in the known vertical direction of the reaction R_a. This done, we select an arbitrary pole O (Fig. 119*b*) and draw rays 1 and 2 as shown. Ray 3 cannot yet be drawn because the position along EF' of the apex F of the polygon of forces is not known. However, through any point b on the line of action of the force P we can now draw two sides of the funicular polygon parallel, respectively, to rays 1 and 2. Prolonging these sides until they intersect, at points a and c, the known lines of action of the reactions R_c and R_a, respectively, we obtain the apexes a and c of the funicular polygon and can draw its closing side ac after which the missing ray 3 in the polygon of forces can be drawn through point O parallel to this closing side ac of the funicular polygon. The intersection of ray 3 with the vertical line EF' determines the apex F of the closed polygon of forces. The vectors $\overline{EF}$ and $\overline{FD}$ represent, respectively, the reactions R_a and R_c.



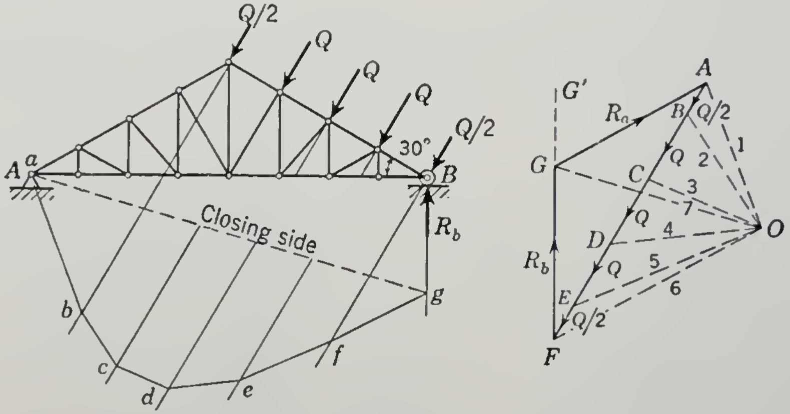
(*a*) (*b*)

FIG. 119

It should be noted that, while any position of the point F along the line EF' (Fig. 119*b*) can satisfy the condition of a closed polygon of forces, only the point F as obtained above can simultaneously satisfy the condition of a closed funicular polygon.

3. Determine graphically the reactions R_a and R_b at the supports A and B of the truss loaded and supported as shown in Fig. 120*a*.

Solution. Since the truss is in equilibrium under the action of the applied forces Q together with the reactions R_a and R_b, all these forces considered together must build a closed polygon of forces and their funicular polygon must be closed also. Proceeding on this basis, we begin with the construction of the polygon of forces (Fig. 120*b*). The vectors representing the active forces Q are laid out in order, and from point F the vertical line FG' is extended in the known direction of the reaction R_b. Since, however, we know neither the magnitude of this reaction nor the direction of the reaction R_a, the closed polygon of forces cannot yet be completed. We know only that from some point G on the vertical line FG' the vector $\overline{GA}$ representing the reaction R_a must close the polygon. An arbitrary pole O is now selected and the rays 1, 2, 3, 4, 5 and 6 drawn, as shown in Fig. 120*b*. The ray 7 cannot yet be drawn, since the apex G of the polygon of forces is not known at this stage of the construction.



(*a*) (*b*)

FIG. 120

We now begin the construction of the funicular polygon (Fig. 120*a*). As we have already seen, a general characteristic of the funicular polygon is that its various apexes must lie on the lines of action of the given system of forces. Now in this problem it happens that one of the forces, namely the reaction R_a, is not known in direction; all that is known regarding it being that its line of action must pass through point A. Hence it follows that, if we are to succeed in completing the construction of the closed funicular polygon (which we must do to find the direction of the missing ray 7 in Fig. 120*b*), we must begin at point A in the plane of action of the forces. Beginning at point A, the sides *ab*, *bc*, *cd*, *de*, *ef*, and *fg* of the funicular polygon are now drawn and the closing side *ag* established. The missing ray 7 in the polygon of forces can now be drawn parallel to the closing side *ag* of the funicular polygon. The intersection of ray 7 with the line FG' determines the apex G of the closed polygon of forces. Thus the vectors $\overline{FG}$ and $\overline{GA}$ completely determine the reactions R_b and R_a, respectively.

4. A horizontal beam is supported by three bars hinged at both ends as shown in Fig. 121*a*. Find graphically the axial forces S_1, S_2, S_3 induced in these bars if the beam is loaded as shown.

Solution. We assume that the sketch in Fig. 121*a* is constructed to scale so that the positions and directions of the applied loads and supporting bars are correctly shown. Since the axes of the supporting bars represent the lines of action of the reactive forces which they exert on the beam, we may regard Fig. 121*a* as a free-body diagram and we see that the beam is in equilibrium under the action of six coplanar forces. Accordingly, these forces must build a closed force polygon and any corresponding funicular polygon must also be closed.

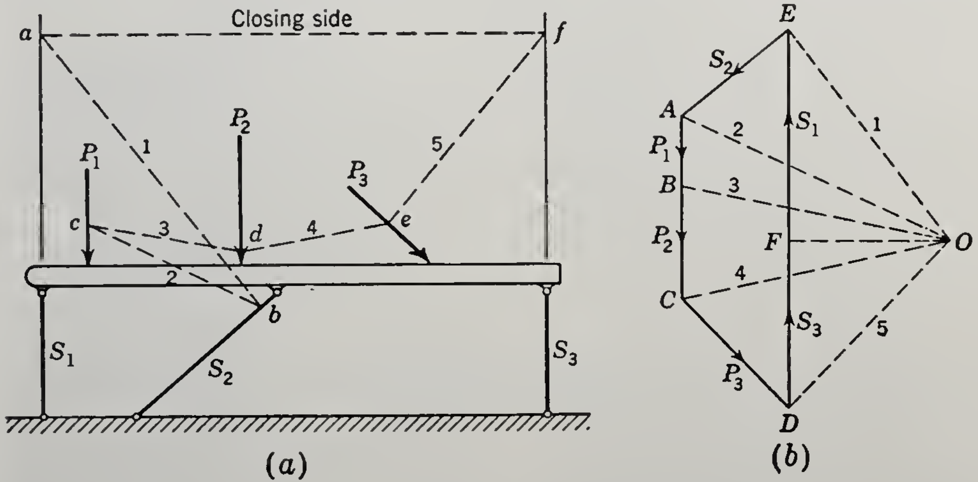


FIG. 121

We begin the construction of the force polygon (Fig. 121*b*) by laying out to scale the vectors $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, representing the applied loads as shown. The three remaining vectors representing the reactions may be added on in any order. We choose to take the two vertical reactions S_3 and S_1 together (and in that order) and then the vector S_2 which must close the polygon. In this way, we are able to close the polygon of forces without yet knowing the individual magnitudes of S_3 and S_1, that is, the position of point F on the vector $\overline{DE}$ representing $S_3 + S_1$ is not yet known.

To locate point F, we choose an arbitrary pole O and draw rays 1, 2, 3, 4, 5 to the known apexes E, A, B, C, D, respectively, as shown. Then beginning at any convenient point *a* on the line of action of S_1 in Fig. 121*a*, we construct a corresponding funicular polygon *abcdef*, as shown. The closing side of this funicular polygon now determines the direction of ray OF in Fig. 121*b* and accordingly the magnitudes of S_3 and S_1.

We see from the directions of the arrows on the vectors in Fig. 121*b* that the vertical bars are in compression and the inclined bar is in tension. The magnitudes of these axial forces in the bars may now be scaled from the force polygon.

PROBLEM SET 3.6

1. Find the resultant R of the coplanar forces shown in Fig. A by constructing a funicular polygon. *Ans.* $R = 141$ lb.

2. Find the resultant of the coplanar forces acting on the gravity dam section shown in Fig. B by constructing a funicular polygon. Forces are shown in kips (1 kip = 1,000 lb).

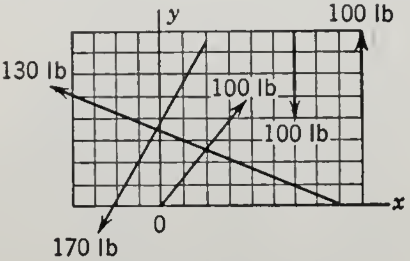

y
100 lb
130 lb
100 lb
100 lb
x
0
170 lb

FIG. A

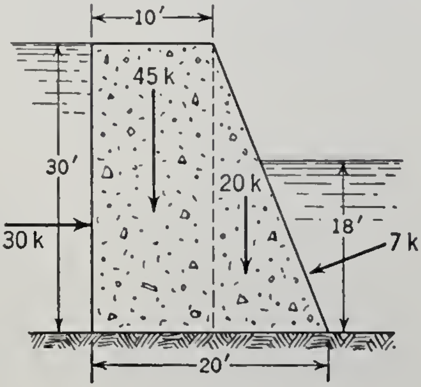

10′
45 k
30′
20 k
30 k
18′
7 k
20′

FIG. B

3. Determine, graphically, the reactions at the supports of a girder AB due to the locomotive loading shown in Fig. C. *Ans.* $R_a = 208.4$ kips; $R_b = 221.6$ kips.

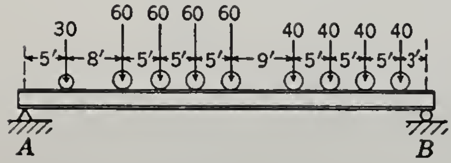

30
60 60 60 60
40 40 40 40
5′
8′
5′
5′
5′
9′
5′
5′
5′
3′
A
B

FIG. C

4. Treating the flexible cable overrunning the pulleys in Fig. D as a funicular polygon, find graphically the forces exerted on the members of the frame by the axles of the pulleys B and D. Each stationary pulley is 1 ft in diameter. *Ans.* $P_b = 4{,}685$ lb; $P_d = 2{,}065$ lb.

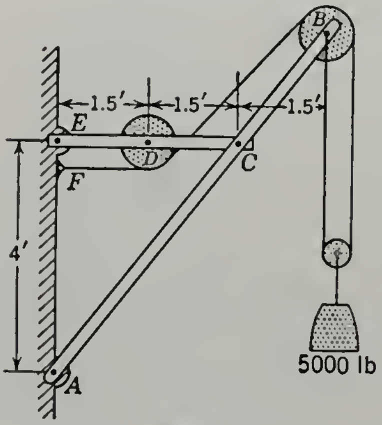

B
1.5′
1.5′
1.5′
E
D
C
F
4′
A
5000 lb

FIG. D

5. Find graphically the reactions at A and B for the beam loaded as shown in Fig. E. *Ans.* $R_a = 86$ lb; $R_b = 77$ lb.

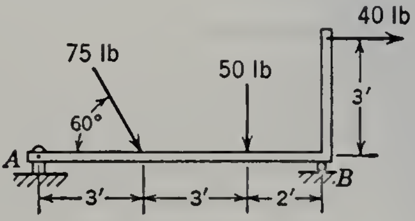

40 lb
75 lb
50 lb
3′
60°
A
B
3′
3′
2′

FIG. E

6. Determine graphically the reactions R_a and R_b at the supports A and B of the horizontal beam AB due to the action of the vertical loads applied as shown in Fig. F. *Ans.* $R_a = 481$ lb; $R_b = 349$ lb.

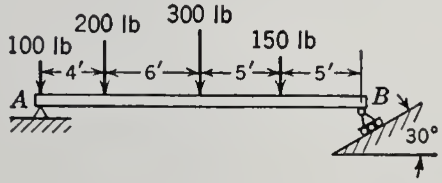

100 lb
200 lb
300 lb
150 lb
4′
6′
5′
5′
A
B
30°

FIG. F

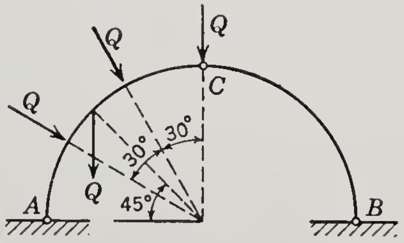

Q
Q
Q
Q
C
30°
30°
45°
A
B

FIG. G

7. Determine graphically the reactions R_a and R_b at the hinged ends A and B of the three-hinged semicircular arch loaded as shown in Fig. G. *Ans.* $R_a = 2.04Q$; $R_b = 1.88Q$.

8. By using the funicular polygon, determine the resultant of the four active forces applied to the beam AB in Fig. H. *Ans.* $M = -1{,}200$ ft-lb.

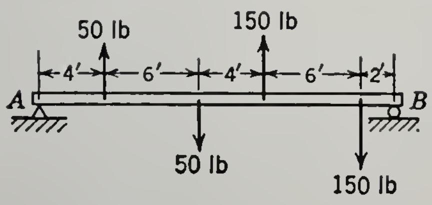

50 lb
150 lb
4′
6′
4′
6′
2′
A
B
50 lb
150 lb

FIG. H

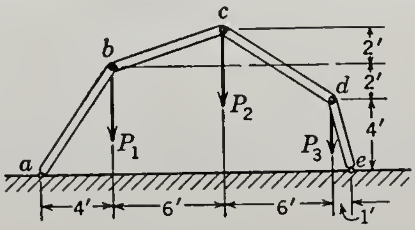

c
b
d
a
e
2′
2′
4′
P_1
P_2
P_3
4′
6′
6′
1′

FIG. I

9. A system of hinged bars in one plane is subjected to vertical loads P_1, P_2, and P_3, applied as shown in Fig. I. Neglecting the weights of the bars, and assuming $P_1 = 100$ lb, find the magnitudes of the other forces if the system is in equilibrium for the configuration shown. *Hint.* Note that *abcde* must be a funicular polygon for the three forces. *Ans.* $P_2 = 85.7$ lb; $P_3 = 286$ lb.

10. Two forces P act on the beam AB, as shown in Fig. J. Determine graphically the reactions R_a and R_b at the supports. *Ans.* $R_a = P/3$, down, $R_b = P/3$, up.

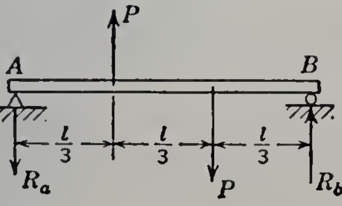

P
A
B
$\frac{l}{3}$
$\frac{l}{3}$
$\frac{l}{3}$
R_a
P
R_b

FIG. J

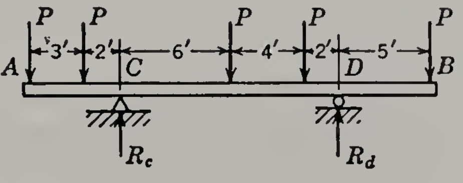

P
P
P
P
P
3′
2′
6′
4′
2′
5′
A
C
D
B
R_c
R_d

FIG. K

11. Determine graphically the reactions R_c and R_d produced at the supports of the horizontal beam AB (Fig. K) due to the action of the vertical loads P applied as shown in the figure. $P = 600$ lb. *Ans.* $R_d = 1{,}300$ lb; $R_c = 1{,}700$ lb.

12. Determine graphically the reactions at the supports A and B of the horizontal beam AB subjected to the action of a system of vertical forces applied as shown in Fig. L. *Ans.* $R_a = 40$ lb, down; $R_b = 340$ lb, up.

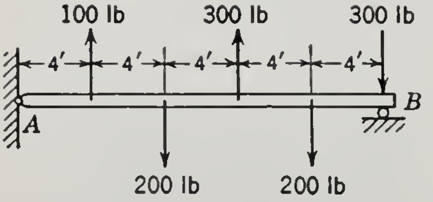


FIG. L

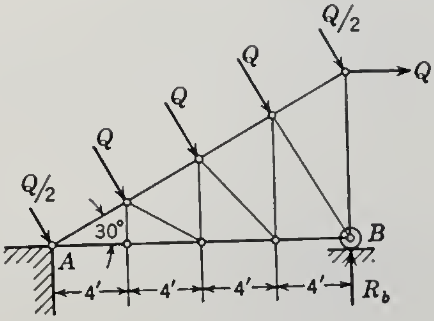


FIG. M

13. Determine graphically the reactions R_a and R_b at the points of support A and B of the truss loaded and supported as shown in Fig. M. $Q = 100$ lb. *Ans.* $R_a = 305$ lb; $R_b = 289$ lb.

14. Determine graphically the forces S_1, S_2, and S_3 in the bars AD, BE, and CF that support the horizontal beam AC loaded as shown in Fig. N. *Ans.* $S_1 = -712$ lb; $S_2 = +566$ lb; $S_3 = -661$ lb.

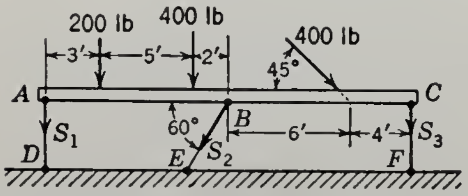


FIG. N

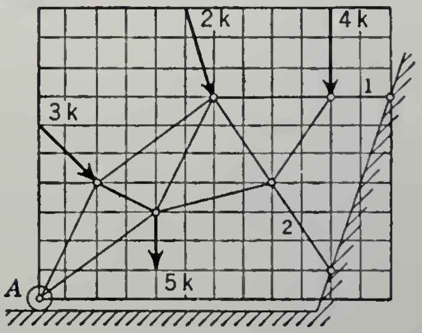


FIG. O

15. Referring to Fig. O, find graphically the reaction R_a and the axial forces S_1 and S_2, if loads act on the truss as shown. The superimposed grid consists of 1-ft squares.

3.7. Maxwell diagrams. We shall discuss here a graphical method of analysis of simple trusses based on the method of joints previously explained in Art. 3.3. Referring to Fig. 122*a*, we have a simple truss ABC supported at A and B and carrying a vertical load P at C. Thus the vertical reactions at A and B are each $P/2$ and we have the entire truss in equilibrium under the action of three parallel forces as shown. Denoting the axial forces in the bars by S_1, S_2, S_3, and considering the equilibrium of each of the joints A, B, C, in succession, we obtain the closed triangles of forces shown in Fig. 122*b* from which the magnitudes of the axial forces S_1, S_2, S_3 can be scaled.

We note now that each of these axial force vectors appears in two different polygons, once for each of the joints at the ends of that member. To avoid this duplication of vectors, the separate polygons of forces, under certain conditions, can be superimposed to form one composite diagram called a Maxwell diagram[1] for the truss. For example, the polygons of forces in Fig. 122*b*, when superimposed, make the composite diagram shown in Fig. 122*c*. Such superposition is desirable, since it reduces the amount of necessary construction and makes a more compact record of the analysis.

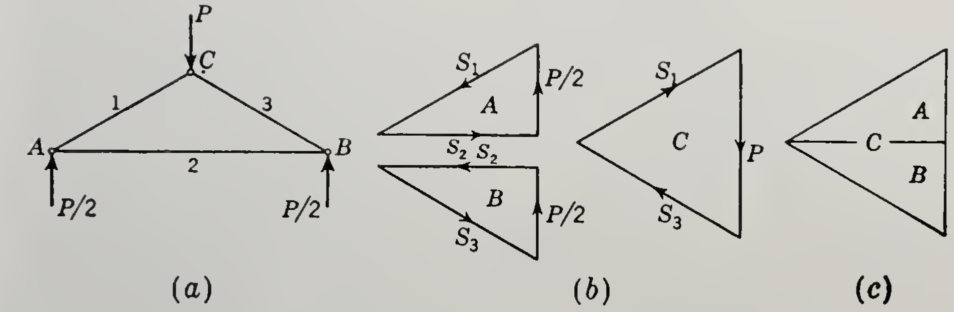


FIG. 122

In constructing separate polygons of forces for the various hinges of a truss, it makes no difference in what order we add the vectors. Each hinge is in equilibrium, and the forces acting on it must build a closed polygon in any order. However, if we wish to superimpose these polygons as in Fig. 122*c*, they must be constructed in a definite manner. It may be noted now that, for each polygon in Fig. 122*b*, the vectors follow each other in the same order as the forces they represent are encountered in going around the corresponding hinge in a *clockwise* direction. This accounts for the fact that all the polygons can be superimposed as shown in Fig. 122*c* without duplicating any vector. In general, we may state the following rule: To construct a Maxwell diagram for a given truss, we must go around every hinge of the truss in the same direction (either clockwise or counterclockwise) and add the vectors in the order in which they are so encountered. The vectors representing the closed polygon for the truss as a whole must be assembled in this same order also.

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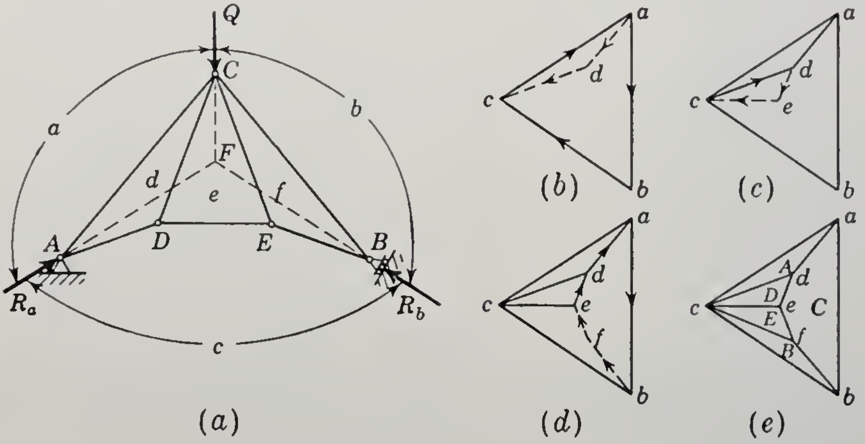
[1] See Clerk Maxwell, On Reciprocal Figures and Diagrams of Forces, *Phil. Mag.*, vol. 26, p. 250, 1864.

In the construction of Maxwell diagrams for the analysis of trusses, a special system of notations, called Bow's notation, is generally used for the designation of the various forces involved. In this system, the spaces between the lines of action of the forces are each given a lower-case letter and each force correspondingly designated by the letters of the two spaces separated by its line of action.

Consider, for example, the simple truss *ABCDE*, supported and loaded as shown in Fig. 123*a*. In accordance with Bow's system of notation, we letter the spaces between the three external forces with *a*, *b*, and *c* and those between the bars of the truss with *d*, *e*, and *f* as shown in the figure. Thus the load Q may be referred to as force *ab*, the reaction R_b as force *bc*, and the reaction R_a as force *ca*. In the same way the force in the bar *AC* may be referred to as *ad*, that in the bar *CE*, as *ef*, etc.



(*a*) (*b*) (*c*) (*d*) (*e*)

FIG. 123

Let us begin now with the construction of a Maxwell diagram for this truss. First, the closed triangle of forces *abc* (Fig. 123*b*) for the entire truss as a free body is constructed, beginning with the vector $\overline{ab}$, representing the known force Q, and laying out the vectors $\overline{bc}$ and $\overline{ca}$ parallel, respectively, to the lines of action of the corresponding forces R_b and R_a. It should be carefully noted that these vectors follow each other in the order in which the corresponding forces are encountered in going around the free body (the entire truss in this case) in a *clockwise* direction. As mentioned before, it is essential in the construction of the Maxwell diagram that this same order be maintained in drawing the remainder of the polygons of forces for the various hinges.

Next, in considering the hinge *A* as a free body, we begin with the already established vector $\overline{ca}$ and construct the closed triangle of forces *cad* by drawing through point *a* (Fig. 123*b*) the vector $\overline{ad}$, parallel to the bar AC, and through point c, the vector $\overline{dc}$, parallel to the bar AD. Thus a new point (point d) is established in the Maxwell diagram. We note again that the three vectors $\overline{ca}$, $\overline{ad}$, and $\overline{dc}$ follow each other in the order in which the corresponding forces are encountered in going around the free body (hinge A) in a clockwise direction.

Proceeding now to a consideration of hinge D as a free body, we begin with the vector $\overline{cd}$ (Fig. 123*c*) which represents the reaction of bar AD on this hinge and which is already established. Drawing, through point d, the vector $\overline{de}$, parallel to bar CD and through point c, the vector $\overline{ec}$, parallel to bar DE, the triangle of forces *cde* for hinge D is obtained and the new point e established.

There being now only two unknown forces at hinge C representing the reactions of bars CE and CB, we may consider this hinge next. The vectors $\overline{ed}$, $\overline{da}$, and $\overline{ab}$ (Fig. 123*d*), representing, respectively, the reactions on this hinge of bars DC, AC, and the action of the load Q, are already established, and we note that they follow each other in the proper order. The polygon of forces *edabf* is completed by drawing, through points b and e, the vectors $\overline{bf}$ and $\overline{fe}$ parallel to the axes of the corresponding bars and a new point f is established.

Either the hinge B or E may be considered next as a free body. Considering B, we note that only one force $\overline{cf}$, representing the reaction on this hinge of the bar EB, remains unknown. Since both points c and f are already established in the diagram (Fig. 123*e*), we may check the accuracy of our construction by noting whether or not the vector $\overline{cf}$ is parallel to the axis of the bar EB as it should be.

Figure 123*e* represents the completed Maxwell diagram. It will be noted that, while no consideration of the equilibrium of the hinge E has been made, the closed triangle of forces *cef* for this hinge, nevertheless, is present in the diagram. The force in any bar of the truss can now be found by scaling the length of the corresponding line in the completed diagram.

Bow's notation is particularly advantageous when we come to decide the sign of these forces, i.e., whether they represent tension or compression. Let us consider, for example, the force in bar DE, which is represented in magnitude by the line *ce* in the Maxwell diagram. Going around joint D in a clockwise direction, the reaction which the bar DE exerts on this hinge will be read as *ec*. Now in the force diagram the vector $\overline{ec}$, from e to c, is directed from right to left, indicating pressure against hinge D, and hence we conclude that the bar is in compression. If, instead, we consider the reaction of this same bar on the hinge E, then reading clockwise around this hinge we have *ce* instead of *ec* and in the force diagram the vector $\overline{ce}$, from c to e, shows that the bar presses on the hinge E, thus indicating compression as before.

EXAMPLES

1. Construct a Maxwell diagram for the plane truss supported and loaded as shown in Fig. 124*a*. The truss is symmetrical with respect to a vertical line through its apex, has interior angles at A and B equal to 30°, and equal panel distances along the lower chord AB. The loads are in kips.

Solution. The first step in the analysis of this truss will be the determination of the vertical reactions at the supports A and B. This may be done either graphically or analytically, and we find $R_a = 15.5$ kips and $R_b = 12.5$ kips, as shown. Having the reactions, we now letter the truss in accordance with Bow's notation and begin in Fig. 124*b* by constructing the closed polygon *abcdefga* for the entire truss as a free body. It will be noted that the vectors in this polygon are added in the order in which the corresponding forces are encountered in going around the entire truss in a clockwise direction. Going around each hinge of the truss in the same clockwise direction, the polygons of forces for these hinges may now be constructed as already explained, and we obtain finally the composite diagram, as shown in Fig. 124*b*. This diagram is drawn to the scale 1 in. = 20 kips, and from it the axial force in any bar of the truss may be found. The student will find it a worthwhile exercise to follow through the diagram and ascertain for each bar whether it is in tension or compression.

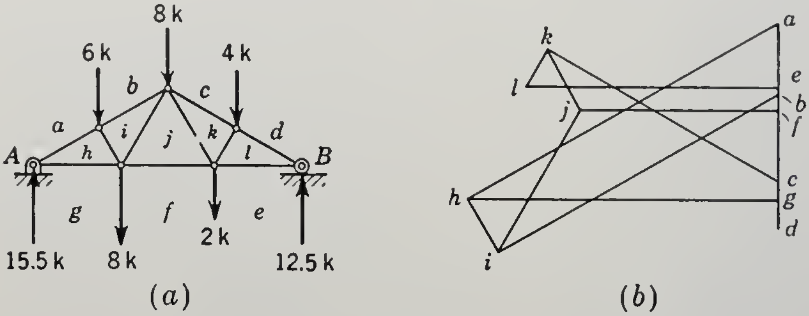
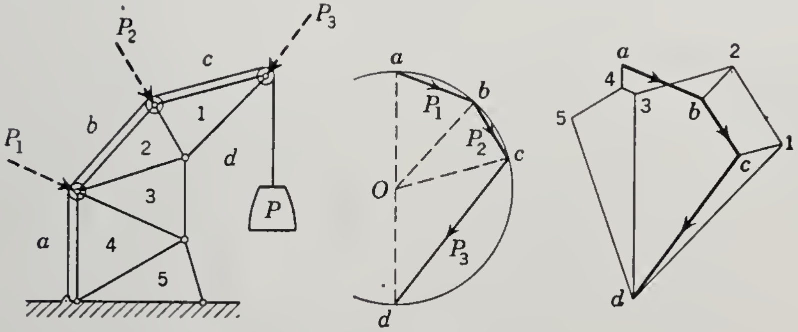


FIG. 124

2. A crane supports a load P which hangs on a flexible cable overrunning small pulleys attached to the top chord joints, as shown in Fig. 125*a*. Find graphically the axial forces produced in the various bars.

Solution. We assume that the diagram of the truss in Fig. 125*a* has been drawn to scale. Then to determine the internal forces in the bars, we need to know first the magnitudes and directions of the external forces P_1, P_2, P_3 exerted on the joints by the axles of the pulleys. These, we find by observing that the cable, under uniform tension P, represents a funicular polygon for the desired forces, as already discussed in Example 1 of Art. 3.6 (see page 142). Accordingly, in Fig. 125*b*, we construct a circle the radius of which represents, to some convenient scale, the magnitude of the load P.

Then drawing rays Oa, Ob, Oc, and Od, parallel to the several portions of the cable, we find the forces P_1, P_2, P_3 exerted on the joints of the truss, as shown. This done, we place these forces on the truss (thereafter ignoring the cable) and letter (or number) the spaces between lines of action of forces in accordance with Bow's notation as shown.



(a) (b) (c)

FIG. 125

Beginning at the top joint and working down in order, we now proceed with the construction of a Maxwell diagram as shown in Fig. 125c. Normally this diagram could be superimposed directly on the construction already made in Fig. 125b, but here it is shown separately for the sake of better clarity. The magnitudes of the axial forces in the bars can now be scaled from the diagram (Fig. 125c).

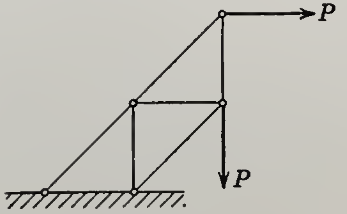


FIG. A

PROBLEM SET 3.7

1. Construct a Maxwell diagram for the truss supported and loaded as shown in Fig. A and determine therefrom the axial force in each bar. Assume $P = 1$ kip.

2. Construct a Maxwell diagram for the truss supported and loaded as shown in Fig. B. Assume $P = 1$ kip, $a = 5$ ft.

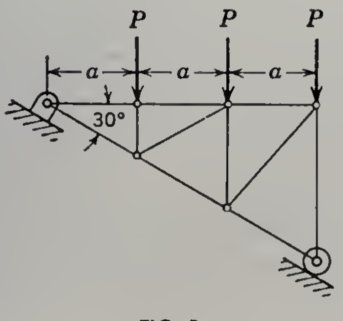


FIG. B

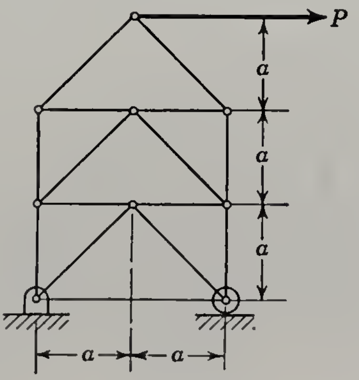


FIG. C

3. Construct a Maxwell diagram for the truss supported and loaded as shown in Fig. C. Assume $P = 1$ kip, $a = 4$ ft.

4. Construct a Maxwell diagram for the truss supported and loaded as shown in Fig. D. Assume $P = 1$ kip, $a = 10$ ft.

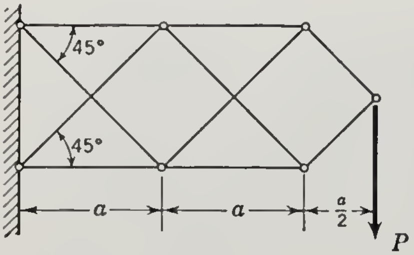


FIG. D

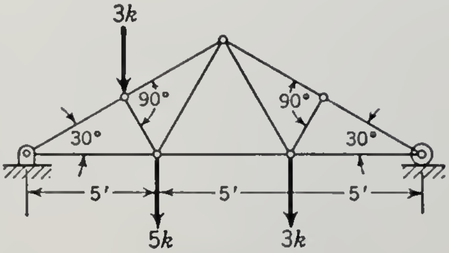


FIG. E

5. Construct a Maxwell diagram for the roof truss supported and loaded as shown in Fig. E.

6. Construct a Maxwell diagram for the truss supported and loaded as shown in Fig. F. All inclined bars are at 45° to AB.

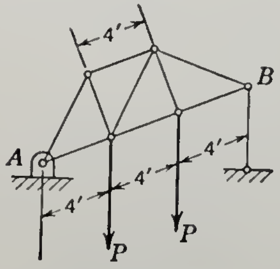


FIG. F

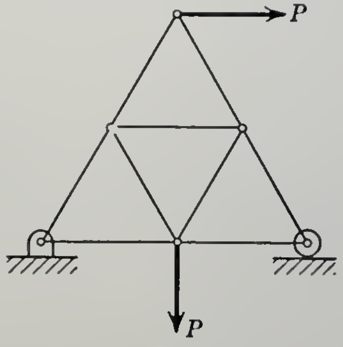


FIG. G

7. Construct a Maxwell diagram for the truss supported and loaded as shown in Fig. G. All bars are of the same length and $P = 1$ kip.

8. Construct a Maxwell diagram for the truss supported and loaded as shown in Fig. H. Make the analysis entirely by graphical methods.

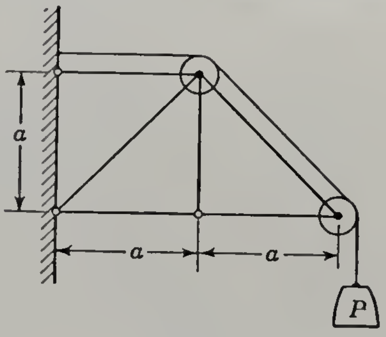


FIG. H

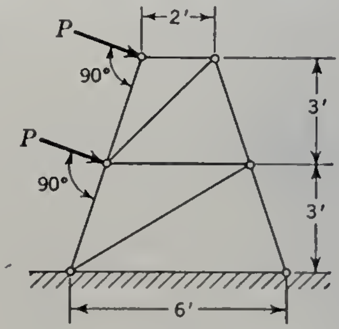


FIG. I

9. Construct a Maxwell diagram for the tower loaded as shown in Fig. I. Assume $P = 1$ kip.

10. Construct a Maxwell diagram for the tower in Fig. I, if the loads P are horizontal.

3.8. Distributed force in a plane. As pointed out in Art. 2.5, there are many cases of distributed force that may be assumed to be confined to one plane. Sometimes the direction of such distributed force varies from point to point in its plane of action and cannot be treated in the same manner as before. The pressure distributed around the rim of a wheel due to a shrunk-fit steel band is one example of such force. Another case is that of hydrostatic pressure on various surfaces such, for example, as water pressure against the curved upstream face of an arch dam, water or steam pressure in pipes, etc.

A general situation is represented in Fig. 126, where some kind of load is exerted against a plane curve AB and varies in magnitude and direction from point to point along this curve Such distributed force will be completely defined by its *line of application* AB together with the *direction* and *intensity* of force at each point on that line. Usually the direction of force is normal to the line of application at each point, in which case only the *intensity*, i.e., the force per unit length of AB needs to be stated. Thus in Fig. 126, the magnitude of the element of force dQ acting on any element of length ds of the line of application is

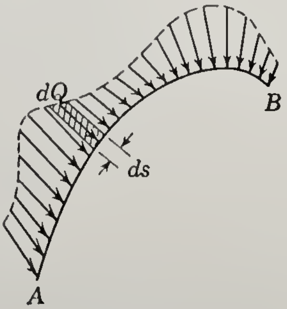


FIG. 126

$$dQ = q\,ds \tag{a}$$

where q is the *intensity* of the distributed force. Since these elements of force, considered as concentrated forces, constitute a coplanar system, the equations developed in Art. 3.1 for defining their resultant must apply. In the case of distributed force, however, it will be necessary to make the summations of projections and moments by the aid of the calculus since we are dealing with infinitesimal quantities.

Various methods of dealing with the equilibrium of bodies under the action of distributed force in a plane will now be considered.

Hoop Tension. If a circular ring or hoop of inside radius a is submitted to the action of internal hydrostatic pressure of uniform intensity q as shown in Fig. 127a, circumferential tension (so-called *hoop tension*) will be produced in the ring, the thickness of which is assumed small compared with its radius.

To find this tension, we begin with a consideration of the equilibrium of one-half of the ring isolated as shown in Fig. 127*b* and wherein either of the circumferential forces S at A or B represents the required hoop tension. It must be self-evident that this tension will be the same at every point around the circumference. To replace the distributed hydrostatic pressure by a single resultant force Q, we consider two elements of the semicircle AB symmetrically situated with respect to the vertical axis Oy and each of length $ds = a\,d\theta$ as shown in the figure. It follows from Eq. (a) that the magnitude of the element of force acting on each of these elements is $qa\,d\theta$. Replacing these elements of force by their horizontal and vertical components, we see that the two horizontal components balance each other and only the two vertical components need be considered. The resultant of these two vertical components is evidently a vertical force of magnitude

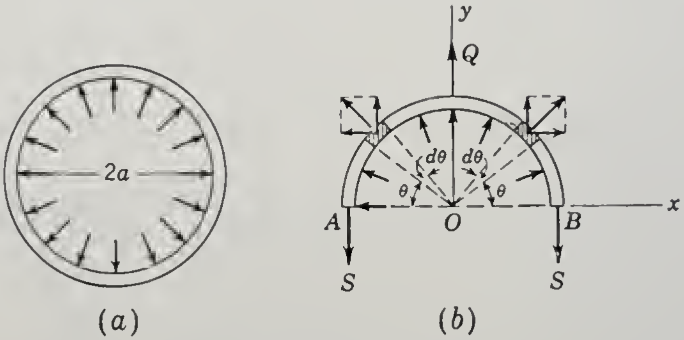


FIG. 127

$$2qa \sin\theta\, d\theta \qquad \text{(b)}$$

acting through point O. A similar argument holds for any other pair of elements of the hydrostatic pressure. Hence the total resultant of the distributed pressure is obtained by summing up the elemental resultants, as represented by expression (b), for all pairs of elements of the arc between A and B. This summation may be expressed as follows:

$$Q = 2qa \int_0^{\pi/2} \sin\theta\, d\theta = 2qa \Big[-\cos\theta \Big]_0^{\pi/2} = 2qa \qquad \text{(c)}$$

Then from symmetry we conclude that the hoop tension is

$$S = \tfrac{1}{2}Q = qa \qquad \text{(d)}$$

We see from this result that the action of uniform pressure around the circumference of the semicircle is equivalent to the same pressure uniformly distributed across the diameter AB. In general it can be proved that uniform normal pressure along any line of application AB (Fig. 128) is equivalent to the same uniform pressure on the chord AB. To show this, we consider one element ds of the curve AB as shown in Fig. 128*b*. Resolving the corresponding elemental force $q\,ds$ into horizontal and vertical components, we obtain, respectively,

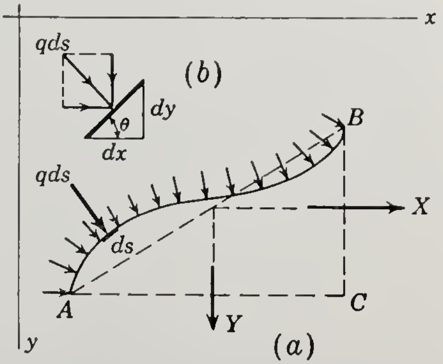


FIG. 128

$$q\,ds\sin\theta = q\,ds\frac{dy}{ds} = q\,dy$$
$$q\,ds\cos\theta = q\,ds\frac{dx}{ds} = q\,dx$$

Then, by summation, the resultant force has the components

$$X = \int_A^B q\,dy = q\int_A^B dy = q\cdot BC$$
$$Y = \int_A^B q\,dx = q\int_A^B dx = q\cdot AC$$

and by Eq. (2), page 25, we obtain

$$Q = \sqrt{X^2 + Y^2} = q\sqrt{\overline{BC}^2 + \overline{AC}^2} = q\cdot AB$$

This shows that the uniform normal distribution of force along the curve AB is equivalent to the same uniform distribution along the chord AB.

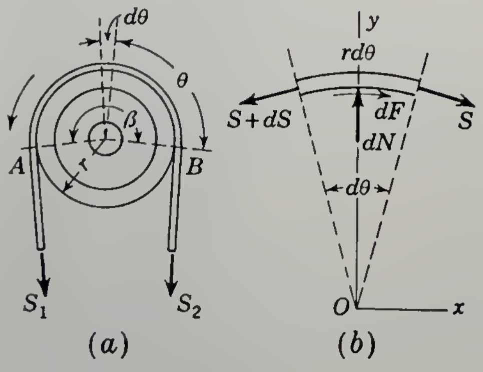


FIG. 129

Belt Friction. We encounter another type of distributed force in the case of the pressure between a belt and the pulley or drum on which it runs. In Fig. 129*a*, for example, a pulley of radius r is driven in the direction shown by virtue of the friction developed between its rim and the encircling belt, which also runs on another pulley (not shown). It is evident that the tension S_1 in the belt on the side leaving the pulley is greater than the tension S_2 on the side approaching, the difference being gradually taken up in friction between the belt and the rim of the pulley. It is desired to find the ratio S_1/S_2 between the maximum and minimum tensions in the belt on the two sides of the pulley when slipping impends.

To do this, we consider the conditions of equilibrium of any element of the belt of length $ds = r\,d\theta$ located at the angle θ from the point of tangency B (Fig. 129*a*). A free-body diagram of this element showing all forces acting upon it is given in Fig. 129*b*. Since these forces represent a system of forces in equilibrium in one plane (the middle plane of the pulley), the algebraic sum of their projections on any axis must be zero. Projecting all forces on the x axis and remembering that $d\theta$ is a very small angle, we obtain

$$S + dF - (S + dS) = 0$$

from which

$$dF = dS \tag{e}$$

That is, the increment of friction developed over the length $r\,d\theta$ of the element is equal to the increment of tension in the belt over the same length.

Projecting all forces onto the y axis and again remembering that $d\theta$ is a very small angle so that $\sin d\theta \approx d\theta$, we obtain

$$dN - S\frac{d\theta}{2} - (S + dS)\frac{d\theta}{2} = 0$$

from which, neglecting small quantities of the second order, we obtain

$$dN = S\,d\theta \tag{f}$$

Equation (f) gives us the element of normal pressure at any point on the belt in terms of the tension S in the belt at that point.

When slipping impends, we have

$$dF = \mu\,dN$$

which, by using Eqs. (e) and (f), becomes,

$$dS = \mu S\,d\theta$$

or

$$\frac{dS}{S} = \mu\,d\theta \tag{g}$$

This expresses the ratio between the increment of tension over the length of the element to the total tension in the belt at the point defined by the angle θ. Integrating Eq. (g) over the entire line of contact AB, of length $r\beta$, we obtain

$$\ln\frac{S_1}{S_2} = \mu\beta \qquad \text{or} \qquad \frac{S_1}{S_2} = e^{\mu\beta} \tag{h}$$

We see that the ratio between the tensions S_1 and S_2 in the belt on the two sides of the pulley increases very rapidly with the magnitude of the central angle β of the line of contact AB. This explains how a man may hold a great load on the end of a rope simply by taking a turn or two of rope around a post. It will also be noted that the ratio S_1/S_2 is independent of the radius r of the pulley.

Center of Pressure. Another problem involving distributed force is illustrated in Fig. 130. The opening to a penstock through the base of a gravity dam is closed by a circular plate AB and it is desired to define the resultant water pressure P on the plate. This is really a problem involving force distributed over an area, but we can easily bring it to an equivalent distribution in the vertical plane of the diameter AB of the plate.

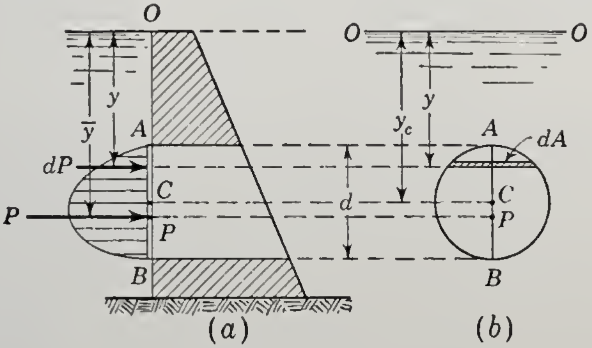


FIG. 130

Looking directly at the plane of the plate (Fig. 130*b*), we consider one element of area dA in the form of a horizontal strip at depth y as shown. Then since the intensity of water pressure $p = wy$ is constant across the length of this strip, the corresponding element of force $dP = wy\,dA$ acts at the mid-point of the strip, i.e., on the vertical diameter AB. This conclusion holds for each such horizontal strip, and we obtain a series of elemental forces distributed along AB and all lying in the same vertical plane, Fig. 130*a*. The point P through which the resultant pressure acts on the plate is called the *center of pressure* and we need only to find the depth $\bar{y}$ to completely define its position on the vertical diameter AB.

We considered this kind of problem previously in Art. 2.5 and came to the conclusion that the force P was equal to the area of the load diagram (Fig. 130*a*) and acted through its centroid. This is true in this case also, but owing to the fact that the magnitudes of dP do not vary linearly with y because of variation of dA with y also, the load diagram is not easily defined and the calculation of its centroid presents difficulties.

To calculate the resultant pressure without reference to the load diagram, we simply take the sum of all parallel elemental forces dP

and obtain

$$P = \int_A^B wy\,dA = w\int_A^B y\,dA \tag{i}$$

We recognize the integral on the extreme right as the *statical moment* of the area of the plate about the axis OO in Fig. 130*b*. This can be written in the equivalent form

$$\int_A^B y\,dA = Ay_c \tag{j}$$

as discussed in Art. 2.3, and expression (i) becomes

$$P = wAy_c = p_cA \tag{k}$$

where $p_c = wy_c$ is the intensity of water pressure at the centroid C of the plate. Using this formula, the resultant pressure on the plate can be calculated in a very simple manner. It is only necessary to multiply the area of the plate by the intensity of pressure at its centroid. Equation (k) is quite general and can be used to calculate the resultant pressure on any plane area subjected to hydrostatic pressure.

Although Eq. (k) defines the magnitude of the resultant pressure in a very simple way, we still do not know its point of application, i.e., the position of the center of pressure. To find the depth $\bar{y}$ to this point, we use the theorem of moments. Taking point O (Fig. 130*a*) as a moment center and equating the moment of the resultant P to the sum of moments of dP gives

$$P\bar{y} = \int_A^B y\,dP \tag{l}$$

Noting further that $dP = wy\,dA$ and using expression (i) for P, Eq. (l) takes the form

$$\bar{y}\int_A^B wy\,dA = \int_A^B wy^2\,dA$$

from which

$$\bar{y} = \frac{\int_A^B y^2\,dA}{\int_A^B y\,dA} \tag{m}$$

This formula defines the position of the center of pressure entirely in terms of properties of the plate shape and its position with reference to the water surface. Although derived on the assumption of a circular plate, Eq. (m) will be valid for any shape of plate that has a vertical axis of symmetry AB. The evaluation of the integral in the denominator of expression (m) has been discussed. The integral in the numerator of expression (m) is called the *moment of inertia* of the area of the plate with respect to the axis OO from which the distances y are measured (Fig. 130*b*). Moments of inertia of plane areas are fully discussed in Appendix I, page A.1, and we find for this case that

$$I_0 = \int_A^B y^2\,dA = A\left(y_c^2 + \frac{d^2}{16}\right)$$

Thus Eq. (m) becomes

$$\bar{y} = \frac{y_c^2 + d^2/16}{y_c} = y_c + \frac{d^2}{16y_c} \tag{n}$$

and we see that the center of pressure is below the center of gravity by the distance $d^2/16y_c$, where d is the diameter of the plate.

PROBLEM SET 3.8

1. A horizontal cylindrical tank having a semicircular cross section of radius $r = 2$ ft is filled with water (specific weight $w = 62.4$ lb/ft^3) as shown in Fig. A. Find the magnitude and direction of the resultant force R exerted on the left-hand wall AC. Consider a unit length of the tank normal to the plane of the figure. *Ans.* $R = 232$ lb; $\alpha = 57°31'$.

2. Determine the resultant pressure force P exerted on one of the semicircular ends of the tank in Fig. A and the depth $\bar{y}$ to its point of application. *Hint.* See Appendix I, page A.5 for moment of inertia of a semicircle. *Ans.* $P = 332$ lb; $\bar{y} = 1.18$ ft.

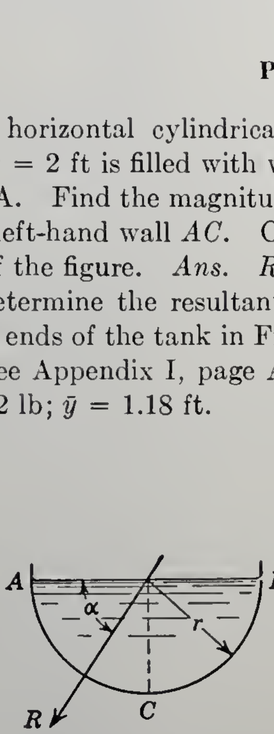


FIG. A

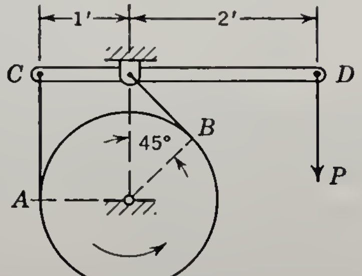


FIG. B

3. A rotating flywheel of radius $r = 1$ ft is braked by the device shown in Fig. B. Calculate the braking moment M produced by a vertical force P applied to the lever at D if the coefficient of friction between the belt AB and the rim of the wheel is $\mu = 0.5$. *Ans.* $M = 1.72P$ ft-lb.

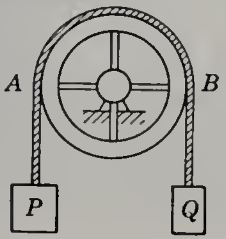


FIG. C

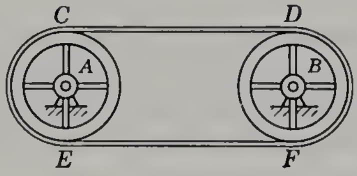


FIG. D

4. Referring to Fig. C, what is the minimum magnitude of the weight Q that can hold the weight $P = 1$ ton in equilibrium if the coefficient of friction between the cord and the pulley is $\mu = 0.4$? Assume that the pulley is locked and cannot turn. *Ans.* $Q = 569$ lb.

5. A belt is stretched over the two pulleys A and B of equal diameters $d = 12$ in. so that, when the pulleys are not rotating, the initial tensions in the portions CD and EF of the belt are equal to 1,000 lb each (Fig. D). Find the maximum possible turning moment that can be transmitted from one pulley to the other if the coefficient of friction between the belt and pulleys is $\mu = 0.5$. Assume that, when moment is being transmitted from one pulley to the other, the tension in the belt is increased on one side by the same amount that it is decreased on the other. *Ans.* $M_{\max} = 7{,}872$ in.-lb.

3.9. Flexible suspension cables. In engineering structures, we sometimes encounter flexible cables or chains suspended between two supports at their ends and subjected to the action of vertical load continuously distributed along their lengths. The distribution of load may, in general, be uniform or otherwise. Two of the most commonly encountered cases of loading are illustrated in Fig. 131. Figure 131*a* represents the case of a flexible chain freely suspended in the gravity field and subjected to the action of its own distributed weight only. Such loading is, of course, uniformly distributed with respect to the curve of the chain itself. Figure 131*b* represents the case of a thin wire cable or cord subjected to the action of a uniformly distributed load attached to it by vertical *hangers*. In the event that this loading is large compared with the weight of the cable itself, we may assume in this case that the load is uniformly distributed with respect to the horizontal span.

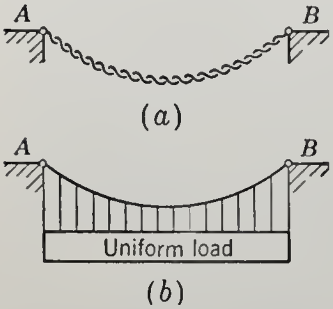


FIG. 131

In discussing such cables as those shown in Fig. 131, two questions at once present themselves: (1) What is the shape of the curve that the chain or cord will assume in the condition of equilibrium, and (2) how does the tension vary from point to point along the curve? To answer these questions, we consider now the general case of a cable $ACDB$ supported at its ends A and B and acted upon by vertical load distributed along the horizontal in any manner as represented by the load diagram $A'abB'$ in Fig.132*a*. With the lowest point C of the curve as an origin, let coordinate axes x and y be selected as shown. Let D be any point on the curve with coordinates x, y. We now consider the portion CD of the cable or chain as a free body (Fig. 132*b*). We may imagine, without altering the conditions of equilibrium, that this portion has been frozen in its configuration of equilibrium before being cut out. In this way, we satisfy the requirement of a rigid body. This free body is in equilibrium under the action of three forces: a vertical force Q representing that portion of the distributed load between C and D and two tensile forces H and S representing the reactions exerted by the unconsidered portions of the chain on either side. The forces H and S are tangent to the curve at C and D, respectively, and the vertical force Q acts through the centroid of that portion of the load diagram between these same two points. Since the three forces must build a closed triangle as shown in Fig. 132c, we may now write

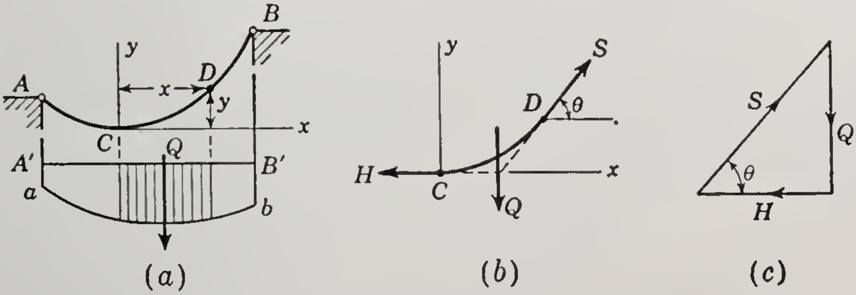


FIG. 132

$$\tan \theta = \frac{Q}{H}$$

which, since $\tan \theta = dy/dx$, becomes

$$\frac{dy}{dx} = \frac{Q}{H} \tag{a}$$

We also have from the triangle of forces in Fig. 132c,

$$S = \sqrt{H^2 + Q^2} \tag{b}$$

Equation (a) is the *differential equation* of the curve of equilibrium assumed by the cable under the action of the load that it carries, while Eq. (b) defines the tension at any point on the curve. The solution of these equations for the two particular cases of loading illustrated in Fig. 131 will now be discussed separately.

Parabolic Cable. Assuming that the cable in Fig. 133 carries vertical load of intensity q uniformly distributed with respect to the horizontal span l, Eq. (a) becomes

$$\frac{dy}{dx} = \frac{qx}{H} \tag{c}$$

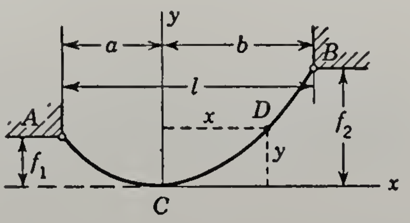


FIG. 133

wherein q/H is a constant. Integrating once, we obtain

$$y = \frac{qx^2}{2H} + C$$

wherein the integration constant C is equal to zero, since for the coordinate axes as chosen in the figure we have $y = 0$ when $x = 0$. Hence

$$y = \frac{qx^2}{2H} \tag{d}$$

Equation (d) is the equation of the curve of equilibrium assumed by the cable, and we see that for a uniform distribution of load along the horizontal span it is a portion of a *parabola* with a vertical axis.

From Eq. (b) we have

$$S = \sqrt{H^2 + (qx)^2} \tag{e}$$

from which we note that the tension in the cable is a minimum at the lowest point C, where it is equal to H, and that it increases toward the ends of the cable, being a maximum at the higher support.

For the forces at the ends A and B of the cable, respectively, we have from Eq. (e)

$$S_a = \sqrt{H^2 + q^2a^2} \qquad S_b = \sqrt{H^2 + q^2b^2} \tag{f}$$

To determine the distances a and b locating the low point C with reference to the supports A and B, we use Eq. (d) successively for the portions AC and BC of the cable and note that, when $x = -a$, $y = f_1$ while, when $x = b$, $y = f_2$. Thus we obtain

$$f_1 = \frac{qa^2}{2H} \qquad f_2 = \frac{qb^2}{2H} \tag{g}$$

Subtracting the first of these equations from the second and using the notation $f_2 - f_1 = h$, we obtain

$$2hH = q(b^2 - a^2)$$

Also we have

$$a + b = l$$

From these last two equations

$$a = \frac{l}{2} - \frac{hH}{ql} \qquad b = \frac{l}{2} + \frac{hH}{ql} \tag{h}$$

It will be noted that, when $h = 0$, that is, when the supports A and B are on the same level, we obtain $a = b = l/2$.

Substituting the above expression for b into the second of Eqs. (g), the following equation for calculating H is obtained:

$$H^2 - \frac{2ql^2}{h^2}\left(f_2 - \frac{h}{2}\right) H + \frac{q^2l^4}{4h^2} = 0$$

from which

$$H = \frac{ql^2}{h^2}\left(f_2 - \frac{h}{2} \pm \sqrt{f_1f_2}\right) \tag{i}$$

In Eq. (i) the minus sign should be used for all cases where the vertex of the parabola corresponding to the configuration of equilibrium of the cable lies between the supports as shown in the figure, while the plus sign should be used for all cases where the vertex of this parabola lies to the same side of both supports. In the particular case where the supports are on the same level, $f_1 = f_2 = f$, $a = b = l/2$, and proceeding as before, we obtain

$$H = \frac{ql^2}{8f} \tag{j}$$

In most practical problems the span l, the sags f_1 and f_2, and the intensity q of the uniformly distributed load will be given so that from Eq. (i) or (j) the minimum tension H in the cable may be found at once, after which any of the quantities previously defined in terms of H can be calculated without difficulty.

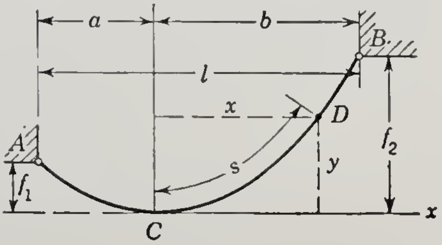


FIG. 134

Catenary Cable. Assuming that the cable in Fig. 134 hangs freely in the gravity field and is subjected only to its own weight uniformly distributed along the curve, Eq. (a) becomes

$$\frac{dy}{dx} = \frac{qs}{H} \tag{k}$$

where q is the weight per unit length of the cable and s is the length of the arc CD. Before this equation can be integrated, it will be necessary to express the length s as a function of the coordinates x and y. To do this, we use the relationship.

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

which, with the above value of dy/dx, becomes

$$ds = \sqrt{1 + \left(\frac{qs}{H}\right)^2}\, dx$$

Integration of this last equation gives

$$\frac{H}{q} \operatorname{arcsinh} \frac{qs}{H} = x + C_1$$

where C_1 is a constant. Since, for the coordinate axes as shown in Fig. 134, we have $s = 0$ when $x = 0$, it is evident that $C_1 = 0$ and the above equation becomes

$$s = \frac{H}{q} \sinh \frac{qx}{H} \tag{l}$$

Substituting the value of s from Eq. (l) into Eq. (k), we obtain

$$dy = \sinh \frac{qx}{H}\, dx \tag{m}$$

which can now be integrated, and we find

$$y = \frac{H}{q} \cosh \frac{qx}{H} + C_2$$

In this case, for the coordinate axes as shown in Fig. 134, we have $y = 0$ when $x = 0$. Hence $C_2 = -H/q$, for which value the above expression becomes

$$y = \frac{H}{q}\left(\cosh\frac{qx}{H} - 1\right) \tag{n}$$

Equation (n) is the equation of the curve of equilibrium assumed by the cable when hanging freely under its own weight and represents a *catenary* with vertical axis.

Remembering that $Q = qs$ and using Eq. (l), we obtain

$$Q = H \sinh\frac{qx}{H}$$

which, when substituted into Eq. (b), gives

$$S = \sqrt{H^2 + Q^2} = H \cosh\frac{qx}{H}$$

Substituting for cosh (qx/H), in the above expression, its value from Eq. (n), we obtain

$$S = H + qy \tag{o}$$

From Eq. (o) we note that, as in the case of the parabolic cable, the tension S is a minimum at the low point C where it is equal to H and that it increases toward the ends of the cable, being a maximum at the higher support. For the tensions at the ends A and B, respectively, we have

$$S_a = H + qf_1 \qquad S_b = H + qf_2 \tag{p}$$

To determine the value of H, we use Eq. (n) successively for the portions AC and BC of the cable, obtaining

$$f_1 = \frac{H}{q}\left(\cosh\frac{qa}{H} - 1\right) \qquad f_2 = \frac{H}{q}\left(\cosh\frac{qb}{H} - 1\right) \tag{q}$$

Equations (q) may be written

$$a = \frac{H}{q}\operatorname{arccosh}\left(\frac{qf_1}{H} + 1\right) \qquad b = \frac{H}{q}\operatorname{arccosh}\left(\frac{qf_2}{H} + 1\right) \tag{r}$$

Adding these last two equations and remembering that $a + b = l$, we obtain

$$\frac{ql}{H} = \operatorname{arccosh}\left(\frac{qf_1}{H} + 1\right) + \operatorname{arccosh}\left(\frac{qf_2}{H} + 1\right) \tag{s}$$

In most practical problems the span l, the sags f_1 and f_2, and the weight q per unit length of cable will be given so that from Eq. (s) the minimum tension

H in the cable can be found by using tables of hyperbolic functions, after which any of the quantities, previously defined in terms of H, may be calculated without difficulty.

PROBLEM SET 3.9

1. A flexible wire cable weighing 2 lb per foot of length overhangs two small pulleys A and B and carries at its ends two equal weights $P = 1$ ton as shown in Fig. A. The distance between the pulleys is $l = 200$ ft, and they are at the same elevation. Treating the weight of the cable as though uniformly distributed with respect to the horizontal span, calculate approximately the sag f at the middle of the span. *Ans.* $f \approx 5$ ft.

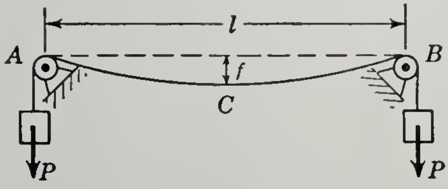


FIG. A

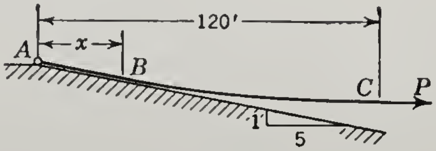


FIG. B

2. A flexible cable AC weighing 2 lb per foot of length is fastened at A and rests on a perfectly smooth plane AB which has a slope of 1 vertical to 5 horizontal (Fig. B). If a horizontal force $P = 900$ lb is applied to the free end C and the horizontal distance from A to C is 120 ft, find the distance x to the point B where the cable breaks contact with the inclined plane. Find also the maximum tensile force in the cable. *Ans.* $x = 30$ ft; $S_a = 930$ lb.

3. A flexible cable AB (Fig. C) is stretched between points A and B until, at the lowest point C, the sag $f = 1$ ft. Determine the tensile force H at point C if the span $l = 60$ ft, $h = 2$ ft, and the weight of the cable per unit length is $q = 5$ lb/ft. Assume that the weight of the cable may be considered as uniformly distributed along its horizontal projection l. *Ans.* $H =$ 1,210 lb.

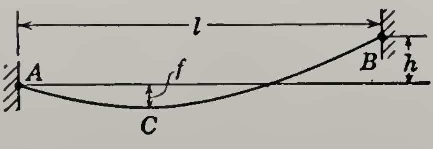


FIG. C

4. The cable shown in Fig. C carries a load $Q = 100{,}000$ lb uniformly distributed with respect to the horizontal span l. Determine the maximum tension in the cable if $l = 100$ ft, $f = 12$ ft, $h = 10$ ft. *Ans.* $S_{\max} = 94{,}700$ lb.

5. For a certain value of P the sag-span ratio for the cable shown in Fig. A is $f/l = \frac{1}{12}$. Find what the sag-span ratio will be if the load P is doubled. *Ans.* $f/l = \frac{1}{25}$.

6. The cable AB (Fig. D) carries a load of 100 lb per foot of horizontal span as shown. Find the maximum tensile force in the cable if the tension at A is just twice as great as that at B. *Ans.* $S_a = 2{,}610$ lb.

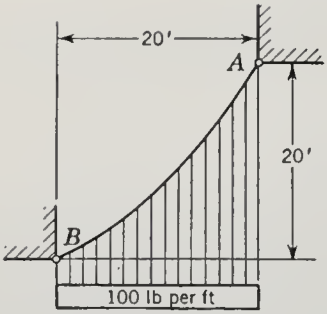


FIG. D

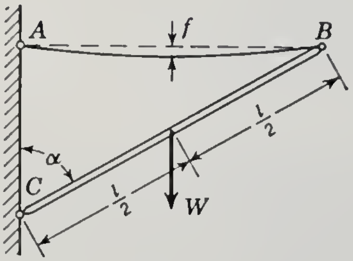


FIG. E

7. A prismatic bar BC of length l and weight W is hinged to a wall at C and supported at B by a flexible cable AB of weight Q, as shown in Fig. E. The points A and B are on the same level. Find the sag f at the middle of the span if $l = 12$ ft, $\alpha = 60°$, $W = 10$ lb, and $Q = 2$ lb. Assume in calculation that the weight Q of the cable is uniformly distributed along the horizontal span AB. *Ans.* $f = 3$ in.

*8. A flexible cable 100 ft long and weighing 5 lb per foot of length is freely suspended at its ends from two supports 50 ft apart and having the same elevation. Find the sag f at the middle of the span. *Ans.* $f = 39.8$ ft.

*9. A flexible cable of uniform weight per unit length rests partly on a horizontal plane and passes over a small pulley at A, as shown in Fig. F. By gradually increasing the force S applied to the end of the cable, the length of contact BC with the plane diminishes to a certain limiting value c at which sliding of the cable along the plane impends. Find this limiting value c if $a = 200$ ft, $f = 20$ ft, and the coefficient of friction between the cable and the plane is $\mu = 0.5$. *Ans.* $c = 147$ ft.

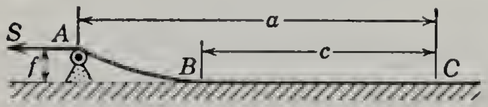


FIG. F

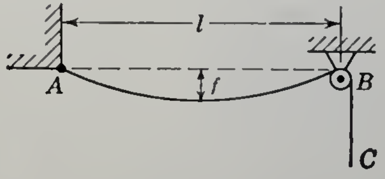


FIG. G

*10. Determine the minimum length L of a flexible cable AC of uniform weight per unit length which can hang in equilibrium as shown in Fig. G. Neglect friction and the dimensions of the pulley B. What is the sag-span ratio? *Ans.* $L_{\min} = 1.14l + 0.80l = 1.94l; f/l = 0.224$.

4

FORCE SYSTEMS IN SPACE

4.1. Concurrent forces in space: method of projections. *Composition.* If several forces in space have the same point of application, their resultant can be found by successive applications of the parallelogram law, as was done in the case of concurrent forces acting in a plane. Let us consider, for example, the case of three forces F_1, F_2, F_3 applied to point O of a body and represented by the vectors $\overline{OA}$, $\overline{OB}$, and $\overline{OD}$, respectively, as shown in Fig. 135*a*. Applying the parallelogram law to the forces F_1 and F_2, we obtain their resultant $\overline{OC}$ acting in the plane AOB as shown. Replacing F_1 and F_2 by their resultant $\overline{OC}$ and again applying the parallelogram law to the partial resultant $\overline{OC}$ and the force F_3, we obtain the resultant R of all three forces, as represented by the diagonal OE of the parallelogram $ODEC$. We see that this resultant of three concurrent forces in space is given by the diagonal of the parallelepiped constructed on the vectors $\overline{OA}$, $\overline{OB}$, and $\overline{OD}$, as indicated in Fig. 135*a*.

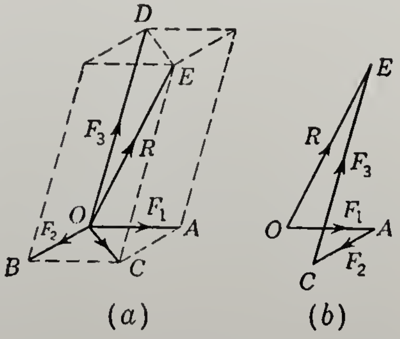
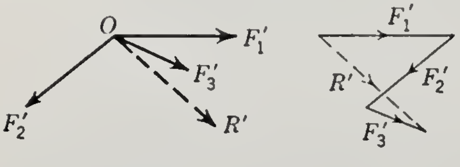


FIG. 135

The same resultant can be obtained also by constructing (in space) the *polygon of forces*, as shown in Fig. 135*b*. We begin with the free vector $\overline{OA}$ representing the force F_1 and from its end construct the vector $\overline{AC}$ representing the force F_2. From C we construct the vector $\overline{CE}$ representing the force F_3. Now by comparison with Fig. 135*a*, it is evident that the free vector $\overline{OE}$ (Fig. 135*b*) representing the geometric sum of the given forces F_1, F_2, and F_3 gives the magnitude and direction of the resultant R of the given forces.

The constructions represented in Fig. 135 can obviously be extended for any number of forces applied at one point. Hence we conclude that the resultant of any number of concurrent forces in space can always be obtained as the geometric sum of the given forces and that its line of action passes through the point of concurrency of the given forces.

If the parallelepiped of forces in Fig. 135a is projected onto any plane, say the horizontal plane, we shall obtain a system of concurrent forces in one plane, as shown in Fig. 136a. Now projecting the space polygon in Fig. 135b onto the same plane, we obtain a plane polygon, as shown in Fig. 136b. The two sets of vectors in Fig. 136, being similar projections of equal parallel lines in space, are mutually equal and parallel. Thus we conclude that the resultant R' of F_1', F_2', and F_3', representing any plane projection of a system of concurrent forces F_1, F_2, F_3, in space, gives us the corresponding projection of the true resultant R. Briefly, *the resultant of the projections is equal to the corresponding projection of the resultant.* This idea is very useful in the solution of problems, since it enables us to work in one plane and to use any and all methods already developed for coplanar systems.



(a) (b)

FIG. 136

The method of projections discussed in Art. 1.4 for the case of concurrent forces in a plane is particularly useful in dealing with concurrent forces in space. From the above discussion, it follows that the projection on any axis of the resultant of several concurrent forces in space is equal to the algebraic sum of the projections of the components on the same axis. Denoting, as before, by X_i, Y_i, Z_i the projections of any force F_i on rectangular coordinate axes x, y, z (Fig. 137) and by X, Y, Z the corresponding projections of the resultant R, we have, from the above statement,

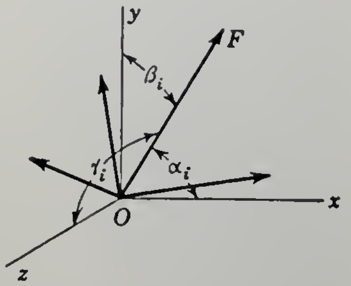


FIG. 137

$$X = \Sigma X_i \qquad Y = \Sigma Y_i \qquad Z = \Sigma Z_i \tag{a}$$

For calculating the projections of a force F_i, we have

$$X_i = F_i \cos \alpha_i \qquad Y_i = F_i \cos \beta_i \qquad Z_i = F_i \cos \gamma_i \tag{b}$$

where α_i, β_i, γ_i are the angles that the force makes with the positive directions of the coordinate axes x, y, z, respectively, as shown. The magnitude of the resultant is obtained from the equation

$$R = \sqrt{X^2 + Y^2 + Z^2} \tag{c}$$

Denoting by α, β, γ, without subscripts, the angles that the resultant makes with the positive directions of the coordinate axes x, y, z, respectively, we obtain these angles from the equations

$$\cos \alpha = \frac{X}{R} \qquad \cos \beta = \frac{Y}{R} \qquad \cos \gamma = \frac{Z}{R} \tag{d}$$

Equations (a) to (d), inclusive, completely define the resultant of any system of concurrent forces in space.

Equations of Equilibrium. If a system of concurrent forces in space is in equilibrium, their resultant must vanish. Referring to Eqs. (c) and (a) we see that this condition requires

$$\Sigma X_i = 0 \qquad \Sigma Y_i = 0 \qquad \Sigma Z_i = 0 \tag{20}$$

where the summations are understood to include all forces in the system. These are the *equations of equilibrium* for concurrent forces in space.

Applying Eqs. (20) to any system of concurrent forces that are in equilibrium, we see that not more than three unknown quantities can be calculated. For instance, if all forces of the system except three are completely known and the directions of these three are known, then their magnitudes can be found by using Eqs. (20). Again, if all forces of the system except one are known, then the three projections of the unknown force can be determined from Eqs. (20).

For a few particular cases some general statements can at once be made on the basis of Eqs. (20). For example: (1) Three concurrent forces that do not lie in one plane can be in equilibrium only if all three forces are zero. (2) If two of four concurrent forces that do not all lie in one plane are collinear in action, then equilibrium can exist only if the other two forces are both zero. The two collinear forces must, of course, be equal in magnitude and opposite in direction. (3) If the lines of action of all but one of any number of concurrent forces in space lie in one plane, then equilibrium can exist only if this one force is zero. (4) If the known lines of action of all but two of any number of concurrent forces in space lie in one plane and one of these two forces is known in magnitude, then the magnitude of the other can always be found without difficulty. The proof of each of these statements is left to the student.

EXAMPLES

1. Three concurrent forces F_1, F_2, and F_3 have the lines of action OA, OB, and OC as shown in Fig. 138, and the magnitudes shown in the following table. Find the magnitude and direction of their resultant R.

Solution. We begin by computing the lengths of the lines OA, OB, and OC, from the observed coordinates of the points A, B, and C, as follows:

$$OA = \sqrt{(2)^2 + (3)^2 + 0} = \sqrt{13}$$
$$OB = \sqrt{(4)^2 + (1)^2 + (4)^2} = \sqrt{33}$$
$$OC = \sqrt{(1)^2 + (-2)^2 + (4)^2} = \sqrt{21}$$

F_i	lb	$\cos \alpha_i$	$\cos \beta_i$	$\cos \gamma_i$	X_i	Y_i	Z_i
F_1	40	0.555	0.832	0	22.2	33.3	0
F_2	10	0.696	0.174	0.696	7.0	1.7	7.0
F_3	30	0.218	−0.436	0.873	6.5	−13.1	26.2
	..	.....	....	Σ	35.7	21.9	33.2

Using these values, the direction cosines of the lines of action of the forces are easily computed and tabulated as shown in the table. Then using Eqs. (b), we find the projections of the given forces as shown in the last three columns of the table. Finally, making the summations of projections and using Eqs. (c) and (d), we obtain

$$R = \sqrt{(35.7)^2 + (21.9)^2 + (33.2)^2} = 53.45 \text{ lb}$$

$$\alpha = \arccos \frac{35.7}{53.4} = \arccos 0.668 = 48°05'$$

$$\beta = \arccos \frac{21.9}{53.4} = \arccos 0.410 = 65°45'$$

$$\gamma = \arccos \frac{33.2}{53.4} = \arccos 0.622 = 51°35'$$

As a check, we have

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

since the chosen coordinate axes are orthogonal.

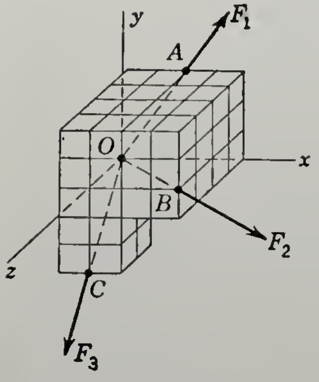


FIG. 138

2. Two equal forces F, lying in a horizontal plane and acting at right angles to each other, are applied at A to the top of a pole AB (Fig. 139). To eliminate bending of the pole a guy wire AC is used, the tension in which can be adjusted by using a turnbuckle D. Determine the proper tensile force S_1 in the wire and the corresponding compressive force S_2 in the pole if each force $F = 100$ lb.

Solution. Let us begin by considering the conditions of equilibrium of the ring A at the top of the pole. If bending of the pole is completely eliminated, the reaction S_2 exerted by the pole on this ring will be a force in the direction of the axis AB of the pole as shown. By virtue of the flexibility of the guy wire the reaction S_1 that it exerts on the ring A will act along the axis AC of this wire and we obtain the free-body diagram as shown in the figure.

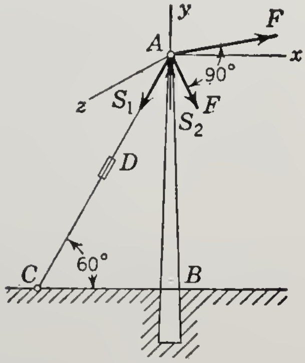


FIG. 139

Taking the xy plane to coincide with the plane ABC and the z axis perpendicular to this plane, we conclude from the third of Eqs. (20) that the plane ABC must bisect the 90° angle between the forces F if the guy wire is to prevent bending of the pole. Projecting all forces at A onto the x axis, we obtain

$$-S_1 \cos 60° + 2F \cos 45° = 0$$

from which $S_1 = 2\sqrt{2}F = 283$ lb. To determine the compressive force S_2 in the pole, we equate to zero the algebraic sum of the projections of all forces at A on the y axis and obtain

$$S_2 - S_1 \cos 30° = 0$$

from which $S_2 = S_1 \cos 30° = 245$ lb.

3. A tripod with legs OA, OB, OC of equal lengths l and hinged together at O is supported on a horizontal floor so that the points of support A, B, and C form an equilateral triangle with sides of lengths $\sqrt{3}\, l/2$, as shown in Fig. 140. Determine the magnitudes of the compressive forces produced in the three legs by a load P hanging from O as shown. Assume that there is sufficient friction between the ends of the legs and the horizontal floor to prevent slipping.

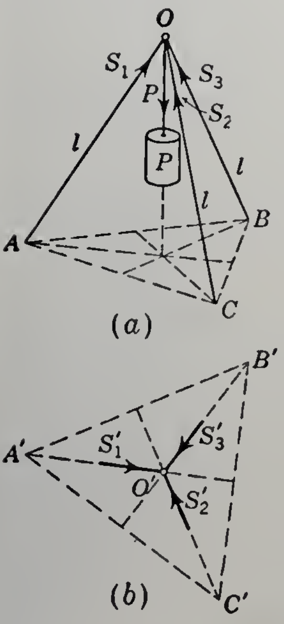


FIG. 140

Solution. We begin by considering the equilibrium of the hinge at O which is acted upon by the four concurrent forces S_1, S_2, S_3, and P as shown in Fig. 140a. Taking point O as the origin of coordinates with the z axis vertical and projecting all forces onto the z axis, we obtain

$$(S_1 + S_2 + S_3) \cos \gamma - P = 0 \tag{e}$$

where γ is the angle that each leg of the tripod makes with the vertical.

From the equality of the lengths of the legs together with the fact that $\triangle ABC$ is equilateral, we conclude that the z axis pierces the plane of the floor at the intersection of the medians of $\triangle ABC$. From this it follows, for the given dimensions, that each leg makes with the z axis an angle $\gamma = 30°$.

Now projecting the system of forces at O onto the horizontal plane of the floor, we obtain the system of forces in one plane as shown in Fig. 140b. Since the original system of forces is in equilibrium, it follows from the discussion on page 170 that the coplanar system of forces represented by this horizontal plane projection of the original system is also in equilibrium. Considering this system of forces in a plane, we see that $S'_1 = S'_2 = S'_3$, from which we conclude that $S_1 = S_2 = S_3 = S$. Returning now to Eq. (e) and remembering that $\gamma = 30°$, we find

$$S = \frac{P}{3 \cos 30°} = \frac{2P}{3\sqrt{3}}$$

4. A bracket made up of six bars hinged together at A and B and to a vertical wall at C, D, E, and F carries a vertical load P at hinge B, as shown in Fig. 141. $ABEF$ lies in a horizontal plane, and points C and D are vertically below E and F, respectively. Calculate the axial force induced in each bar.

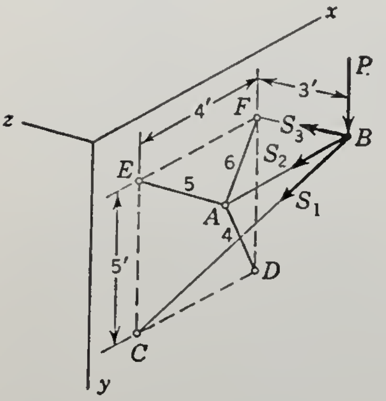


FIG. 141

Solution. As a preliminary step, we compute the length of bar BC, which is

$$BC = \sqrt{3^2 + 4^2 + 5^2} = 5\sqrt{2} \text{ ft}$$

Now considering hinge B as a free body, we see that it is in equilibrium under the action of four concurrent forces: the load P and three forces S_1, S_2, S_3 representing the reactions of the bars CB, AB, and FB, which we assume to be in tension. Equating to zero the algebraic sums of projections of these forces on the coordinate axes x, y, z, we obtain

$$-S_2 - \frac{4}{5\sqrt{2}} S_1 = 0 \qquad +P + \frac{5}{5\sqrt{2}} S_1 = 0 \qquad +S_3 + \frac{3}{5\sqrt{2}} S_1 = 0$$

from which $S_1 = -1.4P$, $S_2 = +0.8P$, $S_1 = +0.6P$. The negative sign for S_1 indicates that bar BC is in compression.

Consideration of the equilibrium of hinge A and the determination of the axial forces in bars 4, 5, and 6 are left to the student.

5. A rectangular block of weight W rests on an inclined plane that makes the angle α with the horizontal, as shown in Fig. 142a. Find the magnitude of a horizontal force P applied to the center of the block and acting in a plane parallel to the inclined plane that will cause motion of the block to impend. Assume that the angle of friction φ for the surface of contact is greater than α and that the proportions of the block are such that sliding will impend before tipping.

Solution. At the instant of impending slipping the block is in equilibrium under the action of the gravity force W, the applied force P, and a reaction R

exerted by the inclined plane. Since these three forces are in equilibrium, we conclude that they must intersect in one point and also lie in one plane. Further, when sliding of the block impends, we know that the reaction R is inclined to the normal to the inclined plane by the angle of friction φ.

Choosing O as the origin of coordinates with the z axis normal to the inclined plane and the x axis horizontal and equating to zero the algebraic sum of the projections of all forces on the z axis, we obtain

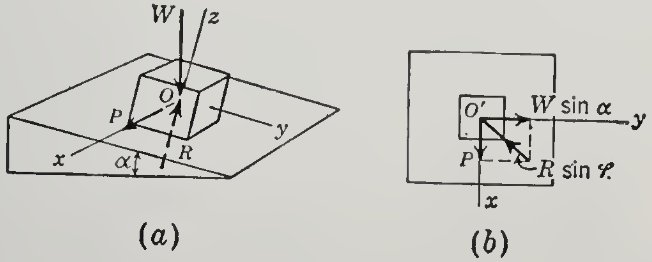


FIG. 142

$$R \cos \varphi = W \cos \alpha \tag{f}$$

Projecting the entire system of forces onto the inclined plane, we obtain the system of coplanar forces in equilibrium as shown in Fig. 142*b*, from which we conclude that

$$R^2 \sin^2 \varphi = P^2 + W^2 \sin^2 \alpha \tag{g}$$

Eliminating R between Eqs. (f) and (g) gives

$$P = W \sqrt{\mu^2 \cos^2 \alpha - \sin^2 \alpha} \tag{h}$$

where $\mu = \tan \varphi =$ coefficient of friction. It will be noted that for the limiting case where $\alpha = \varphi$, Eq. (h) gives $P = 0$. Also for the limiting case where $\alpha = 0$ and the inclined plane becomes a horizontal plane, Eq. (h) gives $P = \mu W$.

We see from the first limiting case that when all available friction is already being used to resist sliding of the block down the plane, then there is no resistance to lateral slipping. This explains, for example, why a rear-wheel-drive automobile can skid so freely from side to side when climbing a grade on wet or icy pavement. For the same reason, a car loses lateral stability if the brakes are too suddenly applied so as to cause the tires to slip.

6. The small pulley in Fig. 143*a* drives the large one in a counterclockwise direction by a V belt overrunning their rims, as shown. The angle of the V in both pulleys is 2α, the total angle of contact on the small pulley is β, and the coefficient of friction between belt and rim is μ. Find the ratio S_1/S_2 between the tensions in the two branches of the belt when slipping impends if the belt contacts the grooves only on its sides and is assumed to be perfectly flexible.

Solution. We use here the same method of analysis already discussed in Art. 3.8 for a flat belt (see page 157). In Fig. 143*b*, we show a free-body diagram for one small element of the belt defined by the angle θ measured from the point of tangency A on the small pulley. The forces acting on this element are (1) tensions S and $S + dS$ acting normal to the two cut faces of the element and inclined by angles $d\theta/2$ to the y axis, as shown; (2) normal forces dN_1 and dN_2 on the sides of the element, inclined by angles α to the z axis, as shown; and (3) friction forces dF_1 and dF_2 lying in the two side planes and parallel to the y axis, as shown. These forces may be regarded as a concurrent system in space that are in equilibrium (uniform running of the belt over the pulleys does not alter the conditions of equilibrium). Thus Eqs. (20) become

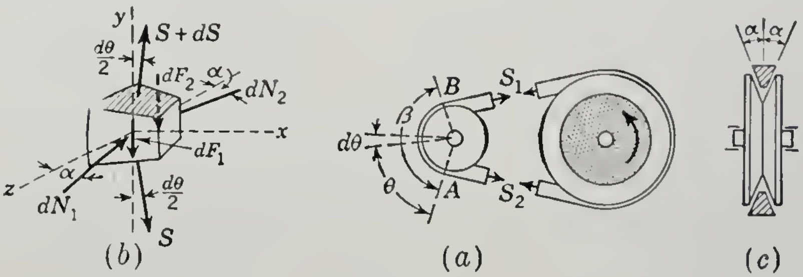


FIG. 143

$$(2S + dS) \sin \frac{d\theta}{2} - (dN_1 + dN_2) \sin \alpha = 0$$

$$(S + dS - S) \cos \frac{d\theta}{2} - (dF_1 + dF_2) = 0$$

$$(dN_2 - dN_1) \cos \alpha = 0$$

From the last equation, we see that $dN_1 = dN_2 = dN$, and when slipping impends, $dF_1 = dF_2 = \mu \, dN$. Substituting these values into the first two equations, noting that $\sin (d\theta/2) = d\theta/2$, $\cos (d\theta/2) = 1$, and neglecting small quantities of second order, these equations reduce to

$$S \, d\theta - 2 \, dN \sin \alpha = 0$$

$$dS - 2\mu \, dN = 0$$

Eliminating dN, we obtain

$$\frac{dS}{S} = \frac{\mu \, d\theta}{\sin \alpha} \tag{i}$$

Integrating expression (i) over the entire line of contact AB, that is, from $\theta = 0$ to $\theta = \beta$, we find

$$\ln \frac{S_1}{S_2} = \frac{\mu\beta}{\sin \alpha} \qquad \text{or} \qquad \frac{S_1}{S_2} = e^{\mu\beta/\sin \alpha} \tag{j}$$

For the case of a flat belt, $\alpha = \pi/2$ and expression (j) coincides with Eq. (h) on page 158. We see that for a small value of α, the V belt is much more efficient than a flat one.

PROBLEM SET 4.1

1. The lines of action of three forces concurrent at the origin O pass, respectively, through points A, B, C, having coordinates

$$\begin{array}{lll} x_a = -1 & y_a = +2 & z_a = +4 \\ x_b = +3 & y_b = 0 & z_b = -3 \\ x_c = +2 & y_c = -2 & z_c = +4 \end{array}$$

The magnitudes of the forces are $F_a = 40$ lb, $F_b = 10$ lb, $F_c = 30$ lb. Find the magnitude and direction of their resultant. *Ans.* $R = 53.6$ lb; $\alpha = 78°35'$; $\beta = 84°25'$; $\gamma = 12°35'$.

2. Referring to Fig. 139 and assuming unequal forces $F_1 = 100$ lb and $F_2 = 80$ lb instead of equal forces F, find the required tension S_1 in the guy wire AC to prevent bending of the pole and the corresponding compression S_2 in the pole. Be sure that the guy wire is placed in such a vertical plane as to eliminate bending of the pole. *Ans.* $S_1 = 256$ lb; $S_2 = 222$ lb.

3. A vertical mast AB of length $l = 16$ ft is guyed by four wires BC, BD, BE, and BF, each of length $l_1 = 20$ ft and forming a regular square pyramid as shown in Fig. A. Determine the compressive force S in the pole if each guy wire carries a tension of 1 ton. *Ans.* $S = 3\frac{1}{5}$ tons.

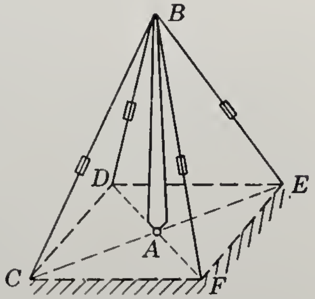


FIG. A

4. A mast AB supported by a spherical socket at A and horizontal guy wires BC and BD carries a vertical load P at B as shown in Fig. B. Find the axial force induced in each of the three members of this system. *Ans.* $S_1 = +0.8P$; $S_2 = +0.6P$; $S_3 = -1.4P$.

5. Repeat the solution of Prob. 4 if point D is 1 ft vertically below the position shown in Fig. B and all other dimensions remain unchanged. *Ans.* $S_1 = +1.03P$; $S_2 = +0.75P$; $S_3 = -1.77P$.

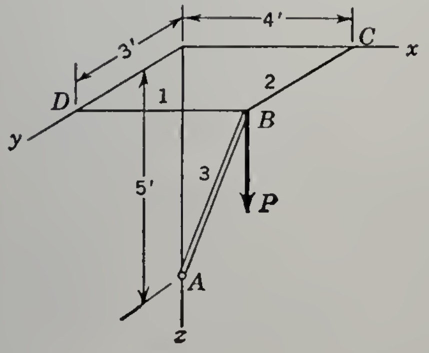


FIG. B

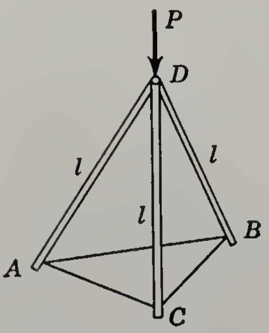


FIG. C

6. In the case of the tripod shown in Fig. C there is no friction between the ends of the legs and the floor on which they rest. To prevent slipping of the legs their ends are connected by strings along the lines AB, BC, and AC. Determine then the tensile force S in each of these strings if each leg makes 30° with the vertical and P is a vertical load. *Ans.* $S = \frac{1}{9}P$.

7. Four balls of equal radii r form a pyramid supported by a smooth horizontal plane surface (Fig. D). The three lower balls are held together by an encircling string as shown. Determine the tensile force S in this string if the weight of each ball is Q and the surfaces of the balls are perfectly smooth. Neglect any initial tension that may be in the string before the top ball is placed upon the other three. *Ans.* $S = 0.136Q$.

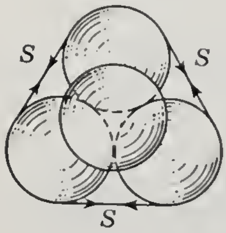


FIG. D

8. Imagine that Fig. D represents three hemispheres, each of radius r and weight $Q/2$ so arranged that they rest with their flat faces on a horizontal plane and support a full sphere of radius r and weight Q above them. In such case, determine the limiting value of the coefficient of friction between the flat faces of the hemispheres and the horizontal plane for which equilibrium will be possible. There is no string to prevent slipping of the hemispheres. Assume the upper sphere to be perfectly smooth. *Ans.* $\mu \geqq \sqrt{2}/5$.

9. A small block A of weight W straddles a bar BC of triangular cross section as shown in Fig. E. If sliding of the block impends when the axis BC of the bar makes the angle β with the horizontal as shown, what is the value of the coefficient of friction μ between the block and the faces of the bar? *Ans.* $\mu = \tan \beta \sin (\alpha/2)$.

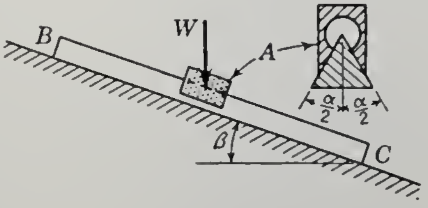


FIG. E

10. Determine the forces produced in the bars 1 to 6, inclusive, of the space truss shown in Fig. F, owing to the action of four vertical loads P applied as shown. $ACBD$ and $A'C'B'D'$ are two squares with parallel sides of lengths a and $2a$, respectively, and the distance between horizontal planes $ACBD$ and $A'C'B'D'$ is $2a$. *Ans.* $S_1 = S_2 = -0.25P$; $S_3 = S_4 = -1.06P$; $S_5 = S_6 = 0$.

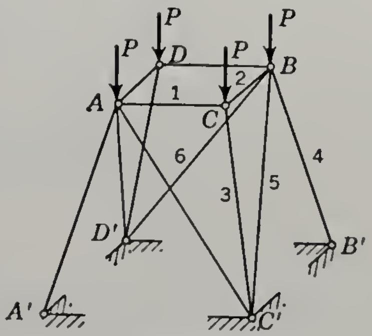


FIG. F

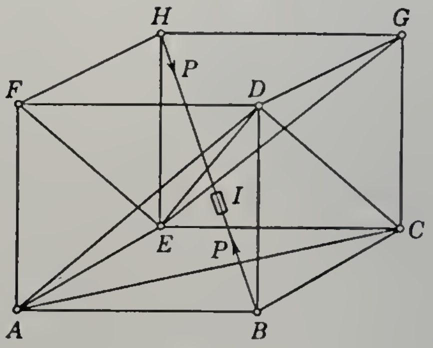


FIG. G

11. The system of hinged bars shown in Fig. G forms a cube. If, by means of a turnbuckle I, a tensile force P is produced in the diagonal bar HB, what forces will be induced in the other bars of the system?

12. A flexible suspension bridge cable is supported at the top of a tower by a saddle, as shown in Fig. H. The portion ACB of the cable is a parabola with vertex at C and the groove in which the cable rests is a 60° V notch as shown. Find the limiting ratio P_1/P_2 for which slipping of the cable in the saddle will impend if the coefficient of friction $\mu = 0.25$. *Ans.* $P_1/P_2 = 2.193$.

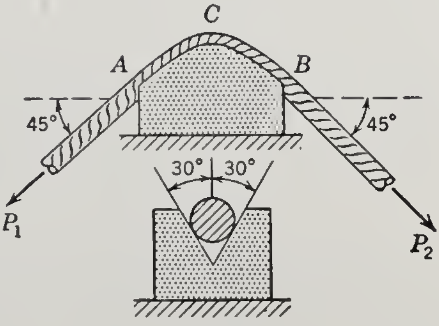
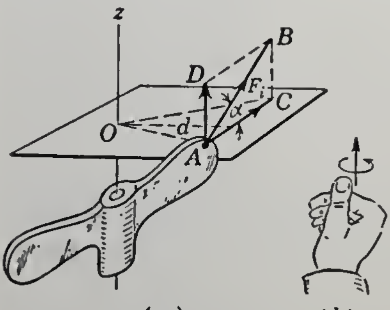


FIG. H

4.2. Concurrent forces in space: method of moments. *Moment of a Force with Respect to an Axis.* The tendency of a force to produce rotation about any fixed axis in a body to which it is applied is measured by the *moment* of the force with respect to that axis. Consider, for example, a body free to rotate about the vertical axis Oz and acted upon by a force F_i applied at point A, as shown in Fig. 144a. Passing a plane OAC through point A and normal to the axis Oz, we resolve the force F_i into rectangular components $F_i \sin \alpha$ normal to this plane and $F_i \cos \alpha$ lying in the plane. Then it is intuitive that only the latter component, lying in the plane, has any tendency to rotate the body around Oz. Accordingly, we define the moment of F_i with respect to the axis Oz as the product of its orthogonal projection $F_i \cos \alpha$ and the distance d of this projection from point O, where the axis pierces the plane. Thus



(a) (b)

FIG. 144

$$(M_z)_i = F_i \cos \alpha \cdot d \tag{a}$$

We see that this is identical with the moment of the projection $F_i \cos \alpha$ with respect to point O and is represented by the doubled area of $\triangle OAC$. The moment $(M_z)_i$ is considered positive if the rotation is in the direction indicated by the fingers of the right hand when the thumb is pointed in the positive direction of the axis Oz (Fig. 144b). This is called the *right-hand rule*, and viewing the plane OAC from the positive end of the axis Oz, we see that it agrees with the *counter-*

clockwise rule introduced before in discussing moment of a force with respect to a point (see page 40).

Extension of the Theorem of Varignon. If a parallelogram of forces $ACBD$ consisting of a force R and its components P and Q (Fig. 145) is projected onto a plane normal to a given axis Oz, we obtain a parallelogram of forces $A'C'B'D'$ from which we conclude that the resultant of the projections of the components P and Q is equal to the corresponding projection of their resultant R. Then it follows at once from Varignon's theorem and the above definition of moment of a force with respect to an axis that the algebraic sum of moments of two concurrent forces P and Q with respect to any axis is equal to the moment of their resultant with respect to the same axis. In Art. 4.1 we have seen that the resultant of any number of concurrent forces in space can be obtained by successive applications of the principle of the parallelogram of forces. Hence, applying Varignon's theorem at each successive step, we conclude finally that the algebraic sum of the moments of any system of concurrent forces with respect to an axis is equal to the moment of their resultant with respect to the same axis.

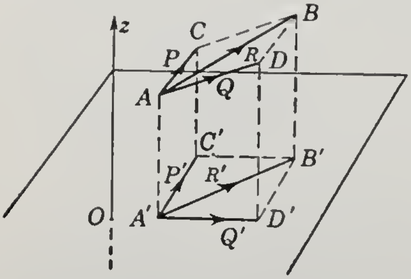


FIG. 145

It can be shown in a similar manner that the above extension of the theorem of moments holds also for the case of parallel forces in space. Thus we may consider the theorem completely general and say that the moment of the resultant of any force system about any axis is equal to the algebraic sum of moments of its components about the same axis.

In analytical investigations of problems of statics, we often use rectangular coordinate axes x, y, z as axes of moments. For this reason expressions for the moments of any force F_i with respect to the three coordinate axes will be helpful. Taking the axes as shown in Fig. 146, we denote by X_i, Y_i, Z_i the projections of any force F_i and by x_i, y_i, z_i the coordinates of its point of application A. Then using the right-hand rule for sign of moment in each case and keeping in mind that the moment of F_i is equal to the algebraic sum of

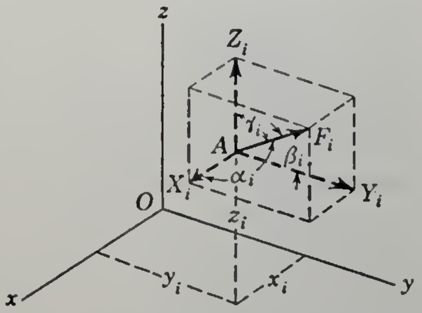


FIG. 146

the moments of its components, we may write

$$(M_x)_i = Z_i y_i - Y_i z_i \qquad (M_y)_i = X_i z_i - Z_i x_i \qquad (M_z)_i = Y_i x_i - X_i y_i \qquad (21)$$

where

$$X_i = F_i \cos \alpha_i \qquad Y_i = F_i \cos \beta_i \qquad Z_i = F_i \cos \gamma_i \qquad \text{(b)}$$

Although derived on the assumption that the force F_i has positive projections, Eqs. (21) are always valid if proper attention is paid to the signs of the projections as defined by Eqs. (b).

Equations of Equilibrium. We have seen above that the algebraic sum of moments of a system of concurrent forces in space with respect to a given axis is equal to the moment of their resultant with respect to the same axis. Further, this sum of moments is zero if the resultant force intersects the axis, is parallel to it, or vanishes itself. These three possibilities are sometimes useful in deciding the question of equilibrium of a system of concurrent forces in space.

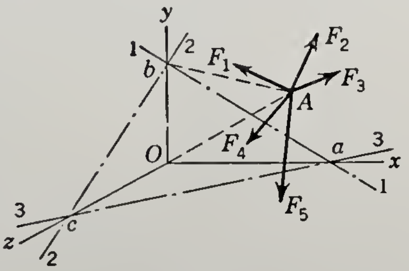


FIG. 147

Consider, for example, the force system concurrent at a point A in space, as shown in Fig. 147, and let 1-1, 2-2, 3-3, be three nonparallel axes defining a plane abc that does not contain point A. Now suppose that the algebraic sum of moments of the forces about the axis 1-1 is zero. Then if there is a resultant, it must either be parallel to, or intersect, this axis. If the sum of moments about the axis 2-2 is also zero, the resultant, if one exists, can only lie along the line Ab, since it cannot be parallel to both 1-1 and 2-2 which are nonparallel. Finally, if the sum of moments about the axis 3-3 is likewise zero, the resultant has to vanish completely, because it was already confined to the line Ab which is neither parallel to, nor intersects, the axis 3-3. Thus the forces must be in equilibrium. These three necessary and sufficient conditions of equilibrium may be expressed algebraically by the following equations:

$$\Sigma(M_1)_i = 0 \qquad \Sigma(M_2)_i = 0 \qquad \Sigma(M_3)_i = 0 \qquad (22)$$

where $(M_1)_i$, $(M_2)_i$, $(M_3)_i$ denote the moments of any force F_i with respect to the axes 1, 2, and 3, as discussed above and in which the summations are understood to include all forces in the system.

In the solution of problems, it is not necessary to adhere rigorously to the three conditions of equilibrium represented by Eqs. (22). We can, for example, use only one of these moment equations together with two of Eqs. (20) developed in Art. 4.1 or various other combinations. In short, if a system of concurrent forces is in equilibrium, the algebraic sum of their moments with respect to any axis must be zero and, likewise, the algebraic sum of their projections on any axis must be zero. We can write any three such equations of equilibrium that will give us the most direct solution of a given problem.

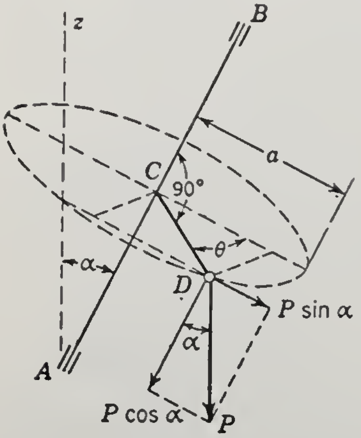


FIG. 148

EXAMPLES

1. A shaft AB inclined to the vertical by an angle α has an arm CD welded to it at right angles, as shown in Fig. 148. Calculate the moment of a vertical force P applied at D with respect to the axis AB if the arm is of length a and the plane ABD makes the angle θ with the vertical plane AzB.

Solution. Through point D, we pass a vertical plane parallel to the plane AzB. In this plane, we resolve P into rectangular components $P\cos\alpha$ and $P\sin\alpha$ as shown. The component $P\cos\alpha$ is parallel to AB and has no moment about this axis. The component $P\sin\alpha$ is perpendicular to AB and its arm, equal to the distance between the two vertical planes, is $a\sin\theta$. Hence, the required moment of P, equal to the sum of moments of its components, is $Pa\sin\alpha\sin\theta$, which according to the right-hand rule is positive.

2. Three bars each of length l are hinged together at D and to fixed points A, B, C, as shown in Fig. 149, where $OA = OB = OC = a$. Find the tension S produced in bar BD by a vertical load P acting at D, if $a = 4$ ft and $l = 5$ ft.

Solution. We begin by passing a vertical plane $OBDE$ through bar BD and intersecting the xz plane along OE. In this plane, we resolve the tensile force S at D into horizontal and vertical components $S\sin\theta$ and $S\cos\theta$, respectively, as shown. Now taking moments of all forces acting at D with respect to the horizontal axis AC (thus avoiding consideration of the axial forces in bars DA and DC), we have, as a condition of equilibrium,

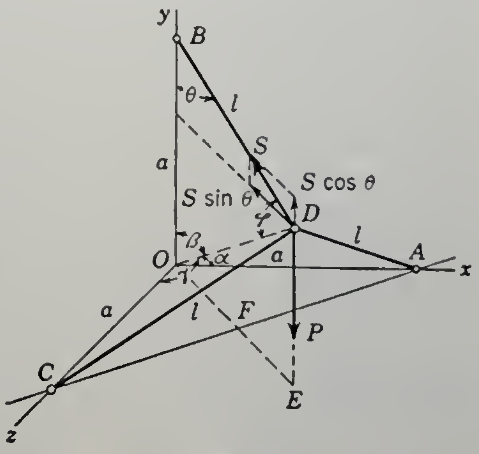


FIG. 149

$$S\sin\theta\cdot DE + S\cos\theta\cdot EF = P\cdot EF \tag{c}$$

wherein, as will be seen from a study of Fig. 149,

$$DE = a - l \cos \theta$$
$$EF = l \sin \theta - \frac{a}{\sqrt{2}}$$

Substituting these expressions in Eq. (c) and reducing, we obtain

$$S = P \frac{(l/a) \sin \theta - 1/\sqrt{2}}{\sin \theta - (1/\sqrt{2}) \cos \theta} \tag{d}$$

The remainder of the problem is purely a matter of finding the magnitude of the angle θ from the geometry of the structure (Fig. 149). To do this, we first note from the symmetry of the system that the direction cosines of OD are all three equal, and since for orthogonal axes x, y, z,

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

we must have $\cos^2 \alpha = \cos^2 \beta = \cos^2 \gamma = \frac{1}{3}$. Thus we find the angles $\alpha = \beta = \gamma = \arccos (1/\sqrt{3}) = 54°45'$.

The lengths a and l being given, we now have two sides and one angle β of $\triangle OBD$, and using a little trigonometry, we finally obtain

$$\sin \theta = \frac{\sqrt{2}\, a}{3l} \left(1 + \sqrt{3 \left(\frac{l}{a}\right)^2 - 2}\right) \tag{e}$$

Eliminating θ between Eqs. (e) and (d), we obtain the value of S entirely in terms of the given lengths a and l. However, for numerical calculation it is simpler to compute θ from Eq. (e) and then S from Eq. (d) without trying to combine them. For example, taking $a = 4$ ft and $l = 5$ ft, Eq. (e) gives $\sin \theta = 0.9953$, $\theta = 84°27'$, $\cos \theta = 0.0967$. Substituting these values together with $l/a = 1.2500$, we obtain $S = 0.580P$. We see from this example that the geometry of space systems can sometimes complicate the problem, even though the statics itself is comparatively simple.

PROBLEM SET 4.2

1. A pulley A of radius a is supported from the face of a vertical wall by two braces AB and AC together with a tie bar AD, as shown in Fig. A. A flexible cord EAF is fastened to the wall at E, passes over the pulley, and carries at its end F a load Q. Find the tensile force S produced in the tie bar AD if $Q = 100$ lb, $a = 6$ in., $b = 4$ ft, $c = 7\frac{1}{2}$ ft. *Ans.* $S = 53$ lb.

2. A ball of radius r and weight Q is hung from the corner of a room by a string AB of length l as shown in Fig. B. Neglecting friction between the ball and the walls, find the tensile force S in the string AB, which in the absence of friction will pass through C. The following numerical data are given: $r = 6$ in., $l = 10$ in., $Q = 100$ lb. *Ans.* $S = 118$ lb.

3. Using the method of moments, find the axial force S produced in the vertical mast AB of the crane which supports a vertical load $P = 5$ tons, as shown in Fig. C. *Ans.* $S = 6$ tons.

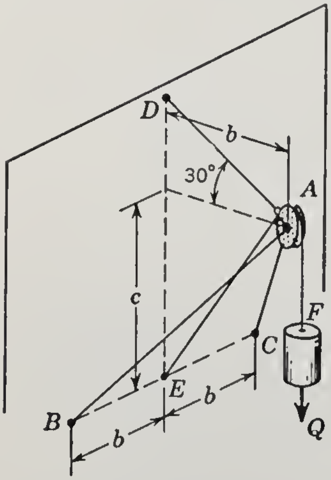


FIG. A

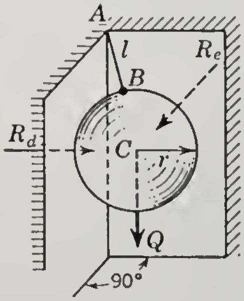


FIG. B

4. Find the tension T in each of the guy wires BD and BE of the crane loaded as shown in Fig. C. *Ans.* $T = 4.38$ tons, each.

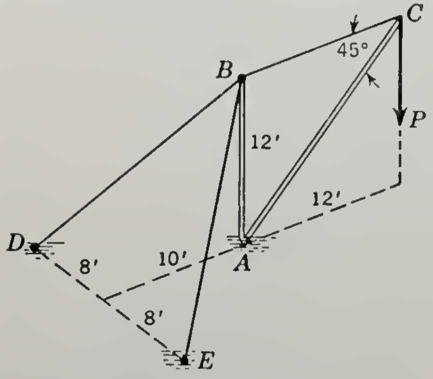


FIG. C

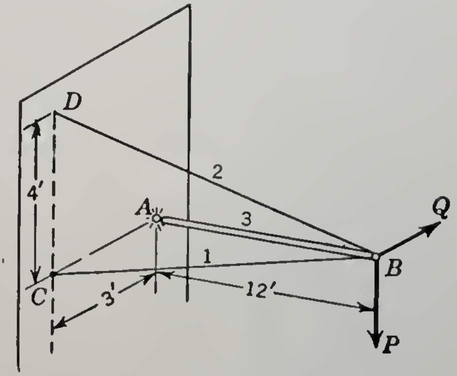


FIG. D

5. A strut AB attached to the face of a vertical wall at A by a spherical hinge stands perpendicular to the wall and is supported by two guy wires, as shown in Fig. D. At B, in a plane parallel to the wall, two forces P and Q act as shown, Q being horizontal and P, vertical. Using the method of moments, find the axial forces produced in the members if $P = 500$ lb and $Q = 400$ lb. *Ans.* $S_1 = 103$ lb; $S_2 = 1{,}625$ lb; $S_3 = -1{,}600$ lb.

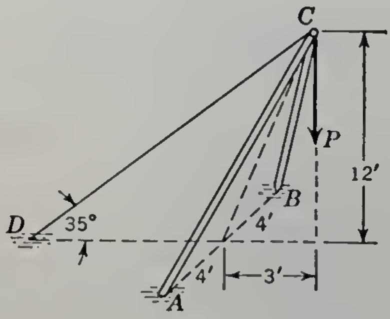


FIG. E

6. Calculate the tension T in the guy wire CD and the compression S in each strut of the shear-leg derrick shown in Fig. E if the vertical load $P = 10$ tons. *Ans.* $T = 3.70$ tons, $S = 1.69$ tons.

7. A space truss as shown in plan and elevation in Fig. F supports three vertical loads $P = 1{,}000$ lb at A, B, and C. Find the axial force S_i in each of the bars 1, 2, 3, if $AB = BC = CA = 3$ ft and $DE = EF = FD = 6$ ft. *Ans.* $S_1 = S_2 = S_3 = -P/4$.

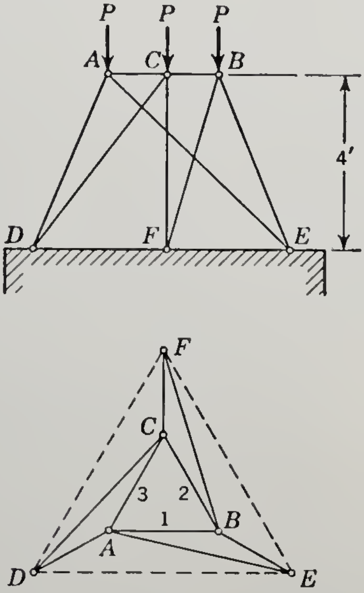


FIG. F

8. Referring again to the space truss in Fig. F, prove that the axial force in each of the diagonal bars DC, EA, and FB, is zero for the condition of loading shown.

9. If the space truss in Fig. F is subjected to only one vertical load P at A, find the axial force S_i in each of the bars 1, 2, 3. Use the same numerical data given in Prob. 7 above. *Ans.* $S_1 = S_2 = 0$; $S_3 = -P/4$.

4.3. Couples in space. In discussing couples in a plane (Art. 2.1) it was shown that the action of a couple on a body is not changed by transformation or displacement of the couple in its plane provided its moment remains unchanged. We shall now prove that the action of a couple on a rigid body to which it is applied will not be changed if the plane of the couple be displaced parallel to itself. Consider, for example, a couple PP acting in a plane M as shown in Fig. 150. To prove that, without changing the action of this couple, it can be displaced from the plane M to the parallel plane N, we proceed as follows: Taking, in the plane N, the line $A'B'$ equal and parallel to AB, we

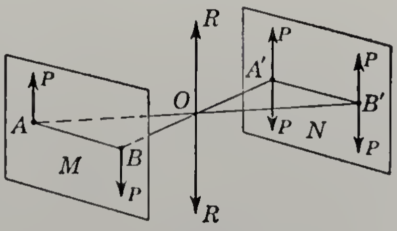


FIG. 150

apply at each of the points A' and B' two equal and opposite forces P equal to the forces acting in the plane M. These forces are in equilibrium and do not change the action of the original couple in the plane M. The two forces P applied at points A and B' and acting upward can now be replaced by their resultant R which acts in the upward direction at the mid-point O of the line AB' as shown. In the same manner the two forces P acting downward at points B and A' can be replaced by their resultant R acting downward at point O. The two forces R acting at point O balance each other and can be removed from the system leaving only the upward force P at A' and the downward force P at B' in the plane N. These two forces constitute a couple in the plane N that is equal and statically equivalent to the original couple in the plane M. Hence it is proved that the action of a couple on a rigid body will not be changed if the plane of the couple be displaced parallel to itself. This statement regarding a couple may be considered as analogous to the theorem of transmissibility of a force (see page 8).

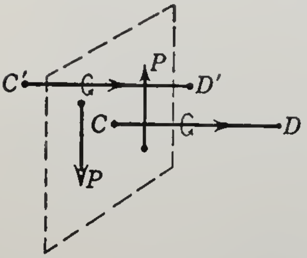


FIG. 151

From the preceding discussion we conclude that a couple is completely defined by three elements: (1) the magnitude of its moment, (2) the aspect of the plane in which it acts, defined by the direction of the normal to this plane, and (3) the direction of rotation in the plane, i.e., the *sense* of the couple. These three elements can be represented completely by a vector as follows: If a couple PP (Fig. 151) is given, we take any point C in the plane of the couple and draw through it a normal to this plane. Along this normal we measure a distance CD which, to a certain scale, represents the magnitude of the moment of the couple and on which we indicate, by an arrow, a direction such that between this direction and the direction of rotation of the couple there exists the same relationship as between the translation and rotation of a right-hand screw. The vector $\overline{CD}$ obtained in this way completely defines the given couple. Since the plane of a couple can be displaced parallel to itself without changing the action of the couple, we conclude that any vector $\overline{C'D'}$ equal and parallel to the vector $\overline{CD}$ represents an equivalent couple. The vectorial representation of a couple is very useful when we have to deal with couples differently oriented in space, and in our further discussions a vector, such as $\overline{CD}$ or $\overline{C'D'}$, will be called a *moment vector* to distinguish it from a *force vector*.[1]

---

[1] We shall take it as axiomatic that the rules of addition of force vectors apply also to moment vectors.

Since the position of a couple in its plane is of no consequence and since the plane of a couple can be displaced parallel to itself without changing the action of the couple, it follows that any system of couples differently oriented in space can be represented by moment vectors that can be taken concurrent at any point in space. Hence it follows at once by analogy to the case of concurrent forces in space that the resultant couple M of any system of couples $M_1, M_2, \ldots, M_n$ may be found by the method of projections.

Consider, for example, the system of couples in space as represented by the moment vectors $M_1, M_2, \ldots$, in Fig. 152, where each couple must be visualized as acting anywhere in a plane normal to the corresponding moment vector. Taking coordinate axes x, y, z, as shown, and denoting by α_i, β_i, γ_i the direction angles of the vector M_i, we see that the projections of this vector are

$$\begin{aligned}(M_x)_i &= M_i \cos \alpha_i \\ (M_y)_i &= M_i \cos \beta_i \\ (M_z)_i &= M_i \cos \gamma_i\end{aligned} \qquad \text{(a)}$$

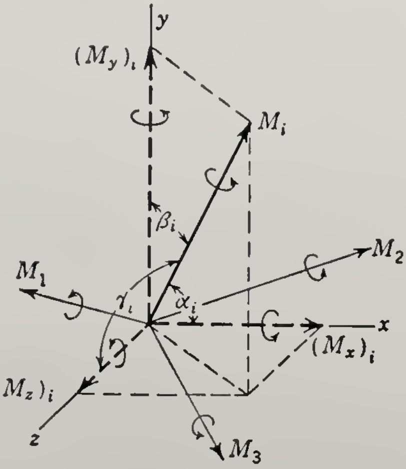


FIG. 152

These three projections or rectangular components of any moment vector M_i represent three component couples which act in the three coordinate planes. For example, $(M_y)_i$ represents a couple in the xz plane which is counterclockwise in sense.

Proceeding in the same manner with the moment vectors M_1, $M_2, \ldots$, and then adding algebraically all corresponding projections, we obtain the three rectangular components of the resultant couple M as follows:

$$M_x = \Sigma(M_x)_i \qquad M_y = \Sigma(M_y)_i \qquad M_z = \Sigma(M_z)_i \qquad \text{(b)}$$

The magnitude of the moment of the resultant couple is given by the expression

$$M = \sqrt{M_x^2 + M_y^2 + M_z^2} \qquad \text{(c)}$$

Denoting by α, β, γ the angles that the normal to the plane of the resultant couple M makes with the coordinate axes x, y, z, respectively, we have, for determining the orientation of this plane, the following equations:

$$\cos \alpha = \frac{M_x}{M} \qquad \cos \beta = \frac{M_y}{M} \qquad \cos \gamma = \frac{M_z}{M} \qquad \text{(d)}$$

If a system of couples in space is in equilibrium, the resultant must be zero, and therefore we conclude from Eq. (c) that the conditions of equilibrium for such a system are expressed by the equations

$$\Sigma(M_x)_i = 0 \qquad \Sigma(M_y)_i = 0 \qquad \Sigma(M_z)_i = 0 \tag{23}$$

These equations of equilibrium for a system of couples in space are completely analogous to Eqs. (20) for concurrent forces in space. However, in using Eqs. (23), it must be kept constantly in mind that we are dealing with moment vectors and not with force vectors. The actual forces of a couple always lie in a plane normal to the moment vector representing them and are not considered in a direct manner at all.

EXAMPLES

1. Five equal forces P act on the corners of a cube with edges of length a, as shown in Fig. 153a. Find the equilibrant of this system of forces.

Solution. The two vertical forces at A and D represent a couple of moment Pa in the front face of the cube. At any convenient point O in space (Fig. 153b), we represent this couple by a moment vector directed along the z axis normal to the face $ABCD$ of the cube. The force at G and the one parallel to it at A represent another couple of moment Pa in the bottom face of the cube. We represent this couple by a moment vector directed along the y axis in Fig. 153b. The resultant of the two moment vectors in Fig. 153b has the magnitude $\sqrt{2}\,Pa$ and is directed normal to the diagonal plane $ABEF$ of the cube. Thus the resultant of four of the five forces in Fig. 153a is a couple of moment $\sqrt{2}\,Pa$ in the plane $ABEF$. To balance this couple, we need one of the same magnitude but opposite sense. The remaining force P acting along BA and an equal but opposite force acting along EF would give such a couple. Hence, the dotted force vector P at F in Fig. 153a is the equilibrant of the system.

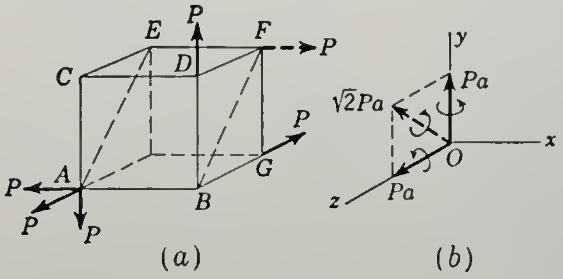


FIG. 153

2. A round bar having a semicircular axis ACB of radius r lies in a horizontal plane and is firmly attached to a vertical wall at its ends A and B by bolted flanges, as shown in Fig. 154*a*. Find the reactions at A and B if the bar is subjected to uniformly distributed twisting moment of intensity m, that is, m in.-lb per inch of circumference acting as shown.

Solution. We divide the bar into elements each of length $r\,d\theta$ so that acting on each element we have a small twisting couple of moment $mr\,d\theta$ which lies in a vertical radial plane normal to the axis of the bar at that point. The corresponding moment vectors then lie along the axis of the bar, as shown in Fig. 154*b*, and their geometric sum is the horizontal moment vector $\overline{ab}$ of length $2mr$. To balance this, we need a moment vector $\overline{ba}$ of the same magnitude. This can be supplied by two couples M_a and M_b each equal to mr and acting in vertical planes through A and B perpendicular to the wall, as indicated in Fig. 154*a*. These couples are the required reactions and they are created by tensions in the upper flange bolts together with compressions in the lower flange bolts.

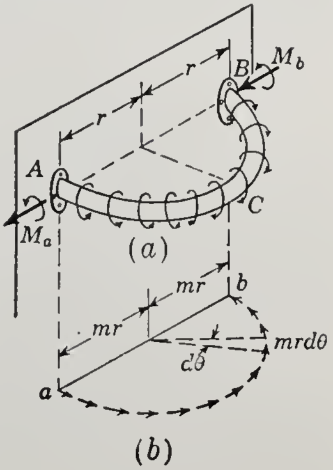


FIG. 154

PROBLEM SET 4.3

1. Three circular disks A, B, and C of radii $r_a = 15$ in., $r_b = 10$ in., and $r_c = 5$ in., respectively, are fastened at right angles to the ends of three rigidly connected arms which all lie in one plane as shown in Fig. A. If couples act on the disks A and B as shown in the figure, find the magnitude of the forces P of the couple that must be applied to the disk C and the angle α that the arm OC must make with the arm OB in order to have equilibrium. *Ans.* $P = 50$ lb; $\alpha = 143°08'$.

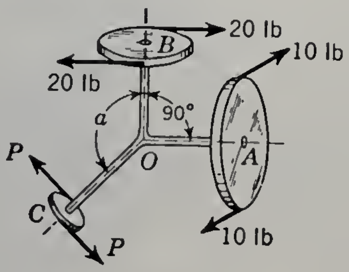


FIG. A

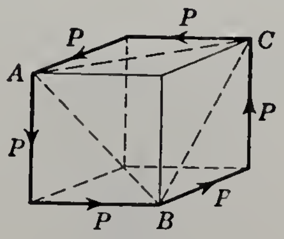


FIG. B

2. Six equal forces $P = 5$ lb act on a cube with edges of length $a = 5$ in., as shown in Fig. B. Determine the resultant of this system of couples. *Ans.* $M = 50\sqrt{3}$ in.-lb in the octahedral plane ABC.

3. Couples of moments M_1, M_2, M_3 act on a prismatic block as shown in Fig. C. What must be the relationship between the magnitudes of moment of these couples if the block is in equilibrium? *Ans.* $\frac{M_1}{\sin \alpha} = -\frac{M_2}{\sin \beta} = \frac{M_3}{\sin \gamma}$.

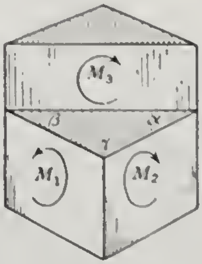


FIG. C

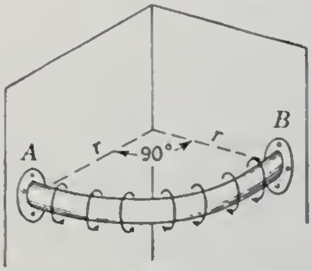


FIG. D

4. A piece of round pipe in the form of a circular quadrant of radius r is attached to two mutually perpendicular vertical walls by flanges A and B as shown in Fig. D. If the pipe is subjected to uniformly distributed twisting moment of intensity m as shown, find the reactions at A and B. Assume that the ends of the pipe can rotate freely inside the flanges so that no twist can be exerted on either flange. *Ans.* $M_a = M_b = mr$.

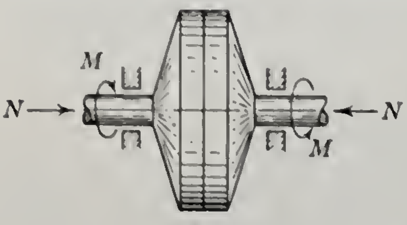


FIG. E

5. Calculate the magnitude of the total torque M that can be transmitted through the disk clutch shown in Fig. E if the disks are of radius r and the coefficient of friction is μ. Assume that the total normal force N is uniformly distributed over the area of contact between the two disks. Neglect friction in the bearings. *Ans.* $M = \frac{2}{3}\mu N r$.

6. Forces act along the edges of a prism as shown in Fig. F. Show that their resultant is a couple, and define it with reference to the coordinate axes xyz as shown. *Ans.* $M = 100$ in.-lb.

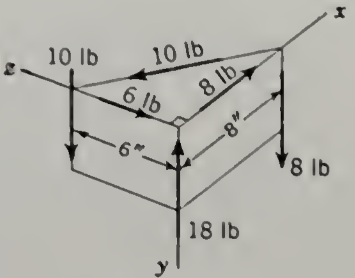


FIG. F

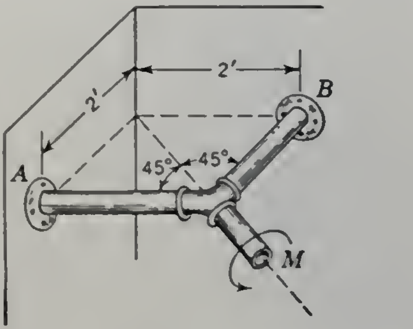


FIG. G

7. A pipe assembly is supported and loaded as shown in Fig. G. Find the reactions at A and B, assuming that the flanges cannot resist twist at A and B. Neglect the weight of the assembly. *Ans.* $M_A = M_B = M/\sqrt{2}$.

4.4. Parallel forces in space. Let us consider here a system of parallel forces $F_1, F_2, \ldots, F_5$ acting on a flat plate as shown in Fig. 155. Using successively the method of composition of two parallel forces discussed in Art. 2.1, we readily find the resultant R_1 of the forces F_1, F_2, F_3, acting upward, and the resultant R_2 of the forces F_4, F_5, acting downward. We see that for any such system we shall always obtain two parallel but oppositely directed partial resultants R_1 and R_2, as shown. If these two forces are unequal in magnitude, they reduce to a *resultant force* $R = R_1 - R_2$, in the direction of the larger force. If they are equal in magnitude but noncollinear, they represent a *resultant couple.* If they are equal in magnitude and collinear, they are in *equilibrium.* Thus a system of parallel forces in space may reduce to (1) a resultant force, (2) a resultant couple, or (3) a state of equilibrium.

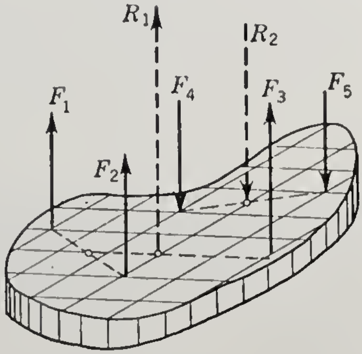


FIG. 155

To distinguish these three possibilities analytically, we refer the forces to a coordinate system x, y, z, and denote the given forces by $Z_1, Z_2, \ldots, Z_n$, as shown in Fig. 156. From the preceding discussion it follows that, if there is a resultant force, it is equal to the algebraic sum of the given forces. The line of action of this resultant force may be found from the condition that its moment with respect to each of the coordinate axes x and y is equal to the algebraic sum of the corresponding moments of all the given forces with respect to the same axis, in accordance with Varignon's theorem. Denoting by x_i and y_i the coordinates of the point of application of any force Z_i and by $\bar{x}$ and $\bar{y}$ the coordinates of the point of application of the resultant Z, this resultant force may be defined analytically by the equations

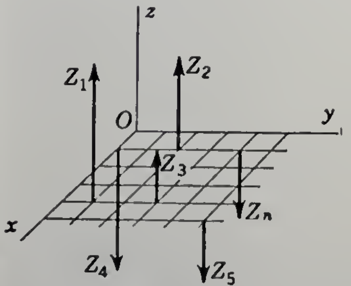


FIG. 156

$$Z = \Sigma Z_i \qquad \bar{x} = \frac{\Sigma(Z_i x_i)}{\Sigma Z_i} \qquad \bar{y} = \frac{\Sigma(Z_i y_i)}{\Sigma Z_i} \tag{24}$$

where the summations are understood to include all forces in the system.

When the algebraic sum of the given forces is equal to zero, the possibility of a resultant force vanishes but there remains the possibility of a resultant couple. To define this couple if it exists, the algebraic sum of the moments of the given forces with respect to each of the coordinate axes x and y must be calculated. If either one of these sums is different from zero, the system reduces to a resultant couple and the calculated sums of moments are its components in the yz and xz planes. Denoting these components by M_x and M_y, respectively, the conditions that the system reduce to a resultant couple may be expressed analytically as follows:[1]

$$\Sigma Z_i = 0 \qquad M_x = \Sigma(Z_i y_i) \qquad M_y = -\Sigma(Z_i x_i) \tag{25}$$

Having the rectangular components M_x and M_y of the resultant couple M, its magnitude and the orientation of its plane can be found by using Eqs. (c) and (d) of Art. 4.3. If either of the above sums of moments is zero, it indicates that the resultant couple lies in one of the coordinate planes.

If both the above sums of moments are zero, the possibility of a resultant couple vanishes also, and we conclude that the conditions of equilibrium for any system of parallel forces in space are expressed by the equations

$$\Sigma Z_i = 0 \qquad \Sigma(Z_i y_i) = 0 \qquad \Sigma(Z_i x_i) = 0 \tag{26}$$

In using Eqs. (26) to determine reactions exerted on a supported body under the action of applied parallel loads, it must be realized that with three equations of equilibrium, we can only find three unknowns, usually the magnitudes of three reactive forces whose lines of action are given.

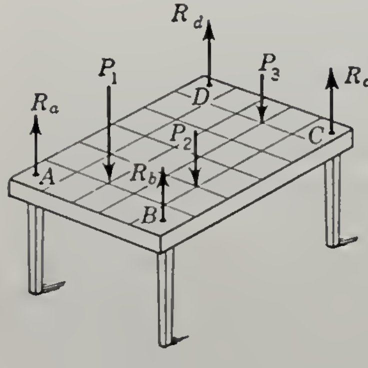


FIG. 157

Consider, for example, the case of an ordinary table with four legs which supports several vertical loads, as shown in Fig. 157. Since the legs will all be under simple compression, they will exert vertical reactions R_a, R_b, R_c, R_d at the four corners of the table. We have then, acting on the table top as a free body, a system of parallel forces in space that are in equilibrium. However, the four unknown reactions cannot be found from Eqs. (26), and the problem is *statically indeterminate.*

Intuitively, we may feel that each leg carries a definite compressive force and that we should be able to determine these axial forces by statics, but this is not the case. There are many ways in which we can divide the total load $P_1 + P_2 + P_3$ among the four legs so as to satisfy Eqs. (26) and we have no way of knowing from statics which is the correct distribution. The correct distribution depends on the flexibility of the top of the table, the floor, the legs, etc., in short, upon elastic deformations of the bodies which we do not consider in statics.

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[1] The minus sign in the expression for M_y is necessary in order to make the right-hand rule for sign of moment compatible with conventional positive coordinates.

EXAMPLES

1. A shaft bent in two right angles rests in open half bearings at A and B and is tied down by a vertical string CE, as shown in Fig. 158. Acting at the free end D is a vertical load P. Find the tensile force S in the string and the reactions R_a and R_b at the bearings if the dimensions are as shown in the figure.

Solution. Neglecting friction, the bearing reactions will be vertical and we obtain a system of parallel forces in space that are in equilibrium. With B as origin and coordinate axes directed as shown, Eqs. (26) become

$$\begin{aligned} R_a + R_b - P - S &= 0 \\ -Pc - R_a d + S(c + d) &= 0 \\ Pa - Sb &= 0 \end{aligned}$$

From the last equation, we see that $S = Pa/b$. Substituting this in the other two equations and solving for R_a and R_b, we find

$$\begin{aligned} R_a &= P\frac{a}{b}\left[1 + \frac{c}{d}\left(1 - \frac{b}{a}\right)\right] \\ R_b &= P\left[1 + \frac{c}{d}\left(1 - \frac{a}{b}\right)\right] \end{aligned} \tag{b}$$

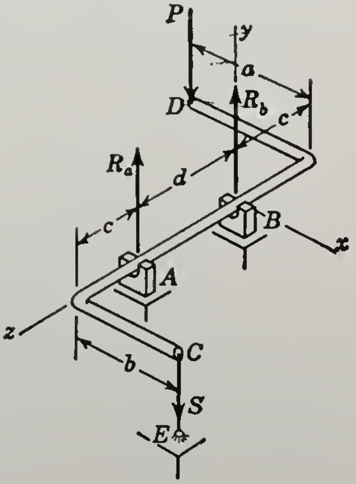

FIG. 158

Numerical values of R_a and R_b for given dimensions of the system can now be calculated from these formulas. We note, since the bearings are open, that negative values of R_a and R_b cannot be produced. A study of expressions (b) shows that the reactions will both be positive, i.e., up, if the quantities within the brackets are positive. This requires

$$\frac{c + d}{c} > \frac{a}{b} > \frac{c}{c + d} \tag{c}$$

Geometrically conditions (c) mean that the line CD joining the two ends of the shaft must cut the central portion of the shaft somewhere between A and B.

2. A movable crane mounted on three wheels, the bearings of which form an equilateral triangle ABC with sides of length a, rests on a horizontal track as shown in Fig. 159. The distribution of the weight Q of the crane itself is such that its center of gravity E is vertically above the centroid of the equilateral triangle ABC. For all possible values of the angle α that the vertical plane of the boom can make with the vertical middle plane of the crane, determine the corresponding maximum values of the load P that can be suspended from point D without causing the crane to tip from the track.

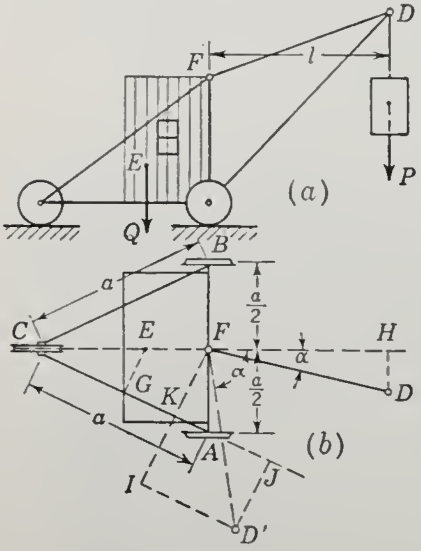

FIG. 159

Solution. For small values of the angle α, as indicated by the position FD of the boom (Fig. 159*b*), the condition of impending tipping of the crane about the axis AB with the wheel C lifting from the track will represent the criterion for the determination of the critical value of the load P, while for larger values of the angle α, as indicated by the position FD' of the boom, the condition of impending tipping of the crane about the axis AC with the wheel B lifting from the track will represent the criterion for the determination of the critical value of the load P. Hence, the possibility of tipping of the crane about each of these axes must be investigated separately and the corresponding critical values of the load P compared to determine within what range of values of the angle α each criterion holds.

Considering first the condition of impending tipping of the crane about the axis AB and remembering that in such case there is no pressure between the wheel C and the track, we find, by equating to zero the algebraic sum of the moments of all forces with respect to the axis AB,

$$P_{cr} \cdot FH - Q \cdot FE = 0$$

Using the given dimensions of the crane, this becomes

$$P_{cr} l \cos \alpha - Q \frac{a}{2\sqrt{3}} = 0$$

From this expression, we find

$$P_{cr} = \frac{Qa}{2\sqrt{3}\, l \cos \alpha} \tag{d}$$

It will be noted that the above value of P_{cr} varies with α from a minimum value of $Qa/2\sqrt{3}\, l$, when $\alpha = 0$, to a value of infinity when $\alpha = \pi/2$. However, before the value $\alpha = \pi/2$ is reached, tipping of the crane about the axis AB ceases to be the criterion of equilibrium, and we therefore proceed now to investigate the condition of impending tipping about the axis AC, which governs for large values of α as indicated by such a position of the boom as FD' in Fig. 159*b*. Remembering that in this case there will be no pressure between the track and the wheel B and proceeding as before, we obtain

$$P_{cr} \cdot D'J - Q \cdot EG = 0$$

or

$$P_{cr}(FI - FK) - Q \cdot EG = 0$$

Using the given dimensions, this becomes

$$P_{cr}\left[l \cos (120° - \alpha) - \frac{\sqrt{3}\,a}{4}\right] - \frac{Qa}{2\sqrt{3}} = 0$$

From this expression, we find

$$P_{cr} = \frac{Qa}{3l \sin \alpha - \sqrt{3}\,l \cos \alpha - \frac{}{2}a} \tag{e}$$

To determine the value of α below which Eq. (d) should be used and above which Eq. (e) should be used, we simply equate the two values of P_{cr} and obtain, for determining this value of α, the following equation:

$$\sin \alpha = \frac{a \pm \sqrt{48l^2 - 3a^2}}{8l} \tag{f}$$

For the particular case where $a = l$, Eq. (f) reduces to

$$\sin \alpha = \frac{1 + \sqrt{45}}{8} = +0.964 \qquad \text{or} \qquad \alpha = 74°30'$$

In this case the negative sign in Eq. (f) has no significance, since, on account of the construction of the crane, it is impossible for α to have a value greater than 90°. For the case where $a = l$, the position $\alpha = 0$ is the most serious, and from Eq. (d) the critical value of the load for this position is found to be $P_{cr} = Q/2\sqrt{3}$.

3. A horizontal cylindrical water trough has a quarter-circular cross section as shown in Fig. 160. If the trough is full, find the resultant water pressure P exerted on the end bulkhead OAB and the coordinates $\bar{x}$ and $\bar{y}$ defining its point of application.

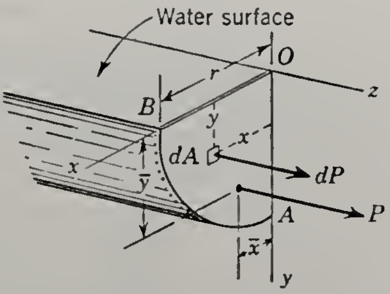

FIG. 160

Solution. The water pressure exerted against the inside face of the bulkhead acts normal to that surface and its intensity increases linearly with the depth. Thus taking an element of area dA with the coordinates x, y, as shown, we have for the corresponding element of pressure

$$dP = wy\,dA \tag{g}$$

where w is the weight per unit volume of water. These elements of pressure represent a system of parallel forces in space and, with proper changes in notation, we may use Eqs. (24) to find their resultant. It is only necessary to replace the summations of Z_i in these equations by corresponding integrals of dP. Thus from the first equation

$$P = \int dP = w\int y\,dA = wAy_c \tag{h}$$

where A is the total area of the bulkhead and y_c is the depth to its centroid C (not shown in the figure). As we observed once before (see page 160), this resultant hydrostatic pressure is simply equal to the product of the area on which it acts and the intensity of pressure at its centroid. For a circular quadrant, we have $A = \pi r^2/4$, $y_c = 4r/3\pi$, and Eq. (h) gives $P = wr^3/3$.

To locate the point of application of this resultant P, we use the second and third of Eqs. (24), again with proper change of notation, and write

$$\begin{aligned} \bar{x} &= \frac{\int x\,dP}{\int dP} = \frac{w\int xy\,dA}{w\int y\,dA} = \frac{I_{xy}}{Ay_c} \\ \bar{y} &= \frac{\int y\,dP}{\int dP} = \frac{w\int y^2\,dA}{w\int y\,dA} = \frac{I_x}{Ay_c} \end{aligned} \tag{i}$$

In these expressions, the quantity $I_{xy} = \int xy\,dA$ is the *product of inertia* of the area of the bulkhead with respect to the xy axes and $I_x = \int y^2\,dA$ is the *moment of inertia* of this area with respect to the x axis. These properties of a plane area are fully discussed in Appendix I (see pages A.5 and A.11). From there, we find for a semicircular quadrant, as shown in Fig. 160, that $I_{xy} = r^4/8$ and $I_x = \pi r^4/16$. Substituting these values into Eqs. (i), we obtain

$$\bar{x} = \frac{3r}{8} = 0.375r \qquad \bar{y} = \frac{3\pi r}{16} = 0.589r$$

It should be noted that the point $(\bar{x},\bar{y})$, called the *center of pressure*, does not coincide with the centroid of the area OAB.

PROBLEM SET 4.4

1. A homogeneous rectangular plate $ABCD$ of width a, length b, and weight Q is supported horizontally by three vertical strings, as shown in Fig. A. Determine the axial forces S_1, S_2, and S_3 in the three supporting strings. *Ans.* $S_1 = S_3 = +Q/2$; $S_2 = 0$.

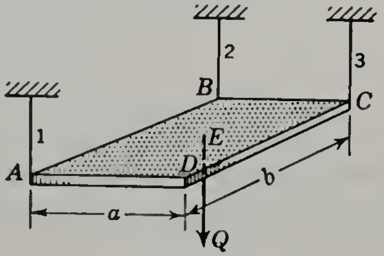

FIG. A

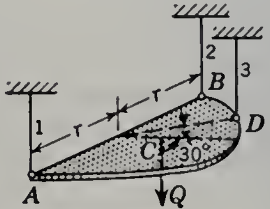

FIG. B

2. A homogeneous semicircular plate of weight Q and radius r is supported in a horizontal plane by three vertical strings as shown in Fig. B. Determine the tensile forces S_1, S_2, and S_3 in these strings. *Ans.* $S_1 = 0.38Q$; $S_2 = 0.13Q$; $S_3 = 0.49Q$.

3. A rigid rectangular frame supported in a horizontal plane by knife-edges at A, B, and C is subjected to vertical loads, as shown in Fig. C. Calculate the reactions at the supports. *Ans.* $R_a = 100$ lb; $R_b = 145$ lb; $R_c = 45$ lb.

4. A homogeneous flat triangular plate of weight W is supported on a horizontal plane by three ball bearings placed under its corners. Prove that the reactions at the points of support will each be equal to $W/3$.

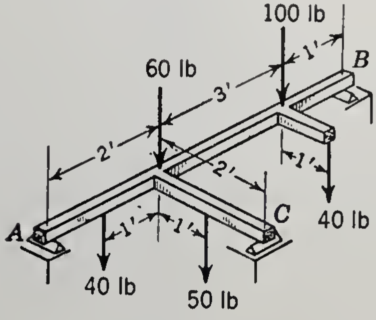

FIG. C

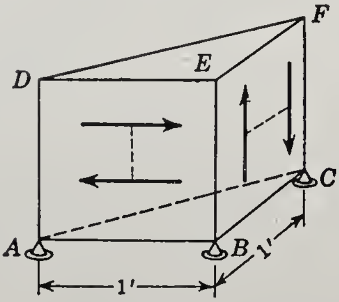

FIG. D

5. A homogeneous half cube weighing 100 lb is supported at points A, B, and C, as shown in Fig. D. Two couples, each of moment $M = 60$ in.-lb, act in the plane faces $ABDE$ and $BEFC$ as shown. Find the magnitudes of the reactions at A, B, and C. *Ans.* $R_a = 28.3$ lb; $R_b = 33.3$ lb; $R_c = 38.3$ lb.

6. Find the resultant of the system of five parallel forces acting as shown in Fig. E. The xy plane is subdivided into 1-in. squares. *Ans.* $Z = -15$ lb; $\bar{x} = 0$; $\bar{y} = 10$ in.

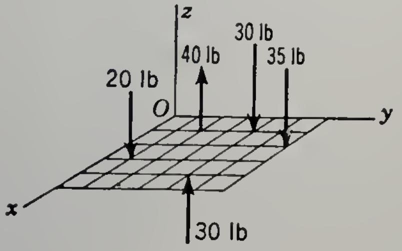

FIG. E

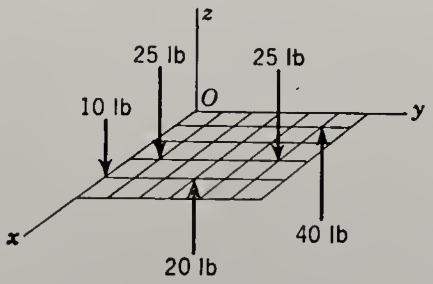

FIG. F

7. Find the resultant of the system of five parallel forces acting as shown in Fig. F. The xy plane is subdivided into 1-in. squares as shown. *Ans.* $M_x = 110$ in.-lb; $M_y = 70$ in.-lb.

8. Three identical bars, each of length l, are nailed together at their midpoints to form a triangular frame which is supported in a horizontal plane as

shown in Fig. G. Find the magnitudes of the reactions at A, B, and C if a vertical load P is applied at E as shown. Neglect the weights of the bars. *Ans.* $R_a = \frac{4}{7}P$; $R_b = \frac{1}{7}P$; $R_c = \frac{2}{7}P$.

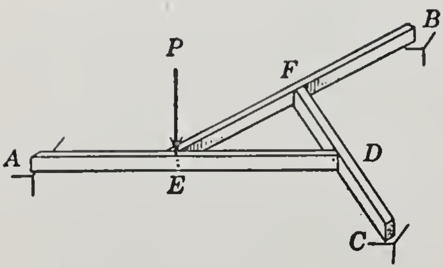

FIG. G

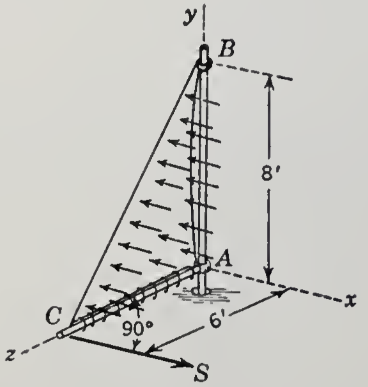

FIG. H

9. A triangular sailcloth attached to a vertical mast AB and a horizontal boom AC is subjected to uniformly distributed wind pressure of intensity p normal to its surface, as shown in Fig. H. Find the tension S in the horizontal stay rope at C if $p = 100$ psf. *Ans.* $S = 800$ lb.

4.5. Center of parallel forces and center of gravity. In Art. 2.3 we have seen that in the case of a given system of parallel forces applied to given points in a rigid body there is one and only one point, called the *center of parallel forces,* through which the resultant always passes regardless of the direction of the parallel lines of action of the forces.

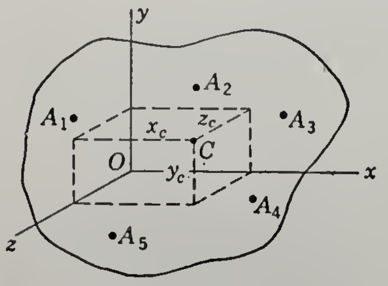

FIG. 161

The location of this center of parallel forces may be determined very conveniently from the condition that the moment of the resultant with respect to any axis must be equal to the algebraic sum of the moments of the given forces with respect to the same axis. Let A_1, A_2, . . . , A_n (Fig. 161) be any system of points in a rigid body, and let x_i, y_i, z_i be the coordinates of any point A_i referred to the rectangular coordinate axes x, y, z. Then to locate the center C of parallel forces for a given system of forces $F_1, F_2, \ldots,$ F_n applied at these points and acting in any direction, let us first imagine that the forces (not shown) act parallel to the yz plane. Then by taking moments with respect to an axis in this plane that is perpendicular to the direction of the forces and using the above statement regarding moments, we obtain for the perpendicular distance from the yz plane to the line of action of the resultant (and consequently to the center of parallel forces) the following expression:

$$x_c = \frac{\Sigma(F_i x_i)}{\Sigma F_i} \tag{27a}$$

Next, we imagine the parallel lines of action of the given forces to be parallel to the xz plane. Then proceeding as before, the perpendicular distance from this plane to the line of action of the resultant (and consequently to the center of parallel forces) is given by the equation

$$y_c = \frac{\Sigma(F_i y_i)}{\Sigma F_i} \tag{27b}$$

Finally, we imagine the given parallel forces to act through their points of application parallel to the xy plane. Then the perpendicular distance from this plane to the line of action of the resultant (and consequently to the center of parallel forces) is given by the equation

$$z_c = \frac{\Sigma(F_i z_i)}{\Sigma F_i} \tag{27c}$$

Since we know that the center of parallel forces is independent of the direction of the forces, we conclude that Eqs. (27) define the coordinates of this point for any direction of the forces.

Since the center of gravity of a body (see Art. 2.3) is the center of parallel gravity forces represented by the weights of the various particles of the body, it follows that the coordinates of the center of gravity of any body can be determined by the use of Eqs. (27). In the case of a body of homogeneous material, we conclude that the position of the center of gravity depends only upon the shape of the body and not upon its density. Thus the center of gravity of a body of uniform density is coincident with the centroid of the volume of space occupied by the body.

It follows from the form of Eqs. (27) that the center of gravity of a body of uniform density which has a plane of symmetry lies in that plane. If the body has two planes of symmetry, the center of gravity lies on the line of intersection of these planes. If the body has three planes of symmetry, the center of gravity lies at the point of intersection of these planes and is completely determined. Thus the center of gravity of a sphere of uniform density lies at the center of the sphere, the center of gravity of a right circular cylinder of uniform density lies at the mid-point of its geometric axis, the center of gravity of a cone of uniform density lies somewhere on its geometric axis, etc.

If a body may be considered as made up of several finite parts the centers of gravity of which, individually, are known, then to locate the center of gravity of the composite body, it is only necessary to determine the coordinates of the center of parallel forces represented by the weights of the several parts applied, respectively, at the known centers of gravity of these parts.

If the bounding planes or surfaces of a body can be defined analytically and the density of the material at each point in the body is known, the coordinates of the center of gravity can always be found by dividing the body into infinitesimal elements and then making the summations indicated in Eqs. (27) by integration.

EXAMPLES

1. Locate the center of gravity of the slender homogeneous wire *AOBD* (Fig. 162) which consists of the straight portions *AO* and *OB*, each of length r at right angles to each other and the portion *BD* representing a quarter of a circle of radius r the plane *OBD* of which is perpendicular to *AO*.

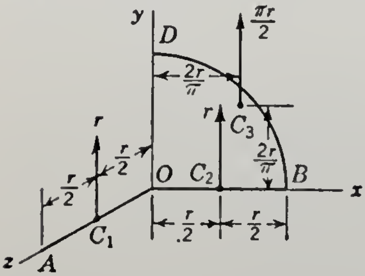

FIG. 162

Solution. Since the wire is slender and of uniform cross section, its center of gravity may be considered as coincident with the centroid of the composite line *AOBD* represented by its axis. Let us imagine then a system of parallel forces having magnitudes numerically equal to the lengths r, r, and $\pi r/2$ of the three portions of the line to be applied at the known centroids C_1, C_2, and C_3. First, letting these forces act parallel to the y axis as shown in the figure and using Eqs. (27*a*) and (27*c*), we obtain

$$x_c = \frac{(\pi r/2)(2r/\pi) + r(r/2)}{2r + \pi r/2} = \frac{3r}{4 + \pi} \qquad z_c = \frac{r(r/2)}{2r + \pi r/2} = \frac{r}{4 + \pi}$$

Next, letting the same forces act parallel to the z axis and using Eq. (27*b*), we obtain

$$y_c = \frac{(\pi r/2)(2r/\pi)}{2r + \pi r/2} = \frac{2r}{4 + \pi}$$

2. Locate the center of gravity of a homogeneous right circular cone of altitude h and radius of base r (Fig. 163).

Solution. Since the cone is of uniform density, we conclude at once that its center of gravity is coincident with the centroid C of its volume, and further, since the geometric axis OB is an axis of symmetry for the cone, we conclude that the centroid lies on this axis. Then it is only necessary to determine the coordinate x_c to have the centroid completely located, and to do this, the method of integration will be very convenient. We proceed as follows: Choosing the rectangular coordinate axes as shown in the figure and selecting as a typical element a thin disk of radius y and thickness dx which is at a distance x from the vertex O of the cone, we see that the volume of this element is $dV = \pi y^2\,dx$ which, since $y:x = r:h$, may be written $dV = (\pi r^2/h^2)\,x^2\,dx$. Then using Eq. (27a) and making the indicated summations by the calculus, we obtain

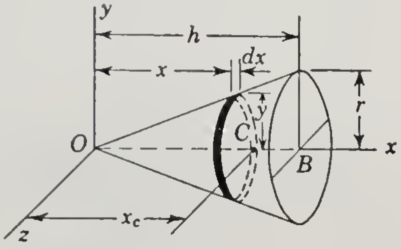

FIG. 163

$$x_c = \frac{\int_0^h \frac{\pi r^2}{h^2}\,x^3\,dx}{\int_0^h \frac{\pi r^2}{h^2}\,x^2\,dx} = \frac{\int_0^h x^3\,dx}{\int_0^h x^2\,dx} = \frac{h^4/4}{h^3/3} = \frac{3}{4}h$$

From this result, we conclude that the center of gravity of a homogeneous right circular cone lies on its geometric axis at a distance of one-quarter of the altitude from the plane of the base.

PROBLEM SET 4.5

1. A homogeneous slender wire 12 in. long is bent in two right angles as shown in Fig. A. Determine the coordinates of its center of gravity. *Ans.* $x_c = 2.0$ in.; $y_c = z_c = 0.67$ in.

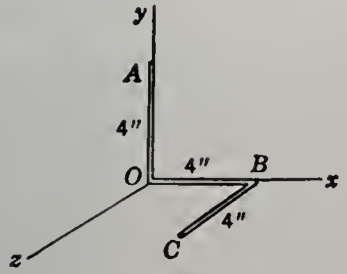

FIG. A

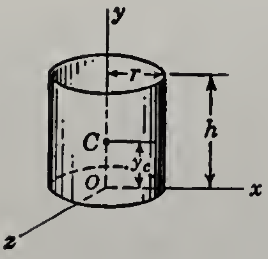

FIG. B

2. Determine the coordinate y_c of the center of gravity C of a right circular cylindrical can of height h and radius of base r if it is made of very thin metal of uniform thickness and density (Fig. B). The can is closed at the bottom and open at the top. *Ans.* $y_c = h^2/(2h + r)$.

3. A steel shaft of circular cross section has a circular steel hub pressed onto it as shown in Fig. C. For the dimensions shown in the figure, determine the distance x_c from the left end of the shaft to the center of gravity C of the composite body. *Ans.* $x_c = 6.28$ in.

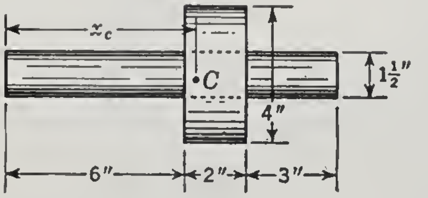

FIG. C

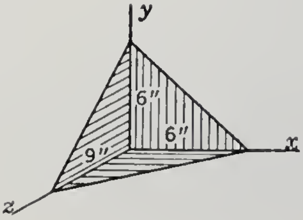

FIG. D

4. The corner of a rectangular box made of thin sheet metal of uniform thickness and density is cut off as shown in Fig. D. Determine the coordinates x_c, y_c, and z_c of the center of gravity of the corner. *Ans.* $x_c = 1.25$ in.; $y_c = 1.25$ in.; $z_c = 2.25$ in.

5. Referring to Fig. E, locate the centroid of the composite area consisting of a square in the xy plane, a triangle in the yz plane, and a circular quadrant in the xz plane. *Ans.* $x_c = 0.365a$; $y_c = 0.292a$; $z_c = 0.219a$.

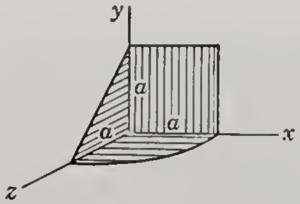

FIG. E

6. Prove that the center of gravity of any homogeneous pyramid with base area A and altitude h lies on the line joining the vertex of the pyramid with the centroid of the area of its base at a distance equal to one-quarter of the altitude from the plane of the base.

7. Determine the height z_c of the center of gravity of a right circular cone above the plane of the base if the density of the material at each point in the cone is proportional to the distance of that point from the plane of the base. *Ans.* $z_c = \frac{2}{5}h$.

8. Determine the distance x_c of the center of gravity of a homogeneous hemisphere of radius r from the plane of its base. *Ans.* $x_c = \frac{3}{8}r$.

9. From a solid homogeneous hemisphere of radius r, it is desired to cut out a cone the base of which coincides with that of the hemisphere. What should be the altitude h of this cone so that its vertex C will be the center of gravity of the remaining material? *Ans.* $h = 0.451r$

10. Determine the distance x_c of the center of gravity of a homogeneous truncated right circular cone from the plane of the base if the radius of the base is r_1, the radius of the top r_2, and the altitude of the truncated portion h.

Ans. $$x_c = \frac{h}{4}\,\frac{(r_1^2 + 2r_1r_2 + 3r_2^2)}{(r_1^2 + r_1r_2 + r_2^2)}.$$

11. A homogeneous body consists of a circular cylindrical portion of radius r attached to a hemispherical portion of radius r as shown in Fig. F. Determine the height h of the cylindrical portion if the center of gravity of the composite body lies at the center C of the circular plane face of the hemisphere. *Ans.* $h = r/\sqrt{2}$.

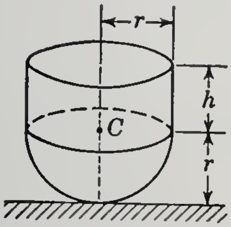

FIG. F

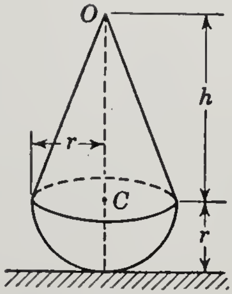

FIG. G

12. A homogeneous body consists of a right circular conical portion attached to a hemispherical portion of radius r as shown in Fig. G. Determine the altitude h of the cone if the center of gravity of the composite body coincides with the center C of the circular base of the cone. *Ans.* $h = \sqrt{3}\, r$.

13. Determine the coordinate y_c of the center of gravity of a steel rivet having the dimensions shown in Fig. H. Assume the head of the rivet to be hemispherical. *Ans.* $y_c = 1.15$ in.

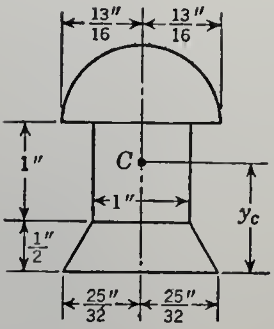

FIG. H

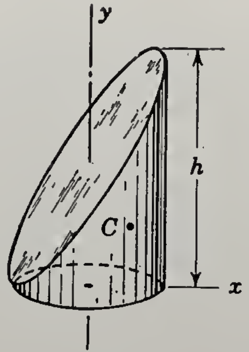

FIG. I

14. Determine the distance x_c of the center of gravity C of a hemispherical thin shell of radius r from the plane of the base. *Ans.* $x_c = r/2$.

***15.** Determine the coordinates x_c, y_c of the centroid C of a homogeneous half of a right circular cylinder of radius r and height h as shown in Fig. I. *Ans.* $x_c = r/4$; $y_c = 5h/16$.

4.6. General case of forces in space. *Composition.* We consider now the most general case of a system of forces in space which are neither concurrent nor parallel. Such a system can always be reduced to a *resultant force,* applied at an arbitrary point, and a *resultant couple.* To show this we consider in Fig. 164*a* several forces F_1, F_2, F_3, applied to a body at points A, B, C, respectively.

Choosing coordinate axes through any convenient point O in the body, we pass a plane through the line of action of each force and the origin O, as shown. In each of these planes, we resolve the corresponding force F_i into an equal parallel force F_i' at O and a couple $F_i'F_i''$, as explained in Art. 2.1, page 67. This operation for the force F_1 is illustrated in Fig. 164*b*. We see that the couple is equal to the moment of the force F_1 with respect to point O and can be represented by the moment vector M_1 normal to the shaded plane and directed in accordance with the right-hand rule, as shown. Repeating this operation for each of the given forces in Fig. 164*a*, we obtain finally a statically equivalent system of forces F_1', F_2', F_3', concurrent at the chosen point O, and a system of couples M_1, M_2, M_3, as shown in Fig. 164*c*. Each force F_i' is equal and parallel to the given force F_i and each moment vector M_i is normal to the plane defined by the line of action of the given force F_i and point O. We already know from Art. 4.1 that the concurrent forces F_1', F_2', F_3' reduce to a resultant force R defined by Eqs. (a) to (d), page 170, and from Art. 4.3 that the system of couples M_1, M_2, M_3 reduces to a resultant couple M, as defined by Eqs. (a) to (d), page 187. Thus any system of forces in space reduces to a resultant force and a resultant couple, as stated at the beginning of this discussion.

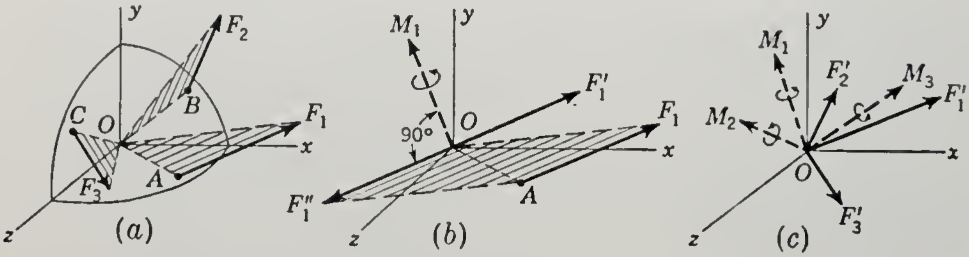

FIG. 164

The resultant force vector R and the resultant moment vector M obtained in the above manner always define a plane through the chosen origin O. We now take a new set of coordinate axes $x'y'z'$ through O such that the plane defined by the vectors $\bar{R}$ and $\bar{M}$ becomes the $x'z'$ plane, as shown in Fig. 165*a*. In this plane, we can always resolve the moment vector $\bar{M}$ into two rectangular components: M', coinciding with the line of action of R, and M'' perpendicular thereto, as shown in the figure. Now at point O_1 on the y' axis, we may introduce two oppositely directed collinear forces R' and R'' each equal and parallel to R without altering the action of the system. If we choose the distance $OO_1 = M''/R$, we see that the original force R at O and the force R'' at O_1 constitute a couple which balances the component M'' of the original couple M. Thus we have left only a force R' applied at O_1 and a couple M' in a plane normal to the line of action of R'. The moment vector $\bar{M}'$ can now be moved to point O_1, and we have finally the simplified system shown in Fig. 165*b*, where the resultant couple acts in a plane normal to the line of action of the resultant force. This simplest possible representation of a system of forces in space is sometimes called a *wrench*, and point O_1 is called the *true center* of the system.[1]

Equations of Equilibrium. From the preceding discussion, we conclude that, in the general case of a system of forces in space, equilibrium can exist only if both the resultant force R and the resultant couple M vanish. Thus the equations of equilibrium, as already obtained in Arts. 4.1 and 4.3, are as follows:

$$\begin{array}{lll} \Sigma X_i = 0 & \Sigma Y_i = 0 & \Sigma Z_i = 0, \\ \Sigma (M_x)_i = 0 & \Sigma (M_y)_i = 0 & \Sigma (M_z)_i = 0 \end{array} \tag{28}$$

These six equations of equilibrium apply to any system of forces, and all cases discussed previously can be obtained from this general case.

If the forces are all parallel and we take the z axis parallel to them, the first, second, and last equations will always be satisfied and we arrive at Eqs. (26) of Art. 4.4.

If the forces all intersect in one point and we choose this point as the origin of coordinates, the last three of Eqs. (28) will always be satisfied and we arrive at Eqs. (20) of Art. 4.1.

If the forces are all in one plane which we can take as the xy plane, the third, fourth, and fifth of Eqs. (28) will always be satisfied and we arrive at Eqs. (18) of Art. 3.2.

In a similar manner it can be shown that for the cases of parallel forces in a plane and concurrent forces in a plane, Eqs. (28) can be reduced to Eqs. (11) of Art. 2.2 and Eqs. (3) of Art. 1.4, respectively.

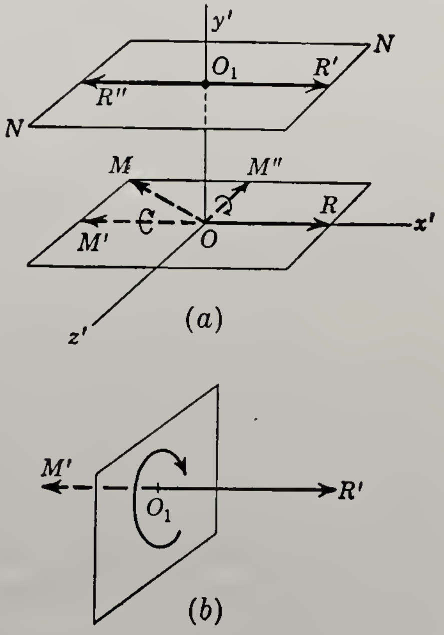

FIG. 165

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[1] Although of academic interest this reduction of a system of forces in space to a *wrench* is not essential.

We see from Eqs. (28) that there are only six independent conditions of equilibrium for the general case of a system of forces in space. This means that in dealing with the equilibrium of a constrained body under the action of applied loads, we cannot determine more than six unknown elements pertaining to the reactive forces. If we encounter more than this number the problem will be *statically indeterminate.*

Even when there are only six unknowns, the problem of solving six simultaneous equations can be rather troublesome. Thus, in writing conditions of equilibrium for a system of forces in space, we always try to select axes upon which to project the forces or about which to take moments in such a way that each equation will contain as few unknowns as possible. It is not at all necessary to adhere to orthogonal coordinate axes x, y, z for projections and moments. If the system of forces is known to be in equilibrium, the algebraic sum of their projections or moments with respect to any axis must vanish. The several examples which follow will serve to illustrate the best procedure to use in each case.

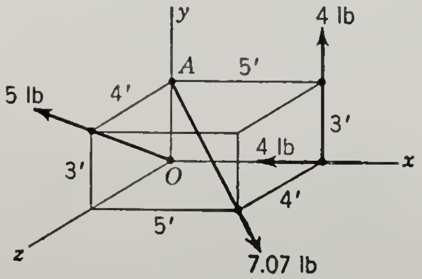


FIG. 166

EXAMPLES

1. Four forces act on a rectangular parallelepiped as shown in Fig. 166. Reduce the system to a resultant force R applied at O and a resultant couple M.

Solution. We first calculate the projections and moments of each force with respect to the coordinate axes and tabulate the data as follows:

F_i	X_i	Y_i	Z_i	$(M_x)_i$	$(M_y)_i$	$(M_z)_i$
4	−4	0	0	0	0	0
5	0	+3	+4	0	0	0
7.07	+5	−3	+4	+12	0	−15
4	0	+4	0	0	0	+20
Σ	+1	+4	+8	+12	0	+ 5

Using the summations from this table in the formulas for resultant force and couple, we obtain

$$R = \sqrt{1^2 + 4^2 + 8^2} = 9 \text{ lb} \qquad M = \sqrt{12^2 + 5^2} = 13 \text{ ft-lb}$$
$$\alpha = \arccos \tfrac{1}{9} = 83°37' \qquad \alpha' = \arccos \tfrac{12}{13} = 22°33'$$
$$\beta = \arccos \tfrac{4}{9} = 63°37' \qquad \beta' = \arccos 0 = 90°00'$$
$$\gamma = \arccos \tfrac{8}{9} = 27°16' \qquad \gamma' = \arccos \tfrac{5}{13} = 67°23'$$

2. A horizontal shaft supported in bearings at A and B has attached at right angles to it two pulleys which are loaded as shown in Fig. 167. Neglecting friction, find the magnitude of the load P necessary to hold the system in equilibrium if $Q = 600$ lb. Determine also the horizontal and vertical components of the bearing reactions as indicated in the figure.

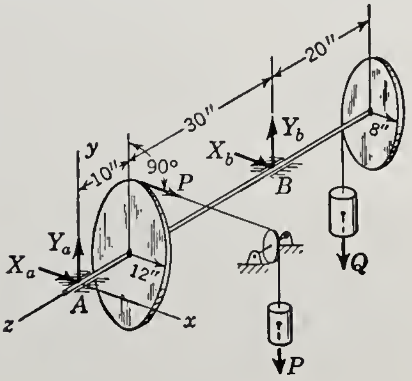


FIG. 167

Solution. We begin by considering the shaft together with the two pulleys as a free body. Neglecting friction, we conclude at once that the bearing reactions at A and B cannot have components in the direction of the axis of the shaft and hence each of these reactions is completely defined by horizontal and vertical components as shown in the figure. Choosing rectangular coordinate axes as shown and using the given dimensions, Eqs. (28) become

$$\begin{aligned} X_a + X_b + P &= 0 \\ Y_a + Y_b - Q &= 0 \\ \cdots\cdots\cdots &\cdots \\ 40Y_b - 60Q &= 0 \\ -10P - 40X_b &= 0 \\ 8Q - 12P &= 0 \end{aligned} \qquad \text{(a)}$$

Then, taking $Q = 600$ lb, we find $P = 400$ lb, $X_b = -100$ lb, $X_a = -300$ lb, $Y_b = 900$ lb, $Y_a = -300$ lb. The negative signs simply indicate that the reactions X_a, X_b, and Y_a act oppositely to the directions assumed in the figure.

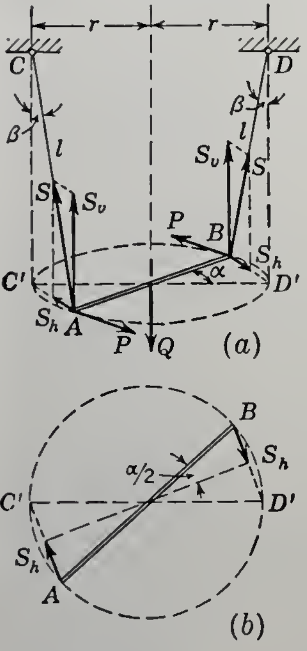


FIG. 168

3. A prismatic bar AB of length $2r$ and weight Q is suspended from a horizontal ceiling by two vertical strings AC and BD of equal lengths l. When subjected to the action of two horizontal forces P forming a couple of moment M, the bar rotates by an angle α in the horizontal plane, as shown in Fig. 168. Determine the magnitude of the moment M of this couple and the tensile force S in each string if the angle α is given.

Solution. Considering the bar AB as a free body, we see that it is in equilibrium under the action of the vertical gravity force Q, the horizontal couple PP and the inclined forces S representing the reactions of the supporting strings on the ends of the bar. Denoting by β the angle that each of the inclined strings makes with the vertical, we now replace each of the forces S by its rectangular components

$$S_h = S \sin \beta \qquad \text{and} \qquad S_v = S \cos \beta$$

which are horizontal and vertical, respectively, as shown in the figure

Equating to zero the algebraic sum of the projections of all forces on a vertical axis, we obtain

$$2S \cos \beta - Q = 0 \tag{b}$$

Noting from Fig. 168*b* that the two horizontal components S_h constitute a couple in the horizontal plane $AC'BD'$ with the arm equal to $2r \cos (\alpha/2)$ and equating to zero the algebraic sum of the moments of all forces with respect to the vertical line of action of the force Q, we obtain

$$M - S \sin \beta \, 2r \cos \frac{\alpha}{2} = 0 \tag{c}$$

Finally, from the figure it is evident that

$$\sin \beta = \frac{2r}{l} \sin \frac{\alpha}{2} \qquad \cos \beta = \sqrt{1 - \frac{4r^2}{l^2} \sin^2 \frac{\alpha}{2}} \tag{d}$$

Substituting the values of $\sin \beta$ and $\cos \beta$ from expressions (d) into Eqs. (b) and (c) above and solving, respectively, for the tensile force S and the moment M of the couple, we obtain

$$S = \frac{Ql}{2\sqrt{l^2 - 4r^2 \sin^2 (\alpha/2)}} \qquad M = \frac{Qr^2 \sin \alpha}{\sqrt{l^2 - 4r^2 \sin^2 (\alpha/2)}} \tag{e}$$

By way of checking these results it will be noted that the expression for S has the dimension of force while the expression for M has the dimension of force times length. Further, for the particular case where $\alpha = 0$, Eqs. (e) reduce to $S = Q/2$ and $M = 0$, as they obviously should. It is interesting to note also that for the case where $\alpha = \pi = 180°$ we obtain again $M = 0$ but $S = Ql/2\sqrt{l^2 - 4r^2}$. In this case the strings are crossed in one plane and a consideration of the equilibrium of the corresponding coplanar system of forces leads to this same result for the tensile force S.

4. A rectangular parallelepiped *ABCDEFGH* with dimensions *a*, *b*, and *c* is supported by six hinged bars arranged as shown in Fig. 169. Determine the forces produced in these bars due to the action of a horizontal force P applied at the corner G and acting along the edge GH.

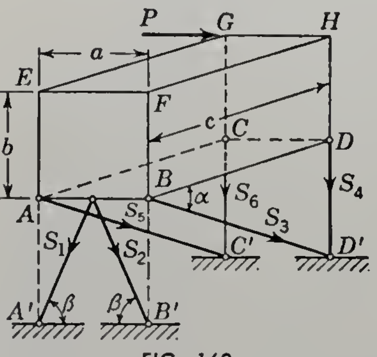


FIG. 169

Solution. We begin by considering the parallelepiped as a free body. Replacing the six supporting bars by the reactions that they exert on the body and remembering that these forces must act along the axes of the bars which produce them, we obtain the free-body diagram as shown in the figure. In making this diagram, we assume all bars to be in tension, as indicated by the arrows. If, later, in calculation we find negative values for any of the forces S_i, this will simply indicate that the corresponding bars are in compression.

Writing, first, an equation of moments with respect to the axis BF which is parallel to or intersected by all bars except bar 5, we obtain

$$-(S_5 \cos \alpha)\, a - Pc = 0$$

from which $S_5 = -Pc/a \cos \alpha$; hence, compression. In the same way, by considering moments of all forces with respect to the axis AE, we conclude that $S_3 = +Pc/a \cos \alpha$, tension. Already knowing the value of S_5 this last result may also be obtained directly by equating to zero the algebraic sum of the projections of all forces on the axis AC.

Now by taking moments of all forces, first with respect to the axis AB and again with respect to the axis $C'D'$, we conclude, successively, that

$$S_4 = -S_6 \qquad \text{and} \qquad S_1 = -S_2$$

Equating to zero the algebraic sum of the projections of all forces on the axis AB, we now obtain

$$-S_1 \cos \beta + S_2 \cos \beta + P = 0$$

from which $S_1 = -S_2 = +P/2 \cos \beta$.

Finally, equating to zero the algebraic sum of the moments of all forces with respect to the axis AC, we obtain

$$(S_1 \sin \beta)\frac{a}{2} + (S_2 \sin \beta)\frac{a}{2} + (S_3 \sin \alpha)\, a + S_4 a + Pb = 0$$

from which, using the previously determined values of S_1, S_2, and S_3, we find $S_6 = -S_4 = P(b + c \tan \alpha)/a$.

Summarizing the results, we have

$$S_3 = -S_5 = \frac{Pc}{a \cos \alpha} \qquad S_1 = -S_2 = \frac{P}{2 \cos \beta} \qquad S_6 = -S_4 = P\,\frac{b + c \tan \alpha}{a}$$

5. Determine the forces produced in the six supporting bars of the rectangular parallelepiped shown in Fig. 170*a* if a horizontal force P acts along EF as shown.

Solution. The solution of this problem can be greatly simplified in the following manner: First we add at hinge A two more forces P that are equal and parallel to the given force P at E as shown in Fig. 170*a*. The addition of these two forces which are in equilibrium does not change the action of the system, but we may now consider that we have a force P at A and a couple consisting of the given force P at E and the opposite force P at A. This couple, the moment of which in the end plane of the parallelepiped is Pb, can now be replaced by a couple of the same moment, as represented by the forces Pb/a applied at hinges C and D as shown in Fig. 170*b* (see Art. 4.3). Since all forces now act at hinges A, C, and D, we obtain by successive considerations of the equilibrium of these hinges the following results: $S_4 = S_6 = Pb/a$, $S_3 = S_5 = 0$, $S_1 = P \csc \alpha$, and $S_2 = P \cot \alpha$. See Fig. 170*b* for directions of reactions.

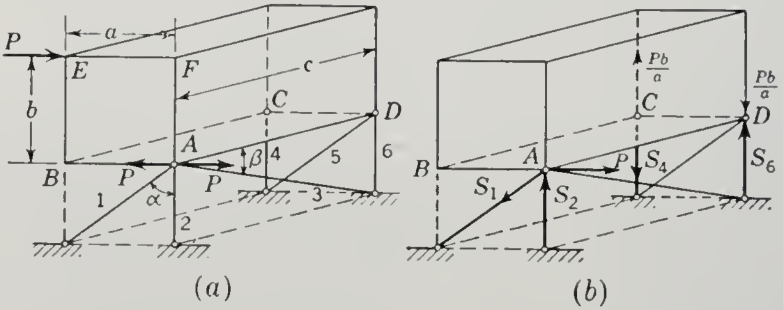


FIG. 170

PROBLEM SET 4.6

1. Three forces act on a rectangular parallelepiped as shown in Fig. A. Reduce the system to a resultant force R applied at O and a resultant couple M. *Ans.* $R = 66$ lb; $M = 823$ ft-lb.

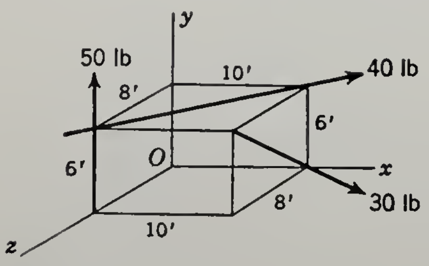


FIG. A

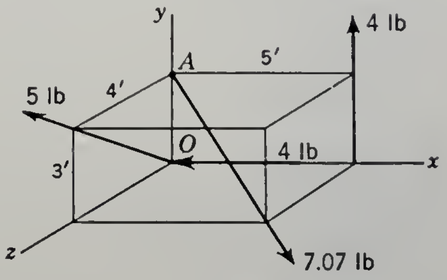


FIG. B

2. Referring to Fig. B, determine the coordinates of a point O_1 in the xy plane at which the resultant force R should be applied in order that the corresponding resultant couple M will lie in a plane normal to the line of action of R. What is the magnitude of moment of this couple? *Ans.* $x_1 = 0.321$ ft; $y_1 = 1.42$ ft; $M = 5.78$ ft-lb.

3. Show that the force system in Fig. B can be reduced to a resultant force R_1 applied at O and a horizontal force R_2 applied at A. What are the magnitudes of these forces? *Ans.* $R_1 = 6.25$ lb; $R_2 = 4.33$ lb.

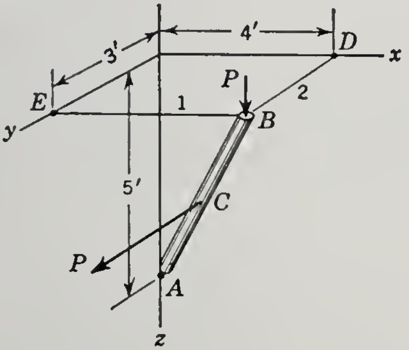


FIG. C

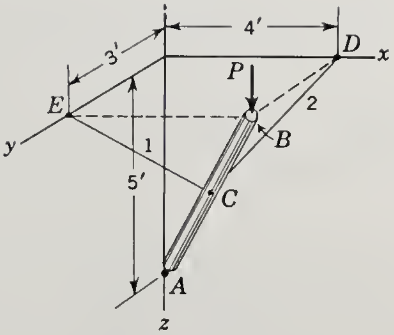


FIG. D

4. Referring to Fig. C, find the tensions S_1 and S_2 in the two horizontal strings 1 and 2 that support the upper end of the mast AB, assuming a spherical hinge at A. The force P at B is vertical, the one at the mid-point C is parallel to the y axis. *Ans.* $S_1 = 0.8P$; $S_2 = 1.1P$.

5. Find the tensions S_1 and S_2 in the strings CD and CE attached to the mast AB at its mid-point C, as shown in Fig. D. The load P is vertical. *Ans.* $S_1 = S_2 = \sqrt{2}\,P$.

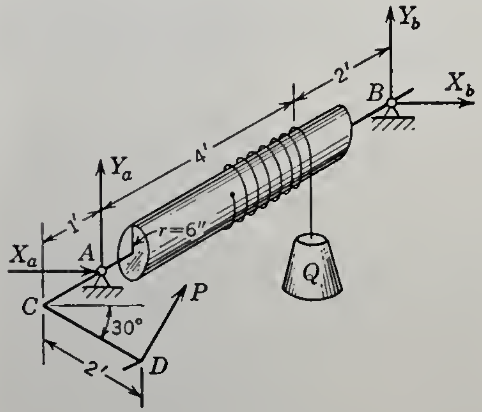


FIG. E

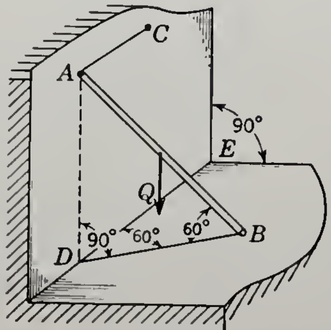


FIG. F

6. To hoist a load Q the windlass shown in Fig. E is used. Assuming that the force P applied to the handle of the crank acts at right angles to the arm CD and in a plane perpendicular to the horizontal axis AB, find the magnitude of this force and the components of the reactions at the bearings if equilibrium exists for the position and dimensions as shown. Neglect friction. *Ans.* $P = 0.250Q$; $X_b = 0.021Q$; $Y_b = 0.703Q$; $X_a = -0.146Q$; $Y_a = 0.081Q$.

7. A prismatic bar AB of weight Q and length l is supported by a smooth horizontal floor and a smooth vertical wall and is held in equilibrium by two horizontal strings AC and BD attached to its ends as shown in Fig. F. Determine the wall and floor reactions R_a and R_b, respectively, and the tensile force in each string. *Ans.* $R_a = Q/4$; $R_b = Q$; $S_a = Q/4\sqrt{3}$; $S_b = Q/2\sqrt{3}$.

8. The rectangular lid $ABDE$ of a chest has a weight $W = 48$ lb and is propped open with a stick DF as shown in Fig. G. Determine the compressive force S in the stick and the components of the hinge reactions at A and B. Neglect friction in the hinges and assume that the lid is free to slide in the direction of the axis AB. *Ans.* $S = 13.86$ lb; $X_a = 6.93$ lb; $X_b = 0$; $Y_a = 12.00$ lb; $Y_b = 24.00$ lb.

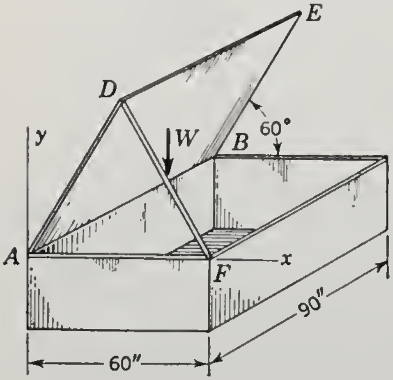


FIG. G

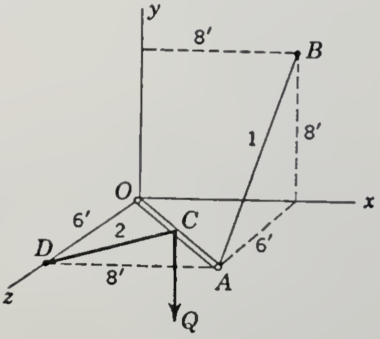


FIG. H

9. A slender bar OA hinged to a fixed point O is supported in a horizontal plane by strings AB and CD as shown in Fig. H. Calculate the tensile forces S_1 and S_2 induced in these strings by a vertical force Q applied to the bar at its mid-point C. *Ans.* $S_1 = S_2 = \frac{5}{8}Q$.

10. A bar AB of length $2r$ is pivoted at its center C and is free to rotate in a horizontal plane as shown in Fig. I. Attached to the end B of this bar is a string that passes through a screw eye at D and carries at its end a load Q as shown. Neglecting friction, find the magnitude of a horizontal force P, applied to the end A of the bar and acting at right angles to its axis, that will hold the bar in equilibrium when it makes an angle α with the position $A'B'$. Assume $r = 4$ ft, $h = 3$ ft, $\alpha = 45°$. *Ans.* $P = 0.659Q$.

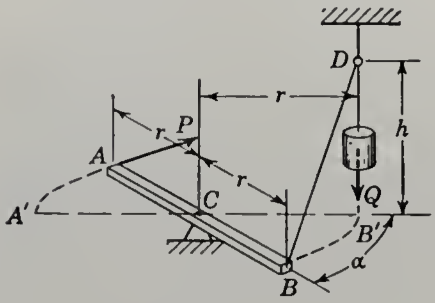


FIG. I

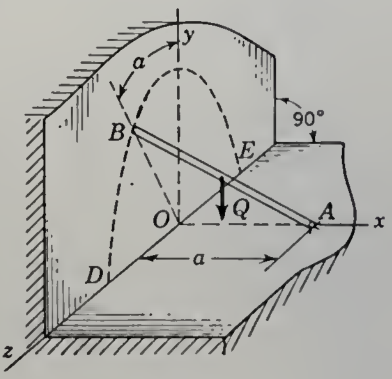


FIG. J

11. A prismatic bar AB of length l and weight Q is pivoted at one end A and supported at its other end B by a vertical wall that is at a distance a from the pivot A (Fig. J). If motion of the bar impends, find the value of the angle α that the plane AOB makes with the vertical xy plane. The coefficient of friction between the end B of the bar and the wall is μ. Neglect friction in the pivot at A.

Hint. When motion of the bar impends, the wall reaction at B must lie in a plane that is normal to the plane of the wall and tangent at B to the circle DBE. *Ans.* $\tan \alpha = \mu a / \sqrt{l^2 - a^2}$.

12. The horizontal shaft of a windmill is supported in bearings at A and B and carries a pulley of radius $r = 12$ in. at C as shown in Fig. K. Determine the force P applied to the rim of the pulley as shown that will maintain equilibrium when the wind pressure on each blade is equivalent to a normal force of 100 lb applied at a distance of 30 in. from the axis of the shaft. Each blade sets at an angle of 30° with the plane $EGFH$, which in turn is at right angles to the axis of the shaft. *Ans.* $P = 500$ lb.

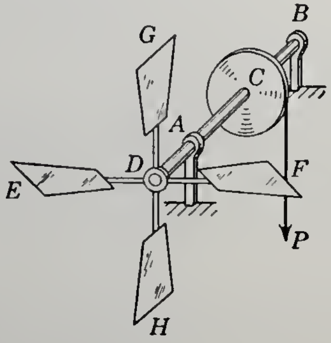


FIG. K

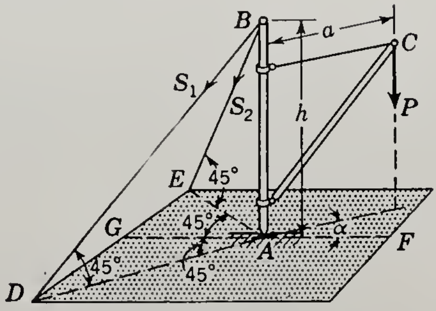


FIG. L

13. A small crane is supported on a horizontal plane by a socket at A and by two guy wires BD and BE and carries a load P at C as shown in Fig. L. Determine the forces S_1 and S_2 in the two guy wires if the vertical plane ABC makes an angle α with the vertical plane BGF. The following data are given: $a = 6$ ft, $h = 10$ ft, $\alpha = 20°$, $P = 1{,}000$ lb. *Ans.* $S_1 = 770$ lb; $S_2 = 359$ lb.

14. A slender bar OA is hinged to a vertical wall at O and held perpendicular thereto by two guy wires BC and DE as shown in Fig. M. A vertical load P acts at A; the weight of the bar itself is negligible. Find the tensions S_1 and S_2 in the guy wires and also the rectangular components of the hinge reaction at O if $P = 100$ lb. *Ans.* $S_1 = 179$ lb; $S_2 = 262$ lb; $X_0 = 286$ lb; $Y_0 = -67$ lb; $Z_0 = -36$ lb.

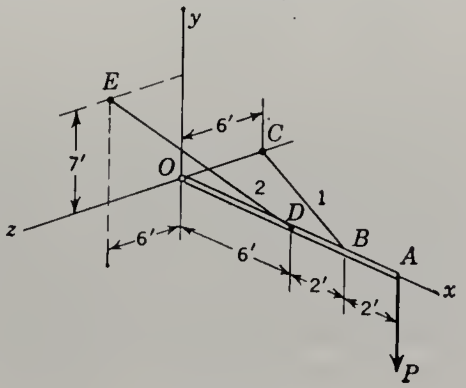


FIG. M

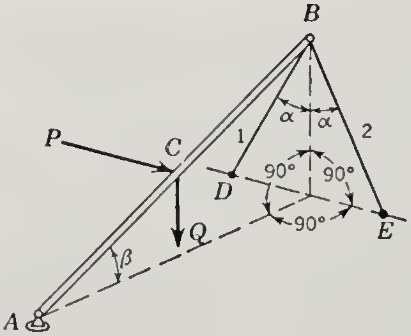


FIG. N

15. A rigid bar AB is hinged to a horizontal floor at A and supported at B by two struts BD and BE as shown in Fig. N. Calculate the axial forces S_1 and S_2 induced in these struts by forces P and Q acting at the mid-point C of the bar. The force Q is vertical; the force P is horizontal and perpendicular to AB. Numerical data are given as follows: $\alpha = 30°$, $\beta = 20°$, $P = 100$ lb, $Q = 50$ lb. *Ans.* $S_1 = 35.57$ lb; $S_2 = -64.42$ lb.

16. A rectangular table top is supported horizontally by means of six hinged bars arranged as shown in Fig. O. Determine the forces in the six bars if a vertical force of 200 lb and a horizontal endwise force of 100 lb act on the table as shown. *Ans.* $S_1 = S_2 = -167$ lb; $S_3 = +167$ lb; $S_4 = S_5 = 0$; $S_6 = -67$ lb.

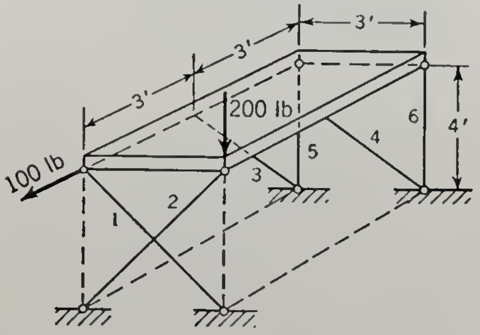


FIG. O

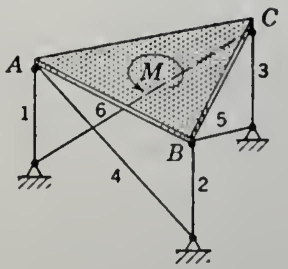


FIG. P

17. An equilateral triangular table top with sides of length a is supported in a horizontal plane by three vertical bars and three inclined bars hinged to its corners as shown in Fig. P. Determine the axial forces that will be produced in these bars owing to a couple of moment M acting in the plane of the table top as shown. Each inclined bar makes an angle of 30° with the horizontal. *Ans.* $S_1 = S_2 = S_3 = +2M/3a$; $S_4 = S_5 = S_6 = -4M/3a$.

18. A triangular prism of weight $W = 300$ lb is supported by six hinged bars as shown in Fig. Q. There is also a horizontal force $P = 400$ lb acting at A' as shown. Calculate the axial force in each supporting bar. *Ans.* $S_2 = S_3 = S_5 = 0$; $S_1 = +100$ lb; $S_4 = -400$ lb; $S_6 = -200$ lb.

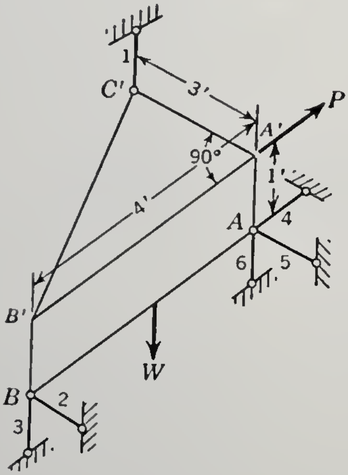


FIG. Q

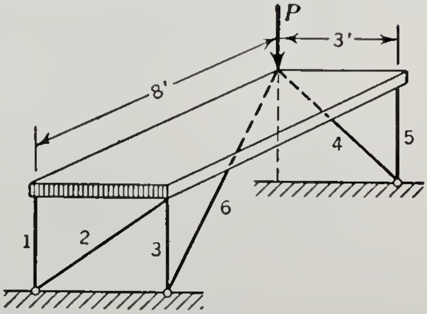


FIG. R

19. Determine the axial force S_i in each of the six bars supporting the horizontal slab shown in Fig. R due to the vertical load P applied as shown. *Ans.* $S_1 = S_5 = -P$; $S_3 = +P$; $S_2 = S_4 = S_6 = 0$.

***20.** In Fig. S, the two bars AB and OD, pinned together at C, form the diagonals of a horizontal square $AOBD$. The ends A and O are attached to a vertical wall by spherical hinges, point B is supported by a guy wire BE, and a vertical load P is applied at D. Find the rectangular components of the reactions at A and O. *Ans.* $X_a = 0$; $Y_a = +P$; $Z_a = -P$; $X_0 = +P$; $Y_0 = -P$; $Z_0 = +P$.

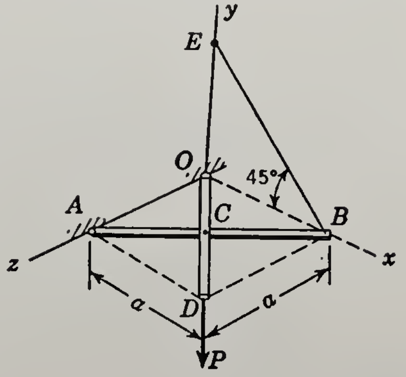


FIG. S

5

PRINCIPLE OF VIRTUAL WORK

5.1. Equilibrium of ideal systems. In engineering applications of statics, we are usually concerned with the question of equilibrium of a rigid body, or system of connected rigid bodies, either completely or partially constrained by various supports and subjected to applied active forces. As we have seen from previous examples, this question may involve either the calculation of reactions in the case of a completely constrained system or the defining of possible configurations of equilibrium of the system when it has some freedom of motion. It is to this second kind of problem that we shall now give our attention.

Virtual Displacements. In Fig. 171, we begin with the case of a lever AB hinged at O so that it is free to rotate about the z axis normal to the plane of the figure and acted upon by forces P and Q also acting in this plane. The position of the lever is completely defined by the angle θ that it makes with the x axis and the problem is to find the value of θ for which equilibrium can exist if the forces P and Q are specified. Since rotation about the z axis is the only kind of motion that the bar can make, we say that it has *one degree of freedom.* The angle θ, defining the position of the lever, is called the *coordinate* of the system.

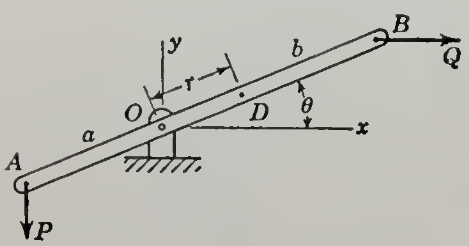


FIG. 171

In our further discussion, it will be necessary to consider infinitesimal displacements of the lever from the position defined by θ. These displacements are obtained by giving to the coordinate θ a small increase or decrease $\delta\theta$. When this is done, any point D on the axis of the lever at the distance r from O will describe an infinitesimal arc $r\,\delta\theta$ which can be treated as a rectilinear element perpendicular to AB. Such displacements are called *virtual displacements.* Thus the virtual displacements of points A and B are $a\ \delta\theta$ and $b\ \delta\theta$, perpendicular to the axis of the bar and oppositely directed.

In Fig. 172, we have a system of three rigid bodies consisting of crank, connecting rod, and piston of an engine. The crank can rotate freely about the z axis perpendicular to the plane of the figure through O and the configuration of the system is completely defined by the angle θ that OA makes with the fixed x axis. Thus again we have a system with one degree of freedom and the angle θ can be taken as the coordinate of the system. To define virtual displacements of various points in this case, we give to the coordinate θ an infinitesimal increment $\delta\theta$. Then the virtual displacement of the crankpin A is $r\ \delta\theta$ perpendicular to OA. Having the displacement of point A and observing that point B is constrained to move along the x axis, we can now express the corresponding virtual displacement of B as follows: The distance x of the piston from O is

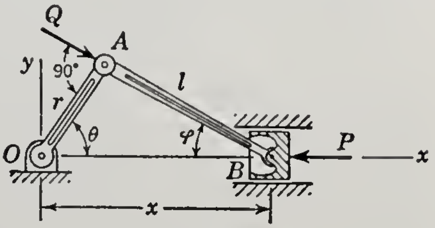


FIG. 172

$$x = r\cos\theta + l\cos\varphi$$

Then the change in x due to an increase $\delta\theta$ in the angle θ is

$$\delta x = -r\sin\theta\ \delta\theta - l\sin\varphi\ \delta\varphi \tag{a}$$

From the triangle AOB (Fig. 172), we have

$$\sin\varphi = \frac{r}{l}\sin\theta \qquad \varphi = \arcsin\left(\frac{r}{l}\sin\theta\right)$$

Substituting these expressions into Eq. (a), we obtain for the virtual displacement of the piston

$$\delta x = -r\ \delta\theta\left[1 + \frac{\cos\theta}{\sqrt{(l/r)^2 - \sin^2\theta}}\right]\sin\theta \tag{b}$$

If, for example, we take $\theta = 60°$ and $l/r = 2$, the virtual displacements of A and B will be $r\ \delta\theta$ and $-1.11r\ \delta\theta$, respectively. From these two examples, we see that the problem of defining virtual displacements of various points of a movable system is purely one of geometry.

Virtual Work. We are now ready to consider the problem of how to define the equilibrium configuration of such systems as discussed above. To do this, we must introduce the concept of *work.* In Fig. 173, the point of application A of a force F_i moves an infinitesimal distance δs_i. Then the force F_i is said to do work which is defined as the product of the displacement δs_i and the projection $F_i \cos \alpha_i$ of the force on the direction of the displacement. Thus using for work the symbol δU_i, we have

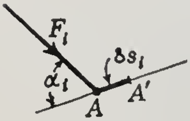


FIG. 173

$$\delta U_i = F_i \cos \alpha_i \, \delta s_i \tag{c}$$

This increment of work has the dimension of force × length and is considered positive when the projection of the force and the displacement have the same sense.

When we make a virtual displacement of a movable system of rigid bodies acted upon by forces, the points of application of the forces move and the forces do a small amount of work called *virtual work*. In dealing with such systems, we will assume that there is no friction in hinges, bearings of an axle, or along sliding surfaces such as the cylinder walls of the piston in Fig. 172 and that the parts of the system are rigid. Such systems are called *ideal systems*. Then only the applied active forces will do work on any virtual displacement of the system and we take as the *principle of virtual work*, the following statement: *If for each virtual displacement of an ideal system, the work produced by the active forces is zero, then the system is in a configuration of equilibrium.* Using expression (c) for the virtual work of a force, this principle can be stated in analytic form by the equation

$$\Sigma F_i \cos \alpha_i \, \delta s_i = 0 \tag{29}$$

where the summation must include all active forces on the system.

Applying Eq. (29) to the lever in Fig. 171, we obtain

$$P \cos \theta \, a \, \delta\theta - Q \sin \theta \, b \, \delta\theta = 0$$

from which

$$\tan \theta = \frac{Pa}{Qb} \tag{d}$$

When the magnitudes of P and Q and the distance a and b are given, expression (d) defines the value of θ for which the bar can be in equilibrium.

Applying Eq. (29) to the engine in Fig. 172, we obtain, for a positive $\delta\theta$,

$$-Qr \, \delta\theta - P \, \delta x = 0$$

where δx is defined by Eq. (b). Taking $\theta = 60°$ and $l/r = 2$, we have already found that $\delta x = -1.11 \, r \, \delta\theta$. Substituting this in the equation of virtual work, we obtain

$$-Qr \, \delta\theta + 1.11 Pr \, \delta\theta = 0$$

from which

$$\frac{Q}{P} = 1.11 \tag{e}$$

It should be noted that in this case we have specified the configuration of the system and then found the ratio between the active forces P and Q required to maintain equilibrium. The advantage of the principle of virtual work in the case of a system like that in Fig. 172, lies in the fact that we can establish a relation between P and Q for equilibrium of the system without having to consider any of the reactive or internal forces such as the reaction at O or the compressive force in the connecting rod.

The first applications of the principle of virtual work to problems of statics were made by Stevinus (1548–1620). Considering the conditions of equilibrium of systems of pulleys like those shown in Fig. 174, he concluded that, in each case, the ratio between the loads P and Q depends only upon the inverse ratio of the vertical displacements of their points of application. Take, for example, case *a*. Giving to point B a small downward displacement δx, we see that point A moves up by the amount $\delta x/2$. Hence, the equation of virtual work becomes

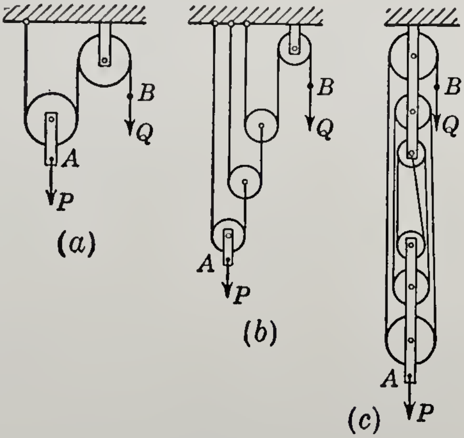


FIG. 174

$$Q\,\delta x - P\,\frac{\delta x}{2} = 0$$

from which $Q = P/2$. Applying a similar reasoning successively to each of the three movable pulleys in case *b*, we find that the upward displacement of point A is equal to one-eighth of the downward displacement of point B. Hence $Q = \frac{1}{8}P$. In case *c* the frame A containing the movable pulleys is suspended on six branches of the rope and its vertical displacement will be only one-sixth that of the end B of the rope. Hence $Q = \frac{1}{6}P$. The weights of the pulleys and of the rope are neglected in all these examples.

An analogous method was used by Galileo (1564–1642) in discussing the equilibrium of two particles connected by an inextensible and perfectly flexible string and resting on smooth inclined planes (Fig.

175). Giving to the particle P an infinitesimal displacement δs_1 along the inclined plane, the particle Q displaces along its inclined plane by the same amount. In calculating the work done by the active forces P and Q on this virtual displacement of the system, only the projection of each force on the corresponding displacement should be considered. Then the principle of virtual work gives in this case

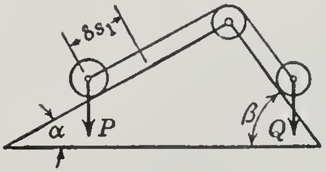


FIG. 175

$$-P\,\delta s_1 \sin\alpha + Q\,\delta s_1 \sin\beta = 0$$

and we obtain

$$\frac{P}{Q} = \frac{\sin\beta}{\sin\alpha} \qquad \text{(f)}$$

By taking the angle α much smaller than the angle β we can lift the load P with a much smaller load Q but the vertical displacement of P will be less than that of Q in the same proportion as Q is smaller than P. It is assumed, of course, that there is no friction.

In the preceding examples, we considered only systems that had some freedom of motion. However, the principle of virtual work can be used also in the case of a completely constrained system. Consider, for example, the case of a beam AC which is hinged at A and supported by a roller at B as shown in Fig. 176*a*. It is required to find the reaction produced at B by a vertical load P applied as shown. Assuming no friction, we conclude that R_b will be a vertical force. To find its magnitude by using the principle of virtual work, we remove the support at B and replace it by the force R_b, as shown in Fig. 176*b*. In this way, we obtain a movable system with one degree of freedom. Defining a virtual displacement of the system by a small angle of rotation $\delta\theta$ of the axis AC about point A, we see that the virtual displacements of the points of application of P and R_b are $x\,\delta\theta$ and $l\,\delta\theta$, respectively, as shown. Thus the equation of virtual work becomes

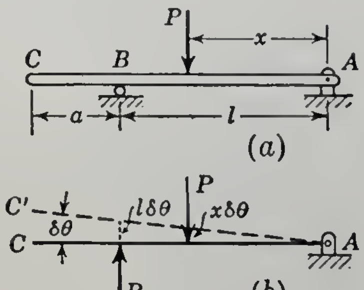


FIG. 176

$$R_b l\,\delta\theta - Px\,\delta\theta = 0$$

from which $R_b = Px/l$.

EXAMPLES

1. Considering the screw press shown in Fig. 177 and neglecting friction in the screw, find the relation between the moment of the couple PP acting in the

plane perpendicular to the axis of the screw and the reaction Q of the compressed body.

Solution. We imagine the compressed body to be replaced by the reactive forces Q that it exerts on the screw and on the frame. These forces should be considered as active forces on the ideal system represented by the frame and screw, which we assume to be absolutely rigid bodies. A virtual displacement of this system compatible with its constraints will be an infinitesimal rotation of the screw by an angle $\delta\theta$. Then the work done by the couple of moment $2Pa$ is $2Pa\ \delta\theta$. Denoting by h the pitch of the screw thread, the corresponding virtual displacement of the end of the screw is $h\ \delta\theta/2\pi$ and the work done on this displacement by the force Q acting on the end of the screw is $-Qh\ \delta\theta/2\pi$. The force Q which acts on the frame produces no work. Then the principle of virtual work gives

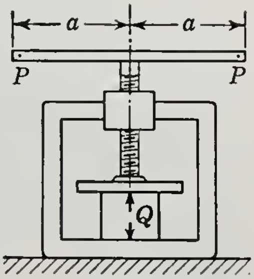


FIG. 177

$$2Pa\ \delta\theta - Q\frac{h\ \delta\theta}{2\pi} = 0$$

from which

$$2Pa = \frac{Qh}{2\pi} \tag{g}$$

2. When equilibrium exists, find the relation between the forces P and Q acting on the differential pulley shown in Fig. 178.

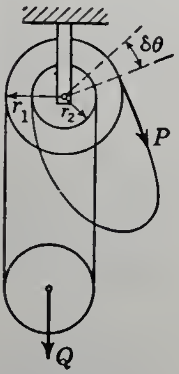


FIG. 178

Solution. If an infinitesimal angle of rotation $\delta\theta$ is given to the fixed pulley, the work produced by the force P is $Pr_1\ \delta\theta$. The corresponding virtual displacement of the load Q is $\frac{1}{2}(r_1 - r_2)\ \delta\theta$. Then from the principle of virtual work we have

$$Pr_1\ \delta\theta - Q\frac{r_1 - r_2}{2}\ \delta\theta = 0$$

from which

$$P = \frac{r_1 - r_2}{2r_1}Q \tag{h}$$

3. A prismatic bar AB of length $2l$ and weight Q passes through a smooth ring at D and presses against a smooth vertical wall at A distance a from the ring D as shown in Fig. 179. Neglecting friction, determine the position of equilibrium of the bar as defined by the angle θ that it makes with the face of the wall.

Solution. Since the weight Q of the bar is the only active force, we conclude at once that the position of equilibrium of the bar must be such that for a virtual displacement compatible with the constraints and defined, say, by an infinitesimal vertical displacement δs_a of point A, the corresponding virtual displacement δs_c of the center of gravity C of the bar is horizontal. The vertical displacement δs_a of the end A of the bar can be resolved into two components $\delta s_a \sin\theta$ and $\delta s_a \cos\theta$, normal and parallel, respectively, to the axis of the bar. The first is obtained by rotation of the bar around D, and the second, by sliding of the bar along its axis. The corresponding components of the displacement δs_c of the center of gravity C are $[(l - c)/c]\, \delta s_a \sin\theta$ and $\delta s_a \cos\theta$, respectively.

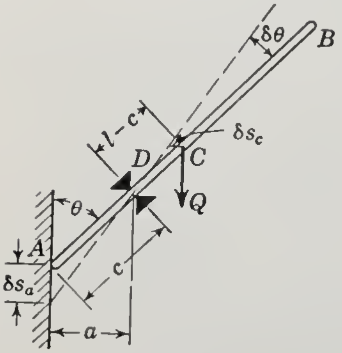


FIG. 179

Remembering now that the algebraic sum of the projections of these latter two components on a vertical axis must be zero, we have

$$\frac{l-c}{c}\sin^2\theta - \cos^2\theta = 0$$

from which

$$\frac{l}{c}\sin^2\theta = 1 \qquad \text{or} \qquad \sin^2\theta = \frac{c}{l}$$

Replacing c by its value $a/\sin\theta$, we obtain finally

$$\sin^3\theta = \frac{a}{l} \qquad \text{or} \qquad \sin\theta = \sqrt[3]{\frac{a}{l}} \tag{i}$$

We note that equilibrium of the bar cannot exist unless we have $a \leqq l$.

4. Two beams AC and CD hinged together at C are supported and loaded as shown in Fig. 180a. Using the principle of virtual work find the magnitude of the reaction R_b for any position of the load P.

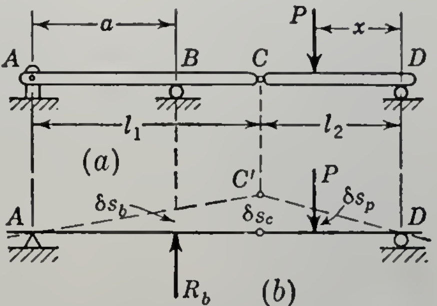


FIG. 180

Solution. To obtain a movable system with one degree of freedom, we replace the support at B by a vertical force R_b, as shown in Fig. 180b. A virtual displacement of this system compatible with the remaining constraints is defined by the infinitesimal vertical displacement δs_c of the hinge C as shown. The corresponding vertical displacements of the points of application of R_b and P, respectively, are

$$\delta s_b = \frac{\delta s_c a}{l_1} \qquad \text{and} \qquad \delta s_p = \frac{\delta s_c x}{l_2}$$

and the equation of virtual work becomes

$$R_b\,\delta s_b - P\,\delta s_p = 0$$

From this, we obtain

$$R_b = P\frac{\delta s_p}{\delta s_b} \tag{j}$$

or, using the above expressions for δs_p and δs_b,

$$R_b = P\frac{xl_1}{l_2a} \tag{k}$$

PROBLEM SET 5.1

1. The pulley arrangement shown in Fig. A is used for hoisting a load Q. Find the ratio between the forces P and Q in the case of equilibrium of the system. The radii of the two steps of the pulley are r_1 and r_2 as shown in the figure. Neglect friction. *Ans.* $P = Q\,r_2/(r_1 - r_2)$.

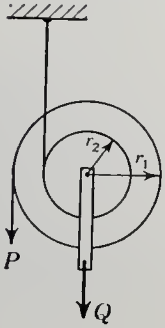


FIG. A

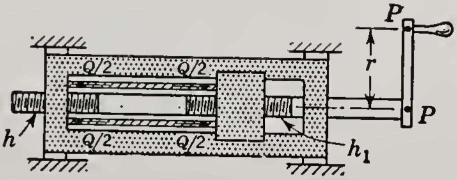


FIG. B

2. A circular cylindrical tube is compressed in the differential screw device as shown in Fig. B. Neglecting friction, find the relation between the total compressive force Q in the tube and the force P applied to the handle of the crank and acting at right angles to the crank in a plane perpendicular to the axis of the screw. The pitches for the two threaded portions of the screw are h and h_1 as shown. *Ans.* $P/Q = (h - h_1)/2\pi r$.

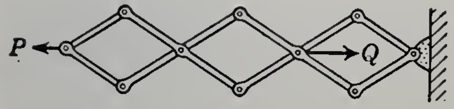


FIG. C

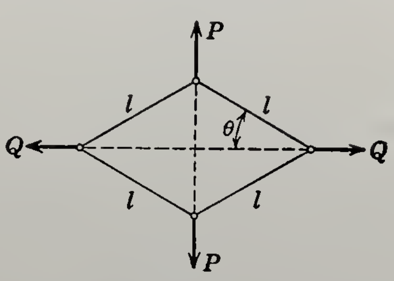


FIG. D

3. Calculate the relation between the active forces P and Q for equilibrium of the system of bars shown in Fig. C. The bars are so arranged that they form three identical rhombuses. *Ans.* $P = Q/3$.

4. Four bars of equal lengths l are hinged together at their ends in the form of a rhombus as shown in Fig. D. Using the principle of virtual work, find the relation between the active forces P and Q for equilibrium of the system in any configuration as defined by the angle θ. Neglect the weights of the bars. *Ans.* $P/Q = \tan\theta$.

5. A slender prismatic bar AB of length l and weight Q stands in a vertical plane and is supported by smooth surfaces at A and B as shown in Fig. E.

Using the principle of virtual work, find the magnitude of the horizontal force P applied at A if the bar is in equilibrium. *Ans.* $P = \frac{1}{2}Q \cot \theta$.

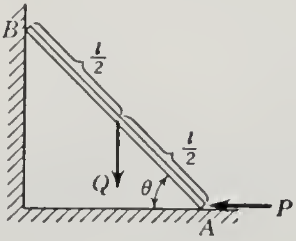


FIG. E

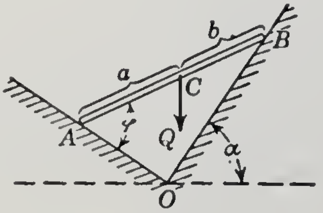


FIG. F

6. A rigid bar AB is supported in a vertical plane by mutually perpendicular smooth surfaces OA and OB, as shown in Fig. F. Using the principle of virtual work, calculate the angle φ defining the configuration of equilibrium of the system. *Ans.* $\tan \varphi = (a/b) \tan \alpha$.

7. Using the principle of virtual work, find the value of the angle θ defining the configuration of equilibrium of the system shown in Fig. G. The balls D and E can slide freely along the bars AC and BC but the string DE connecting them is inextensible. *Ans.* $\tan \theta = \sqrt{3}\, Q/P$.

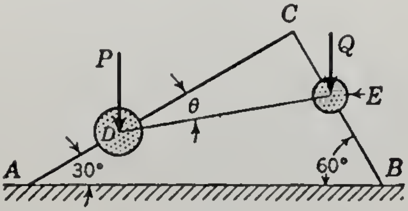


FIG. G

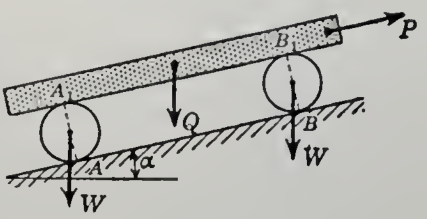


FIG. H

8. A beam supported on two rollers, each of radius r, is moved up an inclined plane by a force P which acts parallel to the plane (Fig. H). Determine the magnitude of P necessary to maintain equilibrium if there is no slipping between either the rollers and the inclined plane or the rollers and the beam. The weight of the beam is Q; the weight of each roller is W. *Ans.* $P = (Q + W) \sin \alpha$.

9. The platform scale (Fig. I) is so arranged that for the horizontal position of the lever COA the relation between P and Q does not depend upon the position of the load Q on the platform FD. If the ratio $GE/GH = 3$, find (a) the ratio OC/OB and (b) the ratio P/Q, for equilibrium of the system. *Ans.* (a) $OC/OB = 3$; (b) $P/Q = OB/OA$.

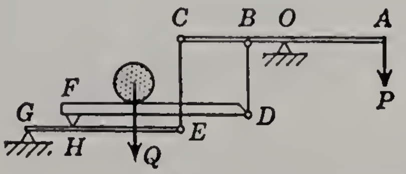


FIG. I

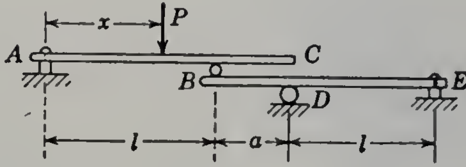


FIG. J

10. Using the principle of virtual work, find the reaction R_d for the system shown in Fig. J for any position of a vertical load P on the beam AC as defined by its distance x from A. *Ans.* $R_d = Px(l + a)/l^2$.

11. Using the principle of virtual work, find the reactions R_b, R_e, and R_f at the roller supports of the compound beam shown in Fig. K, if three equal vertical loads Q act as shown. *Ans.* $R_b = 7Q/2$; $R_e = 5Q/6$; $R_f = Q/3$.

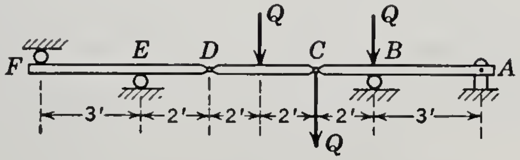


FIG. K

12. Find the horizontal and vertical components of the reactions at A and B of the plane frame loaded as shown in Fig. L. The bars form three equal squares. *Ans.* $X_a = X_b = P/3$; $Y_a = 4P/3$; $Y_b = 2P/3$.

13. Determine the total pressure P exerted by the ball point of the hardness tester shown in Fig. M if the pitch circle radius of the pinion is 2 in. and friction in all moving parts is neglected. *Ans.* $P = 150$ lb.

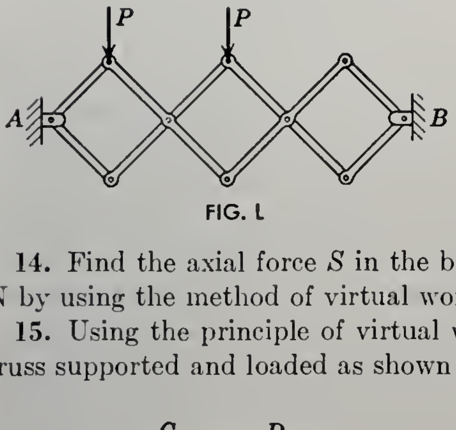


FIG. L

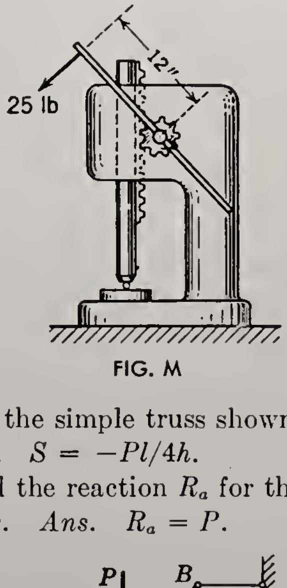


FIG. M

14. Find the axial force S in the bar CD of the simple truss shown in Fig. N by using the method of virtual work. *Ans.* $S = -Pl/4h$.

15. Using the principle of virtual work, find the reaction R_a for the simple truss supported and loaded as shown in Fig. O. *Ans.* $R_a = P$.

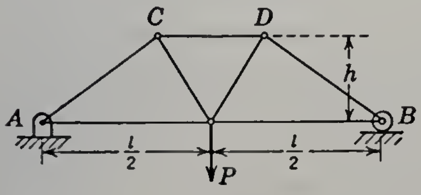


FIG. N

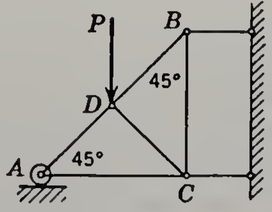


FIG. O

5.2. Efficiency of simple machines. In all previous applications of the principle of virtual work, we have considered only ideal systems without friction. While many of our machines and mechanical devices are of such construction that friction between parts can be neglected, we often encounter cases in which we shall wish to take it into account. In applying the principle of virtual work in these latter cases, it is necessary to consider, in addition to the work of all active forces applied to the system, the work of friction forces.

Consider, for example, the simple mechanical device consisting of an inclined plane, pulley, and cord to be used in raising a block of weight W as shown in Fig. 181. If the plane is not smooth, it will be necessary, in calculating the magnitude of the force P sufficient to cause impending motion of the block up the plane, to consider the effect of friction between the block and plane and the equation of virtual work becomes

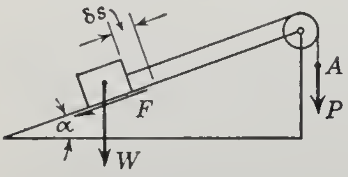


FIG. 181

$$P\ \delta s - W\ \delta s \sin \alpha - F\ \delta s = 0 \qquad \text{(a)}$$

where δs is a vertical displacement of the end A of the cord and F is the friction between the block and plane. Friction in the pulley is neglected. Denoting by μ the coefficient of friction and observing that the normal pressure between the block and the plane is $W \cos \alpha$, Eq. (a) becomes

$$P\ \delta s - W\ \delta s \sin \alpha - \mu W\ \delta s \cos \alpha = 0$$

from which

$$P = W(\mu \cos \alpha + \sin \alpha) \qquad \text{(b)}$$

It is seen, from Eq. (a), that the work $P\ \delta s$ produced by the active force P on the virtual displacement δs is used up not only in raising the block W by the amount $\delta s \sin \alpha$ but also in overcoming friction. The work $W\ \delta s \sin \alpha$ is called the *useful work* and its ratio to the *expended work* $P\ \delta s$ is called the *efficiency* of the machine, in this case an inclined plane used for raising the load W. Thus the efficiency of the inclined plane is

$$\frac{\text{Useful work}}{\text{Expended work}} = \frac{\sin \alpha}{\sin \alpha + \mu \cos \alpha} \qquad \text{(c)}$$

It is seen that the efficiency is unity for $\mu = 0$, that is, for the ideal case without friction, and that it decreases as the coefficient of friction μ increases.

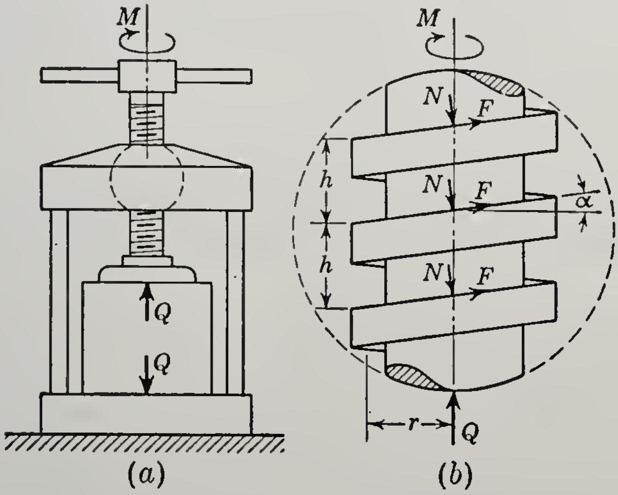
If, instead of raising the load W, we push it downward along the inclined plane by a force P acting parallel to the plane, the magnitude of this force necessary to cause impending motion is obtained from the equation

$$P\ \delta s + W\ \delta s \sin \alpha - F\ \delta s = 0$$

analogous to Eq. (a) above and from which, upon replacing F by its value $\mu W \cos \alpha$, we obtain

$$P = W(\mu \cos \alpha - \sin \alpha) \tag{d}$$

When $\mu \cos \alpha - \sin \alpha = 0$, that is, when $\tan \alpha = \mu$ or, in other words, when the angle of inclination of the plane is equal to the angle of friction φ, we see that the force required for impending motion becomes zero. For values of α greater than the angle of friction φ, the load slides down the inclined plane without the application of any pushing force.



(a) (b)

FIG. 182

In the particular case where $\alpha = \varphi$, the efficiency of the inclined plane as a mechanical device for raising a load W as shown in Fig. 181 is obtained by substituting $\tan \alpha$ for μ in expression (c). Then

$$\frac{\text{Useful work}}{\text{Expended work}} = \frac{\sin \alpha}{\sin \alpha + \tan \alpha \cos \alpha} = \frac{1}{2} \tag{e}$$

The method used above for an inclined plane can be applied also in the case of a screw press (Fig. 182*a*), the screw of which may be considered as an inclined plane wound around a cylinder (Fig. 182*b*). If an infinitesimal rotation $\delta\theta$ is given to the screw, the work done by the applied couple is $M\,\delta\theta$ and the corresponding work of the compressive force Q is $-Qh\,\delta\theta/2\pi$, where h is the pitch of the thread. In calculating the work done by the friction forces F acting between the threads of the screw and the frame (Fig. 182*b*), we assume that the thread is rectangular and that the mean radius of the surface of contact is r. Then the corresponding virtual displacement of the point of application of each friction force F is $\delta s = r\,\delta\theta/\cos \alpha$ and the equation of virtual work becomes

$$M\,\delta\theta - \frac{Qh\,\delta\theta}{2\pi} - \sum F\,\frac{r\,\delta\theta}{\cos\alpha} = 0 \tag{f}$$

in which F is the friction force acting on one element of the surface of contact of the screw thread and the summation is understood to include all such elements. The normal forces N being perpendicular to the displacement δs do not produce work, and friction at the lower end of the screw (Fig. 182a) where it is in contact with the plate of the press has been neglected in writing Eq. (f). Assuming, as before, that the friction force F acting on any element of the thread is proportional to the normal force N acting on the same element and denoting by μ the coefficient of friction, we obtain

$$\Sigma F = \mu \Sigma N$$

Equating to zero the algebraic sum of the projections on a vertical axis of all forces acting on the screw, we obtain

$$\Sigma F \sin\alpha - \Sigma N \cos\alpha + Q = 0$$

Eliminating ΣN between these last two equations, we find

$$\sum F = \frac{\mu Q}{\cos\alpha - \mu\sin\alpha}$$

Substituting this value of ΣF into Eq. (f) and using for the pitch h of the thread the equivalent expression $2\pi r \tan\alpha$, we obtain

$$M - Qr\tan\alpha - \frac{\mu Q r}{\cos^2\alpha - \mu\sin\alpha\cos\alpha} = 0$$

from which the required relation between the applied couple M and the compressive force Q is

$$M = Qr\,\frac{\sin\alpha + \mu\cos\alpha}{\cos\alpha - \mu\sin\alpha}$$

This formula can be simplified by introducing the angle of friction $\varphi = \arctan\mu$, and we obtain

$$M = Qr\tan(\alpha + \varphi) \tag{g}$$

In an ideal case without friction, $\varphi = 0$ and we have

$$M = Qr\tan\alpha$$

In this ideal case the total work produced by the acting couple is transformed into the useful work of the force Q, while in the case represented by Eq. (g) a part of the expended work is wasted by friction. By definition, the efficiency of the screw in general is

$$\frac{\text{Useful work}}{\text{Expended work}} = \frac{\tan \alpha}{\tan (\alpha + \varphi)} \tag{h}$$

If the pitch angle α is large and the angle of friction small, the screw may unwind under the action of the compressive force Q if the applied couple M is removed. To determine in such case the magnitude of the couple necessary to prevent such motion, we again use the principle of virtual work. Changing the direction of the friction forces F we obtain, instead of Eq. (f), the analogous equation

$$M\,\delta\theta - \frac{Qh}{2\pi}\,\delta\theta + \sum F\,\frac{r\,\delta\theta}{\cos \alpha} = 0 \tag{i}$$

where M now represents the moment of the couple required to keep the screw from unwinding. Proceeding as before in the determination of ΣF, we find

$$\sum F = \frac{\mu Q}{\cos \alpha + \mu \sin \alpha}$$

and substituting this value into Eq. (i), we find, for the couple required to prevent the screw from unwinding,

$$M = Qr \tan (\alpha - \varphi)$$

It is seen from this expression that the couple M becomes equal to zero when $\alpha = \varphi$, that is, when the pitch angle of the thread is equal to the angle of friction. Hence we conclude that for $\alpha < \varphi$, the screw is a *self-locking device.* For the particular case where $\alpha = \varphi$, we find from expression (h) that the efficiency of the screw press is $\tan \alpha / \tan 2\alpha$.

When $\alpha = 90° - \varphi$, Eq. (g) gives an infinite value for M and the efficiency (h) becomes equal to zero. In this case the resultants of the forces F and N acting on the thread are within the cones of friction (see page 52) and any increase in the magnitude of the applied couple M only increases the pressure between the screw and the frame.

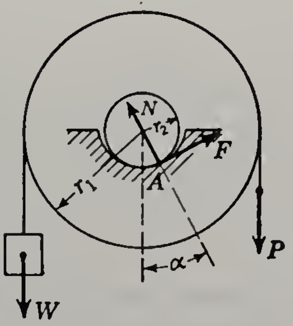


FIG. 183

As a last example here, let us consider the pulley arrangement shown in Fig. 183, which is used for raising a load W. To determine the magnitude of the force P necessary to cause lifting of the load W to impend when friction in the bearings is taken into account, we begin by assuming all forces to act in the middle plane of the pulley. Then, defining by the angle α, as shown in the figure, the position of the point of contact A between the axle and the bearing, we resolve the reaction at this point into a normal component N and a tangential component F representing friction. Giving now an infinitesimal angle of rotation $\delta\theta$ to the pulley, the equation of virtual work gives

$$(P - W)r_1\,\delta\theta - Fr_2\,\delta\theta = 0 \tag{j}$$

where r_1 is the radius of the pulley and r_2, the radius of the axle. For determining the magnitude of F and the position of point A as defined by α, we equate to zero the algebraic sums of the projections of all forces acting on the pulley on vertical and horizontal axes, obtaining, respectively,

$$\begin{aligned} N\cos\alpha + F\sin\alpha - P - W &= 0 \\ F\cos\alpha - N\sin\alpha &= 0 \end{aligned}$$

Substituting in these equations $N = F/\mu$, we find

$$\begin{aligned} \tan\alpha &= \mu \\ F &= \frac{\mu(P + W)}{\cos\alpha + \mu\sin\alpha} \end{aligned}$$

Substituting these values into Eq. (j), we obtain

$$P = W\frac{1 + \mu r_2/r_1\,\sqrt{1 + \mu^2}}{1 - \mu r_2/r_1\,\sqrt{1 + \mu^2}} \tag{k}$$

The efficiency of the pulley then is

$$\frac{1 - \mu r_2/r_1\,\sqrt{1 + \mu^2}}{1 + \mu r_2/r_1\,\sqrt{1 + \mu^2}} \tag{l}$$

If the coefficient of friction μ is small, we can take in place of the efficiency (l) the approximate expression

$$1 - \frac{2\mu r_2}{r_1} \tag{l$'$}$$

In each of the foregoing examples where friction has been taken into account it will be noted that we have had to use, in addition to the equation of virtual work, other equations of statics and it was necessary to deal with several simultaneous equations involving as many unknowns. This is a disadvantage which in some cases may introduce considerable difficulty. In cases where friction is small as in bearings, hinges, etc., the problem can often be simplified by using a method of successive approximations. We begin by neglecting friction entirely and write the equation of virtual work as for an ideal system. In this way we find an approximate relation between the active forces applied to the system, and on the basis of this relation, we can determine all normal pressures at joints and bearings. Multiplying the pressures, found in this way, by the corresponding coefficients of friction, we obtain approximate values of the friction forces which we now consider as additional active forces applied to the system. With these added forces, a more accurate equation of virtual work can be written and the above procedure repeated until the desired degree of accuracy is obtained.

PROBLEM SET 5.2

1. The screw press shown in Fig. A has a single square thread with mean radius $r = 1$ in. and a thread pitch $h = 1$ in. If a couple of moment $M = 100$ ft-lb is applied to the handwheel, what compressive force Q is exerted on the body B? Assume for the thread a coefficient of friction $\mu = 0.20$, and neglect pivot friction at the lower end of the screw. *Ans.* $Q = 3{,}230$ lb.

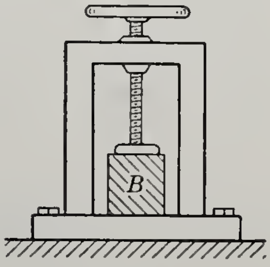


FIG. A

2. Calculate the efficiency of the screw press in Fig. A for the numerical data given in Prob. 1. *Ans.* Eff. = 0.428.

3. Assuming that the screw press shown in Fig. A has a V thread as shown in Fig. B, and again taking $r = 1$ in., $h = 1$ in., and $\mu = 0.20$, find the compressive force Q exerted on the block B due to a moment $M = 100$ ft-lb. applied to the handwheel. *Ans.* $Q = 2{,}960$ lb.

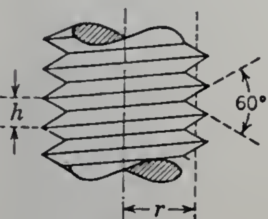


FIG. B

4. Calculate the efficiency of the screw press in Fig. A for the case of the screw with a V thread as shown in Fig. B. Use other data the same as in the preceding problems. *Ans.* Eff. = 0.393.

5. What is the smallest coefficient of friction for which the screw press in Fig. A will be self-locking? (*a*) For a square thread having mean radius $r = 1$ in. and pitch $h = 1$ in. (*b*) For a V thread (Fig. B) having $r = 1$ in. and $h = 1$ in. *Ans.* (*a*) $\mu > 0.159$; (*b*) $\mu > 0.138$.

6. The hand punch shown in Fig. C is so constructed that the jaws A and B always remain parallel. If the coefficient of friction between the slots in these bars and the pins on which they slide is $\mu = \frac{1}{3}$ and friction in other joints is negligible, find the relation between the forces P applied to the handles and the pressure Q exerted by the punch. What is the efficiency of the punch under these conditions? *Ans.* $Q = 3Pl/(2b + a)$; Eff. $= 3b/(2b + a)$

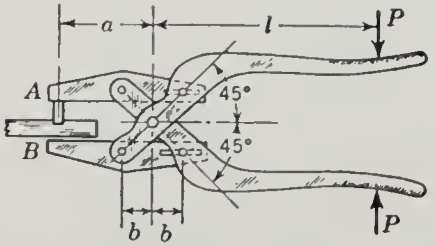


FIG. C

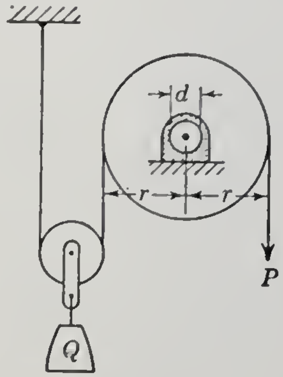


FIG. D

7. A load Q is hoisted by the pulley arrangement shown in Fig. D. Determine the magnitude of the pull P required to raise the load Q if $r = 14$ in., $d = 4$ in., and the coefficient of friction in the journals supporting the large pulley is $\mu = 0.25$. Neglect friction in the movable pulley. What is the efficiency of this device? *Ans.* $P = 0.535Q$; Eff. $= 0.935$.

8. For the system shown in Fig. E, find the force P on the piston D that will be required to produce a specified total pressure Q in each of the chambers A and B. The coefficient of friction on all sliding surfaces is μ and friction in the pin joints is negligible. What is the efficiency of this device? *Ans.* $P = (2Q \tan \alpha)/(1 - \mu \tan \alpha)$; Eff. $= 1 - \mu \tan \alpha$.

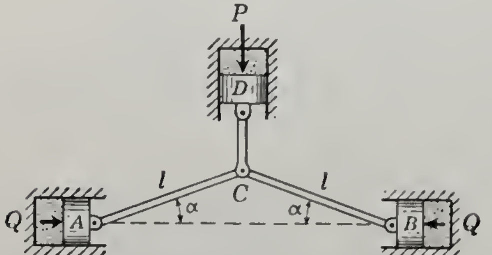


FIG. E

9. A heavy block of weight W is to be raised vertically by a horizontal force P applied to a wedge as shown in Fig. F. Find the magnitude of P required if the coefficient of friction on both faces of the wedge is μ and its weight is negligible. The following numerical data are given: $W = 2{,}000$ lb; $\alpha = 6°$; $\mu = \frac{1}{3}$. What is the efficiency of the wedge in this case? *Ans.* $P = 1{,}572$ lb; Eff. $= 0.134$.

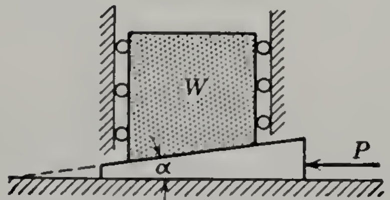


FIG. F

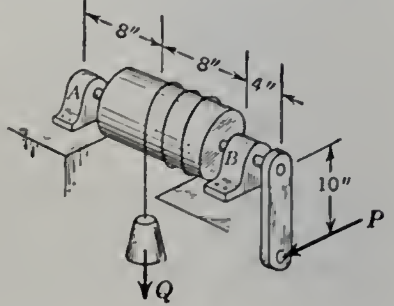
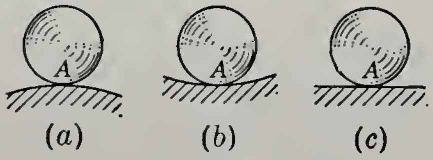


FIG. G

*10. Find the smallest ratio P/Q for equilibrium of the system shown in Fig. G, if the coefficient of friction in the bearings A and B is $\mu = 0.20$. The diameter of the drum is 16 in. and that of the shaft, 2 in. *Ans.* $P/Q = 0.768$.

5.3. Stable and unstable equilibrium. In previous applications of the principle of virtual work, we have been concerned only with establishing the conditions of equilibrium of a body or system of bodies to which active forces are applied. However, this principle is very useful for making further investigation of a configuration of equilibrium of any ideal system to determine whether it is *stable* or *unstable*. The characteristic of a stable configuration of equilibrium in any case is that, if the slightest force which acts only momentarily produces a small displacement from this configuration, the system will return to the original configuration as soon as this disturbing force is removed. On the other hand, in the case of an unstable configuration of equilibrium, the momentary action of a disturbing force may result in completely upsetting the condition of equilibrium so that the system continues to move away from the original configuration even after the force is removed. In certain cases the configuration of equilibrium of a system may be such that, upon the removal of the disturbing force, the system exhibits neither a tendency to return to nor to move farther away from its original configuration and such a configuration of equilibrium is said to be an *indifferent* one.

An example of each of these three classes of equilibrium is shown in Fig. 184. In Fig. 184*a* we have a ball that is placed at the highest point A of a smooth convex supporting surface. That we have, in this case, a condition of equilibrium follows from the fact that the reactive force which is normal to the surface at A is vertical and balances the vertical gravity force of the ball. But we see at once that this position of equilibrium of the ball is *unstable*, since the slightest accidental force may bring the ball into motion downward along the convex supporting surface. In Fig. 184*b*, the ball is placed at the lowest point A of a smooth concave supporting surface, and this position of equilibrium is *stable*, since if any accidental force slightly displaces the ball from this position, it tends to return back to it as soon as the force is removed. In Fig. 184*c* the ball is placed on a smooth horizontal plane, and in this case it is evident that it is in equilibrium in any position and we have *indifferent* equilibrium.



(*a*) (*b*) (*c*)

FIG. 184

To establish, by using the principle of virtual work, that equilibrium exists in each of the three cases represented in Fig. 184, we give to the ball in each case a virtual displacement, i.e., an infinitesimal displacement in which the ball remains in contact with the supporting surface. For such a displacement, we assume that the point of contact of the ball moves horizontally in the plane tangent to the supporting surface at the point of contact. Then the center of gravity of the ball moves also horizontally and the gravity force W does not produce work. Thus we conclude that in all three cases represented in Fig. 184 there is a condition of equilibrium.

To decide if each of the above positions of equilibrium is stable or unstable, a more refined calculation of the work done by the active forces on a virtual displacement of the ball is required. Taking the case shown in Fig. 184*a* and considering an infinitesimal displacement $\overline{AA_1} = \delta s = r\,\delta\varphi$, we see from Fig. 185 that such a displacement involves a lowering of the ball by the amount

$$\overline{AB} = r(1 - \cos\delta\varphi) \approx \frac{r}{2}(\delta\varphi)^2$$

which is a small quantity of the second order if the displacement δs is considered as a small quantity of the first order. It is not necessary to consider this small quantity of higher order in deciding the question of whether or not equilibrium exists. But if the question arises regarding the kind of equilibrium, i.e., whether it is stable or unstable, this second-order quantity must be taken into account. Thus we conclude that, in the case of an unstable equilibrium as represented in Fig. 184*a*, the gravity force of the ball produces on any virtual displacement a positive work and as a result of the slightest displacement a reactive force in the direction of motion appears and the ball has the tendency to move away from its position of equilibrium. In the same manner it can be shown that in the case of a stable equilibrium, as represented in Fig. 184*b*, there will be negative work done by the gravity force on any virtual displacement of the ball. Thus if by some accidental force the ball is slightly displaced from its position of equilibrium, a reactive force opposing this displacement appears and there is a tendency for the ball to return to its initial position.

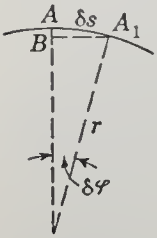


FIG. 185

In calculating the work done by gravity forces, the notion of *potential energy* can be used to advantage. To raise a body of weight W from some fixed level or *reference plane* to a height h above this plane, it is necessary to do work of the amount Wh against the gravity force. This amount of work is called the potential energy of the body by virtue of its new position above the reference plane. That is, it represents the amount of work that the gravity force W can produce if the body is allowed to fall back to its original position. On the basis of this definition, we can state that in the case of an unstable position of equilibrium of the ball, as in Fig. 184*a*, its potential energy decreases on any virtual displacement, while in the case of a stable position of equilibrium, as in Fig. 184*b*, the potential energy increases on any virtual displacement. We can state also that for an unstable position of equilibrium the potential energy of the ball is a *maximum* and for a stable position of equilibrium it is a *minimum.*

The same kind of reasoning can be applied also in the general case of any ideal system to which are applied active forces whose magnitudes depend only upon the configuration of the system.[1] In the investigation of any configuration of equilibrium of such a system, it is necessary to consider all possible virtual displacements from this configuration that are compatible with the constraints. If for all such displacements the work done by active forces, calculated by taking into account small quantities of higher orders, is negative, we have a condition like that represented in Fig. 184*b* and the configuration of equilibrium is stable. If, otherwise, a virtual displacement of the system can be found on which the above-mentioned work is positive, the condition is of the same kind as that represented in Fig. 184*a*, and the configuration of equilibrium under consideration is unstable. A configuration of equilibrium is indifferent if, on each virtual displacement of the system from that configuration, the work of all active forces, calculated by taking into account small quantities of higher orders, is zero.

The notion of potential energy can be used also in the investigation of conditions of equilibrium of an ideal system. We take a certain configuration of equilibrium of the system as a *reference configuration.* Then the work that must be done against the active forces in order to bring the system from this reference configuration to some other configuration represents the potential energy of the system in its new configuration. If, for a certain configuration of equilibrium, the potential energy of a system is smaller than for any adjacent configuration, then the configuration under consideration is one of stable equilibrium. But if an adjacent configuration can be found for which the potential energy of the system is smaller than for the configuration of equilibrium under consideration, then we have an unstable case.

---

[1] Such forces as friction which depend not only on the configuration of a system but also on the direction of motion are excluded from consideration.

EXAMPLES

1. A bar AB of weight W and length $a + b$ has its center of gravity at C and is supported at its ends by two smooth inclined planes as shown in Fig. 186. Prove that the position of equilibrium shown is an unstable one.

Solution. Taking the coordinate axes x and y as shown, we find that the coordinates of the center of gravity C of the bar are

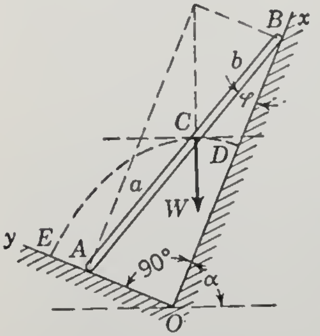


FIG. 186

$$x = a \cos \varphi \qquad y = b \sin \varphi$$

Hence
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and the center of gravity C is constrained to move along the quadrant ED of an ellipse. From the condition of equilibrium it follows that at C the tangent to this ellipse must be horizontal. However, since the curvature of the ellipse is convex upward, any displacement of the bar entails a lowering of the center of gravity C and as in the case of the ball in Fig. 184a the position of equilibrium is an unstable one.

2. A prismatic bar AB of length l and weight Q remains in a vertical plane and is supported at B by a vertical smooth wall and at A by a smooth cylindrical surface OC which is perpendicular to the plane of the figure (Fig. 187). Find the equation of the curve OC, referred to the coordinate axes x and y, if the bar is in equilibrium for any position as defined by the angle θ.

Solution. The conditions of the problem require that each position of the bar be one of indifferent equilibrium. Thus the potential energy of the system must remain constant, and since the reactive forces at A and B cannot produce work, this means that the curve OC must be such as to keep the center of gravity of the bar at the constant elevation $l/2$ above the x axis. We express this condition by the equation

$$\frac{l}{2} \cos \theta + y = \frac{l}{2}$$

which, noting that $\cos \theta = \sqrt{l^2 - x^2}/l$, gives

$$y = \frac{l}{2} - \frac{1}{2} \sqrt{l^2 - x^2} \tag{a}$$

as the desired equation of the curve OC.

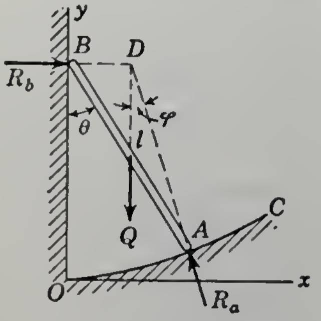


FIG. 187

To appreciate better the value of the method of virtual work in this problem, let us reconsider the analysis, using orthodox statics. For this purpose, the theorem of three forces will be helpful (see page 32). Noting that for equilibrium the three forces Q, R_a, and R_b must meet in one point D and that

R_a must always be normal to the curve OAC, we may write from the figure

$$\frac{dy}{dx} = \tan\varphi = \tfrac{1}{2}\tan\theta \tag{b}$$

where dy/dx is the slope of the curve OAC at A. Then since

$$\tan\theta = \frac{x}{\sqrt{l^2 - x^2}}$$

expression (b) takes the form

$$\frac{dy}{dx} = \frac{x/2}{\sqrt{l^2 - x^2}} \tag{c}$$

which is a differential equation defining the required shape of the curve OAC.

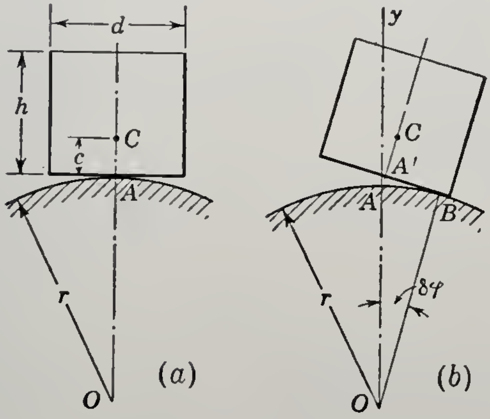


FIG. 188

Integrating Eq. (c), we obtain

$$y = -\tfrac{1}{2}\sqrt{l^2 - x^2} + C \tag{d}$$

where C is a constant of integration. To evaluate this constant, we must observe that the curve which we seek passes through the origin O, that is, $y = 0$ when $x = 0$. Substituting these simultaneous values into expression (d), we find $C = l/2$, and the expression becomes

$$y = \frac{l}{2} - \frac{1}{2}\sqrt{l^2 - x^2}$$

as before. We see that the use of the principle of virtual work eliminates the necessity to solve a differential equation, and this may sometimes be a considerable advantage.

3. An open cylindrical can of diameter d, height h, and center of gravity C rests on top of a spherical surface of radius r, as shown in Fig. 188a. Assuming sufficient friction to prevent slipping, find the relation between c and r consistent with stability.

Solution. If the can is stable, its potential energy must be a minimum; i.e., the center of gravity C must rise slightly for any infinitesimal virtual displacement defined by the angle $\delta\varphi$, as shown in Fig. 188*b*. During this displacement the can rolls on the sphere without slipping. Hence, $A'B$ is equal to the arc $AB = r\,\delta\varphi$ and the elevation of point C above the center O of the sphere is

$$r \cos \delta\varphi + r\delta\varphi \sin \delta\varphi + c \cos \delta\varphi$$

In the configuration of equilibrium (Fig. 188*a*) the elevation of C above O is $r + c$. Hence, for stability, we must have

$$(r + c) \cos \delta\varphi + r\,\delta\varphi \sin \delta\varphi > r + c \tag{e}$$

Since $\delta\varphi$ is very small, we can take

$$\cos \delta\varphi \approx 1 - \frac{(\delta\varphi)^2}{2} \qquad \sin \delta\varphi \approx \delta\varphi$$

Then condition (e) reduces to

$$r(\delta\varphi)^2 > (r + c)\,\frac{(\delta\varphi)^2}{2}$$

from which we find

$$r > c \tag{f}$$

as the required criterion of stability.

For a can of uniform thin material,

$$c = \frac{\pi d h^2}{2\left(\pi d h + \dfrac{\pi d^2}{4}\right)}$$

and the requirement for stability becomes

$$\frac{h^2}{2\left(h + \dfrac{d}{4}\right)} < r \tag{g}$$

4. A ball of weight Q and radius r is in equilibrium at the lowest point of a spherical seat of radius a and has transmitted to it by means of a pin a load P which acts along its vertical diameter (Fig. 189*a*). Assuming that the bodies are absolutely rigid and that there is no friction, determine the criterion of stable equilibrium of the ball.

Solution. Giving to the ball a slight lateral displacement defined by the small angle φ as shown in Fig. 189*b*, we see that the center of gravity of the ball moves up a distance pq while the point of application of the force P moves down a distance mn, as indicated in the figure. Hence the lowest position of the ball will be stable if

$$Q \cdot pq > P \cdot mn \tag{h}$$

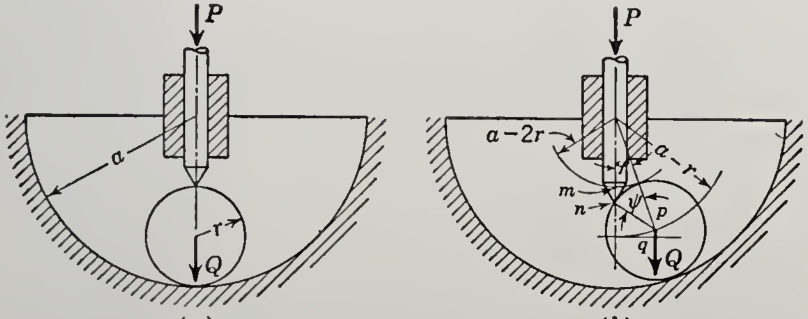
and unstable if

$$Q \cdot pq < P \cdot mn \tag{i}$$

The distance pq between the circle of radius $a - r$ and the horizontal tangent to this circle, for a small angle φ, is

$$pq = \frac{\varphi^2}{2}(a - r) \tag{j}$$

The distance mn between the circle of radius $a - 2r$ and the point of contact of the pin with the ball, for a small angle φ, is



(a) (b)

FIG. 189

$$mn = \frac{\varphi^2}{2}(a - 2r) + \frac{\psi^2}{2}r = \frac{\varphi^2}{2}(a - 2r) + \frac{\varphi^2}{2}\frac{(a - 2r)^2}{r}$$

which reduces to

$$mn = \frac{\varphi^2}{2r}(a - 2r)(a - r) \tag{k}$$

Substituting the values (j) and (k) into (h) above, we obtain the following criterion for stable equilibrium:

$$P < \frac{Qr}{a - 2r} \tag{l}$$

PROBLEM SET 5.3

1. Show that a nonhomogeneous circular cylinder on a horizontal plane (Fig. A) has a position of stable equilibrium when its center of gravity C is vertically below the axis O of the cylinder as shown.

2. A pendulum with two bobs of weights W_1 and W_2 is supported at O, as shown in Fig. B, and can turn freely without friction around this point. Neglecting the weight of the bar, establish the criterion of stable equilibrium for the vertical position of the pendulum. *Ans.* $W_2l_2 > W_1l_1$.

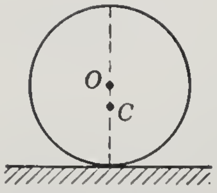


FIG. A

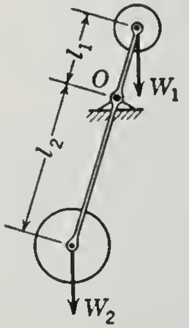


FIG. B

3. A prismatic bar AB of weight W is supported by a spherical hinge at A and rests against a smooth horizontal bar CD which is perpendicular to the vertical plane AEF (Fig. C). Prove that this position of equilibrium of the bar is unstable.

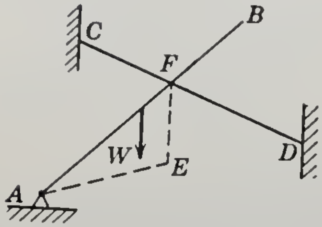


FIG. C

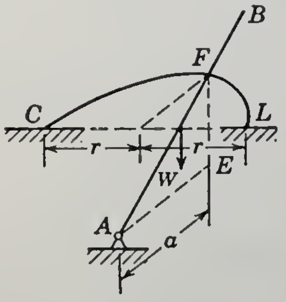


FIG. D

4. Establish the criterion for the stability of the equilibrium of the prismatic bar AB supported as shown in Fig. D by a spherical hinge at A and a smooth horizontal circular ring CD of radius r. *Ans.* $a > r$.

5. A solid of uniform density is made of a hemisphere and a right circular cone which have a common base of radius r (Fig. E). Determine the largest value of the height h of the cone consistent with stability of the body in the vertical position. *Ans.* $h_{\max} = \sqrt{3}\,r$.

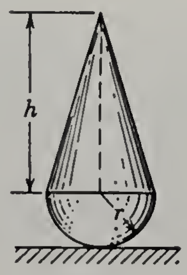


FIG. E

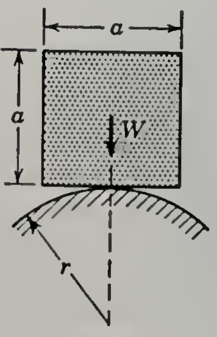


FIG. F

6. A cube of uniform density and lateral dimensions a is balanced on a cylindrical surface of radius r as shown in Fig. F. Establish the criterion for stable equilibrium of the cube assuming that friction is sufficient to prevent slipping. *Ans.* $r > a/2$.

7. Using the principle of virtual work and taking θ as coordinate, determine all possible configurations of equilibrium of the system in Fig. G and discuss the stability of each.

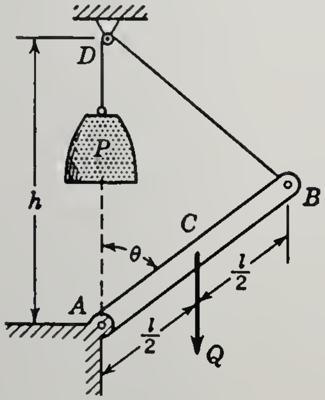


FIG. G

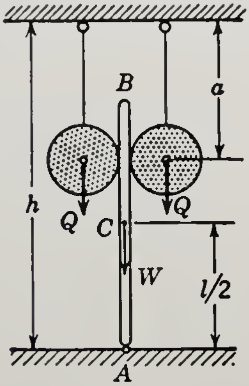


FIG. H

8. A slender prismatic bar AB of length l and weight W is hinged to the floor at A and held in a vertical position by two balls suspended from the ceiling by vertical strings as shown in Fig. H. Each ball is of weight Q and radius r, and other dimensions are as shown in the figure. Neglecting friction and the thickness of the bar AB, establish the criterion of stability of the system in the configuration shown. *Ans.* $Wl/2 < Qa[(h - a)/a]^2$.

9. A hemispherical cup of radius r and having its center of gravity at C rests on top of a spherical surface of radius R as shown in Fig. I. Assuming that there is sufficient friction to prevent slipping, establish the criterion of stability of the cup in the position shown. *Ans.* $R > r[(r/c) - 1]$.

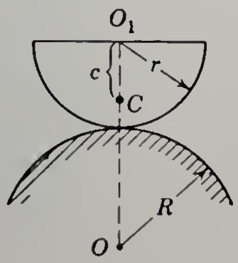


FIG. I

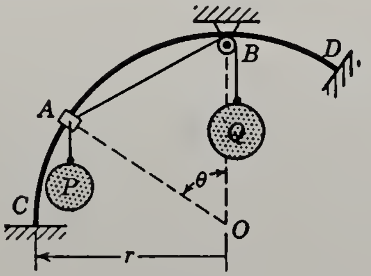


FIG. J

10. Using the principle of virtual work and taking θ as coordinate, determine all possible configurations of equilibrium of the system in Fig. J and investigate the stability of each. The bead A can slide freely on the circular wire CD and the pulley at B is negligibly small.

Part Two

DYNAMICS

6

RECTILINEAR TRANSLATION

6.1. Kinematics of rectilinear motion. *Introduction.* In *statics* we have considered rigid bodies that are at rest. In *dynamics* we shall consider bodies that are in motion. While statics is a very old science, dynamics, on the contrary, is a comparatively new one, its beginning usually being counted from the time of Galileo (1564–1642). Delayed development in the case of dynamics was due principally to difficulties inherently connected with experiments that were necessary to establish the foundation of the science. In statics there are only two kinds of units to be dealt with: that of *length* for measuring the dimensions or distances between bodies and that of *force* for measuring the actions and reactions between them. Precision instruments for the measurement of length and force are relatively simple and were well developed in early times. In dynamics, however, we need, in addition to the units of length and force, a unit for measuring *time.* Such time-measuring devices as the pendulum clock and balance-wheel watch are comparatively new and were completely unknown in Galileo's time.[1] It is quite natural then that the difficulties connected with the measurement of small intervals of time should have impeded the development of the science of dynamics.

For convenience, dynamics is commonly divided into two branches called *kinematics* and *kinetics.* In kinematics we are concerned only with the space-time relationship of a given motion of a body and not at all with the forces that cause the motion. If we see that a wheel rolls along a straight level track with uniform speed, the determination of the shape of the path described by a point on its rim and of the position along this path that the chosen point will occupy at any given instant are problems of kinematics.

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[1] The first pendulum clock was developed by Huygens in 1657, and the balance-wheel watch still later by Robert Hooke.

In kinetics we are concerned with finding the kind of motion that a given body or system of bodies will have under the action of given forces or with what forces must be applied to produce a prescribed motion. If a constant horizontal force is to be applied to a given body that rests on a smooth horizontal plane, the prediction of the way in which the body will move is a problem of kinetics. Again, the determination of the constant torque that must be applied to the shaft of a given rotor in order to bring it up to a desired speed of rotation in a given interval of time is a problem of kinetics.

The whole science of dynamics is based on the natural laws governing the motion of a *particle* under the action of a given force. A *particle* is defined as a material point without dimensions but containing a definite quantity of matter. Strictly speaking, there can be no such thing as a particle, since a definite amount of matter must occupy some space. However, when the size of a body is extremely small compared with its range of motion, it may, in certain cases, be considered as a particle. Stars and planets, for example, although many thousands of miles in diameter, are so small compared with their range of motion that they may be considered as particles in space. In the same way the dimensions of a rifle bullet are so small compared with those of its trajectory that ordinarily it may be considered as a particle.

Whenever a particle moves through space, it describes a curve that is called the *path*. The path of a particle may be either a space curve, called a *tortuous path*, or a plane curve, called a *plane path*. In the simplest case the path will be a straight line, and the particle is said to have *rectilinear motion*.

Displacement. Taking the line of motion as the x axis (Fig. 190), we can define the *displacement* of the particle by its x coordinate, measured from the fixed reference point O. We shall consider this displacement as positive when the particle is to the right of the origin O and as negative when to the left. As the unit of length by which the displacement is measured, we shall use most commonly the *foot* or *inch*.

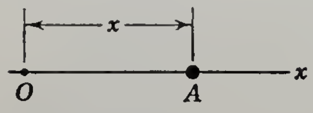

FIG. 190

As the particle moves, the displacement varies with time, and the motion of the particle is completely defined if we know the displacement x at any instant of time t. Analytically, this relation can be expressed by the general *displacement-time equation*

$$x = f(t) \tag{30}$$

where $f(t)$ stands for any function of time. Depending upon how the particle moves along the x axis, Eq. (30) will take different forms.

Consider, for example, motion of the particle as represented by the equation

$$x = c + bt \tag{30a}$$

In this equation, the constant term c represents an *initial displacement* at the time $t = 0$, while the constant b shows the rate at which the displacement increases. This motion of a particle is called *uniform rectilinear motion.* Besides being a very simple case, it is one commonly encountered in engineering problems.

As a second example of rectilinear motion of a particle, let us consider Eq. (30) in the form

$$x = \tfrac{1}{2}gt^2 \tag{30b}$$

Here the displacement x is represented as being proportional to the square of the time. Such motion, as is known from elementary physics, we have in the case of freely falling bodies.

A third example of rectilinear motion may be represented by taking Eq. (30) in the following form:

$$x = re^{-kt} \tag{30c}$$

where r and k are constants and e is the natural logarithmic base. Such motion, in which the displacement decreases exponentially with time, can be encountered in dealing with a particle projected into a highly viscous medium which finally brings it to rest.

Instead of analytical expressions for Eq. (30), it is often useful to represent the displacement-time relationship graphically. Taking time t as absissa and displacement x as ordinate, the curve represented by Eq. (30) can be plotted for any particular case. This gives us a so-called *displacement-time diagram.* Such diagrams for the particular cases represented by Eqs. (30a) to (30c) are shown in Fig. 191. These curves represent graphically the same information as given by the analytic expressions.

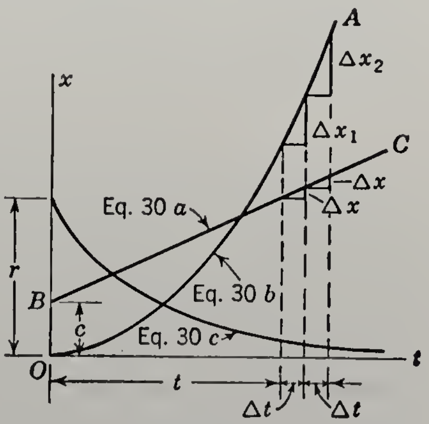

FIG. 191

Velocity. In discussing velocity of a particle having rectilinear motion, we begin with the case of uniform rectilinear motion as represented graphically by the straight line BC in Fig. 191 and we see that for equal intervals of time Δt the particle receives equal increments of displacement Δx. Thus the velocity v of uniform motion is given by the equation

$$v = \frac{\Delta x}{\Delta t} \tag{a}$$

This velocity is considered positive if the displacement x is increasing with time and negative if it is decreasing with time. Using the foot as the unit of displacement and the second as the unit of time, the unit of velocity will be the *foot per second,* usually written *fps.*

In the more general case of nonuniform rectilinear motion of a particle as given by Eq. (30*b*), represented graphically in Fig. 191 by the curve OA, we see that in equal intervals of time Δt the particle receives unequal increments of displacement Δx_1 and Δx_2. Since in this case the increment of displacement increases with time, we have *accelerated motion.* If Δx denotes the increment of displacement received during the interval of time Δt, we introduce the notion of *average velocity* during this time by the equation

$$v_{av} = \frac{\Delta x}{\Delta t} \tag{b}$$

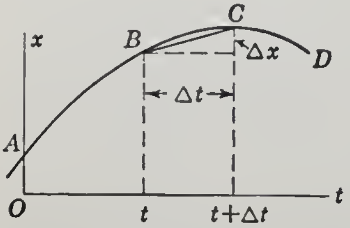

FIG. 192

As the interval of time Δt is taken smaller and smaller, the motion during the interval becomes more and more nearly uniform so that the ratio $\Delta x/\Delta t$ approaches more and more closely to the velocity at any particular instant of the time interval. Taking the *limit* approached by the ratio $\Delta x/\Delta t$ as Δt is indefinitely diminished, we obtain the *instantaneous velocity* of the particle for the moment t. Thus

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = \dot{x}^* \tag{31}$$

The foregoing discussion may also be given a convenient graphical interpretation. If the curve AD in Fig. 192 is a displacement-time diagram and we are considering the interval of time from t to $t + \Delta t$, the introduction of the notion of average velocity during that interval is equivalent to replacing the actual motion by a uniform motion as represented by the straight line BC. It is evident that the slope of this line represents the average velocity as expressed by Eq. (b). As the interval Δt becomes smaller and smaller, the portion BC of the curve representing the true motion deviates less and less from the straight line BC and in the limit this straight line becomes the tangent to the displacement-time diagram at B and its slope gives the instantaneous velocity v at the instant t. Thus we see that the variation in the velocity of a nonuniformly moving particle is represented geometrically by the variation in slope of the displacement-time curve.

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* The notation $\dot{x}$ to denote the derivative dx/dt is one introduced by Newton and will be used throughout this book.

From Eq. (31) we see that an analytical expression for the *velocity-time* relationship of a moving particle can always be obtained by differentiating the displacement-time equation (30). In so doing we distinguish not only the magnitude of the velocity but also its direction. If the derivative of Eq. (30) is positive, i.e., if the displacement increases with time, the velocity is positive and has the positive direction of the x axis; otherwise it is negative. For the particular cases of rectilinear motion of a particle that we have represented by Eqs. (30*a*), (30*b*), and (30*c*), we obtain by Eq. (31):

$$v = \frac{d}{dt}(c + bt) = b \tag{31a}$$

$$v = \frac{d}{dt}ct^2 = 2ct \tag{31b}$$

$$v = \frac{d}{dt}re^{-kt} = -rke^{-kt} \tag{31c}$$

Variations in velocity with respect to time can also be represented graphically. Taking velocity v as ordinate and time t as abscissa, we may plot from Eq. (31) the so-called *velocity-time diagrams.* For the three particular cases under discussion, these diagrams are shown in Fig. 193. In the case of Eq. (31*a*), we see that the velocity-time curve is a horizontal straight line indicating motion with constant velocity b. This agrees with the constant slope of the corresponding displacement-time curve in Fig. 191. From Eq. (31*b*) we obtain, for the velocity-time diagram, a straight line with constant slope $2c$ which indicates that the slope of the corresponding displacement-time curve in Fig. (191) increases at a uniform rate. In the final case, Eq. (31*c*) gives a more general velocity-time curve in which the velocity varies exponentially with time. Velocity-time diagrams such as those shown in Fig. 193 are especially helpful in studying various cases of rectilinear motion.

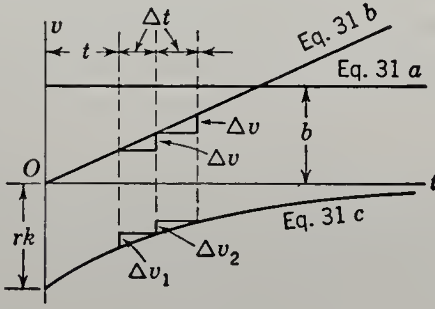


FIG. 193

If we have the velocity-time diagram for any rectilinear motion of a particle, the distance traveled by the particle during any interval of time from $t = t_1$ to $t = t_2$ will be represented by the area under the curve between these two ordinates of time. To prove this statement, we consider the velocity-time diagram shown in Fig. 194. From Eq. (31) we may write

$$dx = v\,dt \tag{c}$$

which expresses the small increment of displacement through which the particle moves during the time interval dt. From the figure we see that this expression also represents the area of the infinitesimal strip of width dt and height v of the velocity-time diagram. Summing up all such increments of displacement between the instants t_1 and t_2, the large shaded area is obtained as the total distance traveled by the particle during the specified time interval. Thus we see that the velocity-time diagram is particularly valuable in that from it we can obtain not only the velocity of motion at any instant but also the distance traveled during any given interval of time.

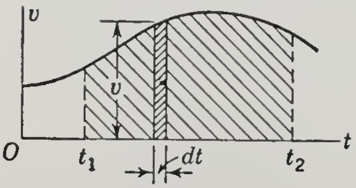


FIG. 194

Acceleration. If the rectilinear motion of a particle is nonuniform, its velocity is changing with time and we have *acceleration.* From Fig. 193 we see that in the case of uniform motion the velocity remains constant and we have zero acceleration. For the case represented by Eq. (31*b*) in Fig. 193 the particle receives equal increments of velocity Δv in equal intervals of time Δt. Thus we have motion with *constant acceleration* as given by the equation

$$a = \frac{\Delta v}{\Delta t} \tag{d}$$

As in the case of velocity, we distinguish not only the magnitude of acceleration but also its direction. Acceleration is considered positive when the velocity obtains positive increments in successive intervals of time and negative if the velocity is decreasing. The acceleration of a particle may be positive when its velocity is negative, or vice versa. Using the foot and second as units of length and time, respectively, and observing that the increment of velocity Δv in Eq. (d) is in units of feet per second, we conclude that the unit of acceleration is the *foot per second per second*, usually written ft/sec^2.

In a more general case like that represented by Eq. (31*c*) in Fig. 193 we note that in equal intervals of time Δt the increments of velocity

Δv_1 and Δv_2 are unequal. Thus we have motion of a particle with *variable acceleration.* To obtain the acceleration of the particle at any instant t in such case, we proceed as before in defining instantaneous velocity. We take a small interval of time from the instant t to $t + \Delta t$ and use the notion of *average acceleration*

$$a_{\text{av}} = \frac{\Delta v}{\Delta t} \tag{e}$$

over that interval. To obtain from this the acceleration at the instant t we go to the limit and find[1]

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \ddot{x} \tag{32}$$

From Eq. (32) we see that in any case of rectilinear motion of a particle the acceleration-time relationship can be expressed analytically by taking the second derivative with respect to time of the displacement-time equation (30). For the three cases under consideration we obtain, respectively,

$$a = \frac{d^2}{dt^2}(c + bt) = 0 \tag{32a}$$

$$a = \frac{d^2}{dt^2}ct^2 = 2c \tag{32b}$$

$$a = \frac{d^2}{dt^2}re^{-kt} = rk^2e^{-kt} \tag{32c}$$

It will be noted that these relationships are represented graphically by the slopes of the velocity-time diagrams shown in Fig. 193.

EXAMPLES

1. Point A', starting from A_0, moves with constant speed around the circumference of a circle of radius r, as shown in Fig. 195. Develop expressions for the displacement, velocity, and acceleration of its projection A on the x axis, that is, on the diameter OA_0 of the circle.

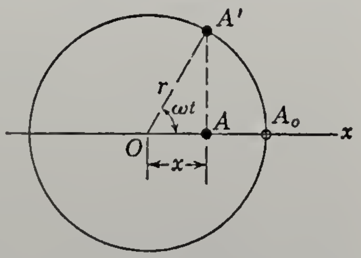


FIG. 195

Solution. Let ω represent the angle in radians swept out by the radius vector OA' per unit of time. This is called the *angular velocity* of OA' and is expressed in units of radians per second ($\sec^{-1}$). Then, from the trigonometry of the figure, we have

$$x = r\cos\omega t \tag{f}$$

---

[1] The notation $\ddot{x}$ will be used hereafter to denote the second derivative of the displacement x with respect to time t.

which is the required displacement-time equation. This motion of point A is called *simple harmonic motion.* We see that the time required for point A to make one complete oscillation along its path is the same as that required for the uniformly rotating radius vector OA' to make one complete revolution. This time interval, equal to $2\pi/\omega$, is called the *period* of the simple harmonic motion and is denoted by τ.

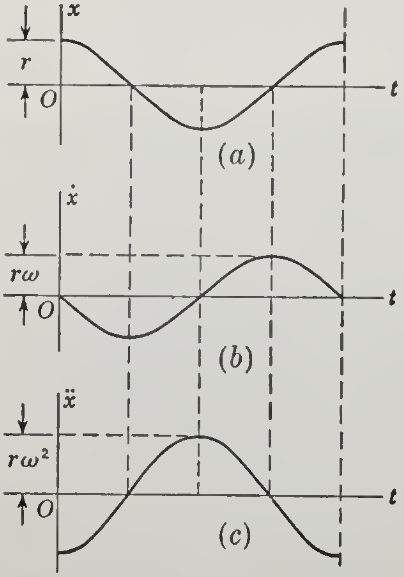


FIG. 196

Differentiating expression (f) with respect to time, we obtain for the velocity-time equation

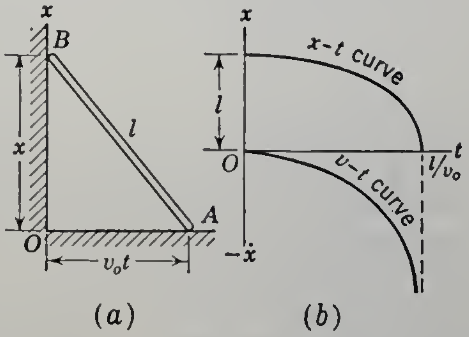
$$\dot{x} = -r\omega \sin \omega t \tag{g}$$

Differentiating expression (g) again with respect to time gives, for the acceleration-time equation,

$$\ddot{x} = -r\omega^2 \cos \omega t \tag{h}$$

From Eqs. (f), (g), (h), we can now plot displacement-time, velocity-time, and acceleration-time diagrams as shown in Fig. 196. From these diagrams we see that simple harmonic motion is periodic, i.e., after the time interval $\tau = 2\pi/\omega$, the curves repeat themselves. Comparing the three diagrams in Fig. 196, we see that the velocity is a maximum or a minimum when the displacement is zero, and vice versa. We also note that the acceleration is proportional to the displacement but always of opposite sign. Simple harmonic motion is a very important type of rectilinear motion and the student should become thoroughly familiar with its various characteristics.

2. A slender bar AB of length l which remains always in the same vertical plane has its ends A and B constrained to remain in contact with a horizontal floor and a vertical wall, respectively, as shown in Fig. 197*a*. The bar starts from a vertical position, and the end A is moved along the floor with constant velocity v_0 so that its displacement $OA = v_0t$. Write the displacement-time, velocity-time, and acceleration-time equations for the vertical motion of the end B of the bar.



(a) (b)

FIG. 197

Solution. Choosing, for the x axis, the vertical line of motion OB with the origin at O, we see from the figure that the displacement x of the end B is given by the equation

$$x = \sqrt{l^2 - (v_0t)^2} \tag{i}$$

Equation (i) is the desired displacement-time equation and two differentiations with respect to time give the velocity-time and acceleration-time equations as follows:

$$\dot{x} = -\frac{v_0^2 t}{\sqrt{l^2 - v_0^2 t^2}} \tag{j}$$

$$\ddot{x} = -\frac{v_0^2 l^2}{(l^2 - v_0^2 t^2)^{\frac{3}{2}}} \tag{k}$$

These expressions can be used only in the interval $0 < t < l/v_0$ since when $t = l/v_0$ the bar is in a horizontal position and further motion of A to the right is impossible, since it was assumed that B is always in contact with the wall.

The displacement-time and velocity-time diagrams, plotted from Eqs. (i) and (j), are shown in Fig. 197*b*. Since Eq. (i) may be written in the form

$$\frac{x^2}{l^2} + \frac{t^2}{(l/v_0)^2} = 1$$

we conclude that the displacement-time curve is the quadrant of an ellipse with semiaxes equal to l and l/v_0.

3. A particle starting from rest moves rectilinearly with constant acceleration a and acquires in time t a velocity v, having traveled a total distance s. Develop formulas showing the relationships that must exist between any three of these quantities.

Solution. The required relationships can be found from a study of the velocity-time diagram shown in Fig. 198. In this diagram, the acceleration a is represented by the slope of the straight line OA, the velocity v by the ordinate BA, the time t by the abscissa OB, and the distance s by the area OAB. Thus from the geometry of the right triangle OAB, we may write

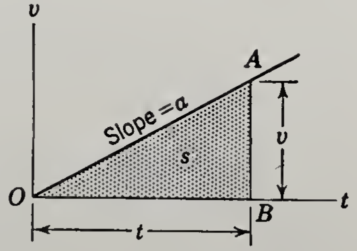


FIG. 198

$$v = at \qquad s = \tfrac{1}{2}vt \tag{l}$$

From these two equations, we can also obtain the relationships

$$s = \tfrac{1}{2}at^2 \qquad v = \sqrt{2as} \tag{m}$$

Formulas (l) and (m) are very useful in the numerical solution of problems involving the rectilinear motion of a particle with constant acceleration, but it must be remembered that they do not account for any initial displacement or initial velocity.

PROBLEM SET 6.1

1. The rectilinear motion of a particle is defined by the displacement-time equation $x = x_0 + v_0 t + \frac{1}{2}at^2$. Construct displacement-time and velocity-

time diagrams for this motion and find the displacement and velocity at time $t_2 = 2$ sec. The following numerical data are given: $x_0 = 10$ in., $v_0 = 5$ ips, $a = 20$ in./sec². *Ans.* $x_2 = 60$ in.; $v_2 = 45$ ips.

2. The velocity-time relationship of a moving particle is given by the equation $\dot{x} = \frac{1}{2}ct^2$, where $c = 8$ ft/sec³. Determine the displacement of the particle at the instant $t_3 = 3$ sec if there was no initial displacement. *Ans.* $x_3 = 36$ ft.

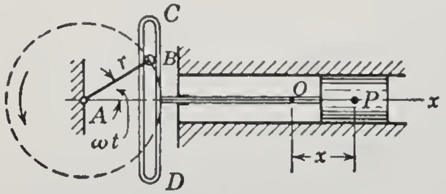


FIG. A

3. If the crank of the engine shown in Fig. A rotates $\omega = 4\pi$ radians/sec and the crank radius $r = 10$ in., find the maximum velocity and maximum acceleration of the piston. *Ans.* $|\dot{x}_{\max}| = 40\pi$ ips; $|\ddot{x}_{\max}| = 160\pi^2$ in./sec².

4. A rope AB is attached at B to a small block of negligible dimensions and passes over a pulley C so that its free end A hangs 5 ft above the ground when the block rests on the floor (Fig. B). The end A of the rope is moved horizontally in a straight line by a man walking with a uniform velocity $v_0 = 10$ fps. (*a*) Plot the velocity-time diagram for the motion of the block B. (*b*) Find the time t required for the block to reach the pulley if $h = 15$ ft and the pulley is negligibly small. *Ans.* (*a*) $v = v_0^2t/\sqrt{h^2 + v_0^2t^2}$; (*b*) $t = 3.16$ sec.

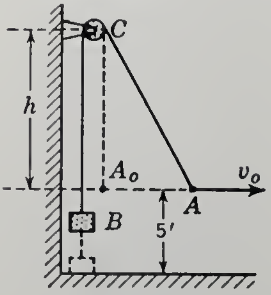


FIG. B

5. A particle starts from rest and moves along a straight line with constant acceleration a. If it acquires a velocity $v = 10$ fps after having traveled a distance $s = 25$ ft, find the magnitude of the acceleration. *Ans.* $a = 2$ ft/sec.²

6. A bullet leaves the muzzle of a gun with velocity $v = 2{,}500$ fps. Assuming constant acceleration from breech to muzzle, find the time t occupied by the bullet in traveling through the gun barrel, which is 30 in. long. *Ans.* $t = 0.002$ sec.

7. A ship while being launched slips down the skids with uniform acceleration. If 10 sec is required to traverse the first 16 ft, what time will be required to slide the total distance of 400 ft? With what velocity v will the ship strike the water? *Ans.* $t = 50$ sec; $v = 16$ fps.

8. Water drips from a faucet at the uniform rate of n drops per second. Find the distance x between any two adjacent drops as a function of the time t that the trailing drop has been in motion. Neglect air resistance and assume constant acceleration $g = 32.2$ ft/sec². *Ans.* $x = gt/n + g/2n^2$.

9. A stone is dropped into a well and falls vertically with constant acceleration $g = 32.2$ ft/sec.². The sound of impact of the stone on the bottom of the well is heard 6.5 sec after it is dropped. If the velocity of sound is 1,120 fps, how deep is the well? *Ans.* 577 ft.

10. The rectilinear motion of a particle is defined by the displacement-time equation $x = x_0(2e^{-kt} - e^{-2kt})$, in which x_0 is the initial displacement, k is a constant, and e is the natural logarithmic base. Sketch the displacement-time and velocity-time curves for this motion and find the maximum velocity of the particle. *Ans.* $\dot{x}_{\max} = -kx_0/2$, when $t = \ln 2/k$.

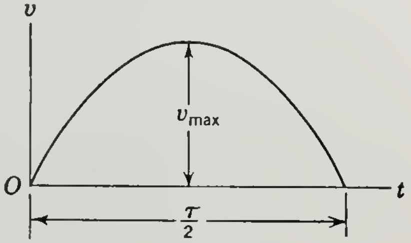


FIG. C

11. If the velocity-time diagram for the rectilinear motion of a particle is the half wave of a sine curve as shown in Fig. C, find the total distance x that the particle travels during the half-period time interval $\tau/2$. *Ans.* $x = \tau v_{\max}/\pi$.

12. If the velocity-time curve shown in Fig. C is a parabola with vertical axis, find the distance traveled by the particle during the time interval $\tau/2$. *Ans.* $x = \tau v_{\max}/3$.

13. An automobile starting from rest increases its speed from 0 to v with a constant acceleration a_1, runs at this speed for a time, and finally comes to rest with constant deceleration a_2. If the total distance traveled is s, find the total time t required. *Ans.* $t = s/v + (v/2)\,(1/a_1 + 1/a_2)$.

14. The greatest possible acceleration or deceleration that a train may have is a, and its maximum speed is v. Find the minimum time in which the train can get from one station to the next if the total distance is s. *Ans.* $t_{\min} = s/v + v/a$.

6.2. Principles of dynamics. *Newton's Laws.* In Art. 6.1, we have discussed the *kinematics* of rectilinear motion of a particle without regard to its cause. We now turn our attention to the question of the *kinetics* of rectilinear motion, i.e., the relationship between the kind of motion of a particle and the forces producing it. As a basis of this discussion of kinetics, we take several axioms called the *principles of dynamics.* These axioms are broad generalizations of Kepler's observations on the motion of heavenly bodies and of carefully conducted experiments with the motion of earthly bodies. The first reliable experiments were made by Galileo who discovered the first two laws of motion of a particle.[1] The complete set of principles and their final formulation are the work of Newton and they are commonly called *Newton's laws of motion.* We now proceed with a detailed discussion of these laws.

FIRST LAW. *Every body continues in its state of rest or of uniform motion in a straight line except in so far as it may be compelled by force to change that state.* This law, sometimes called the *inertia law,* was first discovered by Galileo. It was at variance with the teaching of ancient philosophers who maintained that, to produce uniform rectilinear motion of a particle, the action of a constant force in the direction of motion would be required. The statement was at variance also with everyday experience, which shows that a body projected along a horizontal surface has its velocity gradually diminished and finally comes to rest. Galileo perceived that this change in velocity of a body moving along a horizontal plane is due to friction forces and air resistance. By reducing these forces we can approach more and more closely to a uniform motion and in the ideal case, in which resistances are entirely eliminated, the body will move with a constant velocity along a straight line. In the particular case where this velocity is zero, the body remains at rest if forces do not act upon it. This implies *absolute rest,* but in practical problems we may consider the surface of the earth as immovable and refer the motion of particles to the earth. Solutions obtained on the basis of such an assumption will usually be in satisfactory agreement with experiments and observations. Sometimes this assumption is not sufficiently accurate, and the motion of the earth must be taken into account. Then a system of coordinates defined by fixed stars can be taken as an immovable system and the motion of the earth with respect to these stars considered.

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[1] See Galileo Galilei, "Two New Sciences," trans. by H. Crew and A. deSalvio, Macmillan, New York, 1933.

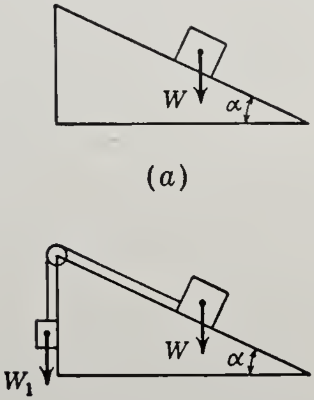
From the first law it follows that any change in velocity of a particle is the result of a force. The question of the relation between this change in velocity and the force that produces it is answered by the second law of dynamics which follows:

SECOND LAW. *The acceleration of a given particle is proportional to the force applied to it and takes place in the direction of the straight line in which the force acts.* This law also was first discovered by Galileo. In his famous experiments with falling bodies, he found, by measuring stretches passed in successive equal intervals of time, that this motion has a constant acceleration. Thus it was established that a constant force acting on a given body produces a constant acceleration. It was found also that this acceleration is independent of the substance of the body. That is, for example, all bodies falling under the action of gravity have the same acceleration. This conclusion was held by Galileo and later verified by Newton by means of numerous elaborate experiments in which he used pendulums made of various materials and derived the magnitude of the acceleration produced by gravity from the periods of oscillation of these pendulums. This acceleration due to gravity is usually denoted by g and from the experiments of Galileo and Newton its value was found to be 32.2 ft/sec^2 = 386 in./sec.^{2}

To slow down the velocity of motion of a falling body and thereby eliminate the difficulty in measuring short intervals of time, Galileo came to the idea of using for his experiments a body moving along an inclined plane (Fig. 199*a*). He found that this motion has also a constant acceleration. By varying α, the angle of the inclined plane, he was able to investigate many cases of uniformly accelerated motion and, by taking every precaution to minimize friction, found that between the acceleration a of a body moving along the inclined plane and the acceleration g of a freely falling body there exists the relation

$$a = g \sin \alpha \tag{a}$$

From Eq. (a) we see that as α approaches $\pi/2$ we obtain the case of a freely falling body, while as α approaches zero we obtain the case of uniform motion of the body with zero acceleration.



(*a*)

(*b*)

FIG. 199

Considering the forces acting on the moving particle W in Fig. 199*a*, Galileo made the observation that, in the case of motion along the inclined plane, not the gravity force W but a smaller force must be taken, namely, a force equal to the weight W_1 (Fig. 199*b*) which would be necessary to keep the body on the inclined plane in equilibrium. Considering an ideal case in which there is no friction, this force is

$$W_1 = W \sin \alpha \tag{b}$$

Comparing expressions (a) and (b), we see that with the variation of the angle α the force acting on the moving particle varies in the same proportion as the acceleration. In this way Galileo discovered that the acceleration of a particle is proportional to the acting force. That is, if any force produces a known acceleration of a given body, double this force, when acting on the same body, will produce double the acceleration, and so on.

This observation, made by Galileo on the basis of his experiments with uniformly accelerated rectilinear motion of a particle, was generalized by Newton and taken by him as the second law of dynamics. In this law, as formulated above, there is nothing said about the motion of the particle before it was acted upon by the force. The acceleration of a particle, produced by a given force, is independent of the motion of the particle. A given force acting on a particle produces the same acceleration regardless of whether the particle is at rest or in motion and regardless of the direction of motion.

Likewise, there is nothing said as to how many forces are acting on the particle. If any system of concurrent forces is acting on a particle, then each force produces exactly the same acceleration as it would if acting alone. Thus the resultant acceleration of a particle is obtained as the geometric sum of the accelerations produced separately by each of the forces that acts upon it, and since the acceleration produced by each force is in the direction of the force and proportional to it, the resultant acceleration will be in the direction of, and proportional to, the resultant force.

By using the first two laws formulated above, we can investigate the motion of a single particle subjected to the action of given forces. In more complicated cases where we must deal with a system of particles or rigid bodies, the mutual actions and reactions between them must be considered, and this can be done by using the law of action and reaction which we have already made use of in our problems of statics. Newton's formulation of this law reads as follows:

THIRD LAW. *To every action there is always an equal and contrary reaction, or the mutual actions of any two bodies are always equal and oppositely directed.*

If one body presses another, it is, in turn, pressed by the other with an equal force in the opposite direction. If one body attracts another from a distance, this other attracts it with an equal and opposite force. For instance, the sun attracts the earth with a certain force and therefore the earth attracts the sun with exactly the same force. All this holds not only for forces of gravitation but also for other kinds of forces such as magnetic attraction, electric forces, or pressure forces between two bodies in contact as we have considered in various problems of statics. A magnet attracts a piece of iron no more than the iron attracts the magnet.

General Equation of Motion of a Particle. By using the second law of dynamics, we can establish the general equation of motion of a particle. We know from Galileo's experiments that the gravity force W, if acting alone, produces an acceleration of the particle equal to g. If, instead of the force W, a force F acts on the same particle, then from the second law it follows that the acceleration a produced by this force is in the direction of the force and is in the same ratio to the known acceleration g as the force F is to the gravity force W, that is,

$$\frac{a}{g} = \frac{F}{W}$$

from which we obtain

$$\frac{W}{g} a = F \tag{33}$$

Equation (33) is the *general equation of motion* of a particle from which its acceleration at any instant can be obtained provided the acting force F at this instant is known.

It is seen from Eq. (33) that for a given magnitude of the force F the acceleration produced is inversely proportional to the factor W/g. This factor measures the degree of sluggishness with which the particle yields to the action of an applied force and is a measure of the *inertia* of the particle. It is called the *mass* of the particle and is generally denoted by m. Thus using the notation

$$m = \frac{W}{g} \tag{c}$$

the general equation of motion of a particle becomes

$$ma = F \tag{33$'$}$$

From Eq. (c) we see that the mass m of a particle has the dimension of force ÷ acceleration. Taking the pound as the unit of force, the foot as the unit of length, and the second as the unit of time, we find that mass will be measured in units of pounds × seconds^2 ÷ feet, usually written lb-sec^2/ft. This derived unit of mass is sometimes called the slug from the word sluggishness of which it is a measure and sometimes the gee-pound because it is the gravity force W divided by the gravitational acceleration g.

In the above discussion it was assumed that the acceleration g due to the force of gravity is constant. However, accurate measurements show that the value of g depends on locality, increasing with latitude and decreasing with elevation above sea level. For a given locality the value of g expressed in units of feet per second per second is represented to a high degree of accuracy by the formula

$$g = 32.089(1 + 0.00524 \sin^2 \varphi)(1 - 0.000000096h) \tag{d}$$

in which φ is the latitude and h the elevation in feet. It is seen that as φ varies from 0 to $\pi/2$ the total variation in g is only about 0.5 per cent, while as h varies from 0 to 25,000 ft, the corresponding variation in g is about 0.25 per cent. Such fluctuation is unimportant in most engineering problems and we can assume with sufficient accuracy that g is constant. In our calculations we shall usually take $g = 32.2$ ft/sec^2 $= 386$ in./sec^2.

In applying Eq. (33) to cases where greater accuracy is required, we must observe carefully the meaning of the various symbols appearing therein.

The quantity W designates the gravity force acting on the particle, and this force, like the acceleration due to gravity, varies slightly from one locality to another. This variation can be noted when we are measuring the force on the basis of the elongation that it produces in the spring of a dynamometer. A more accurate method of detecting the variation in gravity force, however, is by observing the period of oscillation of a pendulum which, as we shall see later (Art. 8.5), is inversely proportional to the square root of the gravity force acting on the pendulum. From Galileo's experiments we know that the acceleration of a particle is proportional to the active force. Thus W and g in Eq. (33) are varying with locality in the same proportion and their ratio, which defines the mass of the body, remains constant.

The inconvenience resulting from variation in gravity is sometimes eliminated by a change in our system of units. In mechanics we deal with four kinds of quantities, *length, time, force,* and *mass,* whose relationships are expressed by the equation of motion (33) or (33′). We can take any three of these quantities as primary units and then the fourth will be derived from the equation of motion. In engineering mechanics we assume length, time, and force as our fundamental units (gravitational system of units). This leads to the derived unit of mass called the slug, or gee-pound, as discussed above.

In physics and astronomy the units of length, time, and mass are usually taken as fundamental units (absolute system of units). As the unit of mass, the quantity of matter in a definite volume of a definite substance under definite physical conditions is usually taken. For instance, in physics, the unit of mass is taken as the mass of a cubic centimeter of water at the temperature of its maximum density and under specified pressure. This unit of mass is called the *gram.* On this basis it is seen from Eq. (33′) that the corresponding derived unit of force is that force which by acting on a unit of mass produces unit acceleration. Taking the gram as the unit of mass, the centimeter as the unit of length, and the second as the unit of time, we find that the unit of force is that force which by acting on 1 g of mass produces an acceleration of 1 cm/sec^2. This derived unit force is called a *dyne.* Sometimes in the absolute system of units the *pound*[1] is taken as the unit of mass, the foot as the unit of length, and the second as the unit of time. Then the corresponding derived unit of force is seen to be that force which by acting on a unit of mass (1 lb) produces an acceleration of 1 ft/sec^2. This derived unit of force is called the *poundal.*

6.3. Differential equation of rectilinear motion. The general equation of motion for a particle (Eq. 33) can be applied directly to the case of rectilinear translation of a rigid body, since all particles of the body have the same motion and we consider the entire body as a particle concentrated at its center of gravity. Whenever such a body or particle moves under the action of a force applied at its center of gravity and having a fixed line of action, acceleration of the body is produced in the same direction, and if any initial velocity of the body is also directed along this line, we obtain the case of rectilinear translation. Taking the line of motion as the x-axis and using for acceleration the notation $\ddot{x}$ (see Eq. 32) and for the resultant acting force the notation X, the differential equation of rectilinear motion of a particle becomes[1]

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[1] The platinum cylinder kept in the Tower of London and called the Imperial standard pound.

$$\frac{W}{g}\ddot{x} = X \tag{34}$$

This equation can be used for solving two kinds of problems: (1) The motion of the particle is given, i.e., the displacement x is expressed as a certain function of time t and it is required to find the force X necessary to produce such a motion. (2) The force X is given, and it is required to find the motion of the particle, i.e., to find for x such a function of t that Eq. (34) will be satisfied.

Problems of the first kind are very simple. In each particular case it is only necessary to substitute for x in Eq. (34) the given function of time. Then by successive differentiations the force X is obtained. The second kind of problem requires the integration of Eq. (34), which cannot always be rigorously solved. Various particular cases for which such integration can be made will be considered in subsequent articles. For the present we shall limit our attention to the solution of some problems of the first kind.

EXAMPLES

1. For the engine represented in Fig. 200, the combined weight of the piston and piston rod $W = 100$ lb, the crank radius $r = 10$ in., and the uniform speed of rotation $n = 120$ rpm. Determine the magnitude of the resultant force acting on the piston: (*a*) when it is in an extreme position; (*b*) when in its middle position.

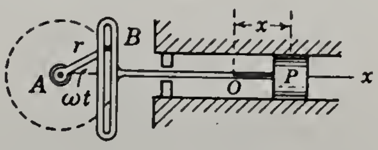

FIG. 200

Solution. As discussed in Art. 6.1, the piston has a simple harmonic motion represented by the displacement-time equation

$$x = r\cos\omega t \tag{a}$$

in which $\omega = 2\pi n/60 = 4\pi$ radians/sec. By double differentiation of Eq. (a) we obtain

$$\ddot{x} = -r\omega^2\cos\omega t \tag{b}$$

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[1] Since in engineering mechanics we use the gravitational system of units, we usually adhere to the notation W/g for mass, since this emphasizes its dimensional characteristic.

Substituting the value (b) for $\ddot{x}$ in Eq. (34), we find

$$X = -\frac{W}{g} r\omega^2 \cos \omega t \tag{c}$$

from which we see that the resultant active force is a function of time. For any instant at which the piston is in the extreme left position, it is seen from the figure that the angle ωt will be equal to $i\pi$ where i is any odd integer and hence, for this position, $\cos \omega t = -1$. Substituting this value in Eq. (c), and using the given data we obtain $X = 409$ lb.

For the middle position of the piston the angle ωt must be $i(\pi/2)$, where i is again any odd integer, and we see that $\cos \omega t = 0$. From this we conclude that for this position the resultant force is zero.

2. A balloon of gross weight W is falling vertically downward with constant acceleration a. What amount of ballast Q must be thrown out in order to give the balloon an equal upward acceleration a? Air resistance should be neglected.

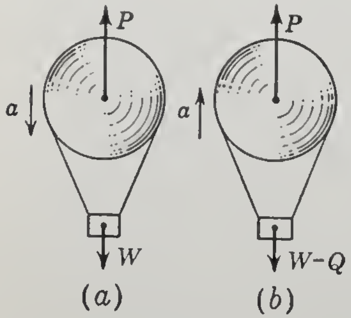

FIG. 201

Solution. Let us consider first the case where the balloon is falling (Fig. 201*a*). The active forces are the total weight W, including the ballast, which acts vertically downward and the buoyant force P, representing the weight of the displaced volume of air. Equation (34), then, becomes

$$\frac{W}{g} a = W - P \tag{d}$$

Considering next the case where the balloon is rising, after the ballast has been thrown out (Fig. 201*b*), Eq. (34) becomes

$$\frac{W - Q}{g} a = P - (W - Q) \tag{e}$$

Eliminating the buoyant force P from Eqs. (d) and (e), we obtain

$$Q = \frac{2Wa}{g + a} \tag{f}$$

It is interesting to check the logic of this equation by considering two extreme conditions. If the balloon were in equilibrium in the first case (i.e., falling with an acceleration $a = 0$), we would conclude at once that no ballast would have to be thrown out to cause it to rise with the same zero acceleration. Putting $a = 0$ in Eq. (f), we obtain $Q = 0$. On the other hand, if, in the first case, the balloon had no buoyancy at all so that it was falling with an acceleration $a = g$, we would conclude that it would be physically impossible to throw out sufficient ballast to cause it to rise with the same acceleration.

Putting $a = g$ in Eq. (f), we obtain $Q = W$, which result, of course, implies that it is impossible to throw out sufficient ballast to cause the balloon to rise.

3. Two equal weights W and a single weight Q are attached to the ends of a flexible but inextensible cord overhanging a pulley as shown in Fig. 202. If the system moves with constant acceleration a as indicated by the arrows, find the magnitude of the weight Q. Neglect air resistance and the inertia of the pulley.

Solution. We have here a system of two particles and may write two equations of motion. Denoting by S the tension in the cord, we have for the weight on the left

$$\frac{W}{g} a = S - W$$

and for the weights on the right

$$\frac{W + Q}{g} a = W + Q - S$$

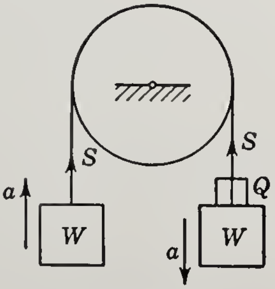

FIG. 202

Eliminating the tensile force S from these two equations, we find

$$Q = \frac{2Wa}{g - a} \tag{g}$$

From this equation we note that to produce an acceleration $a = g$ of the system would require a weight $Q = \infty$.

4. In Fig. 203, a piston of weight W, constrained to move vertically, is raised and lowered by a cam ACB that moves horizontally with constant speed v_0. The face of the cam has the shape of a full cosine wave of length l and maximum height h as shown. Find the greatest speed v_0 that the cam may have without losing contact with the cam follower D, throughout the cycle. Neglect friction in the sleeve which guides the rise and fall of the piston.

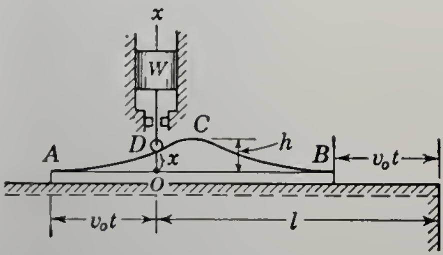

FIG. 203

Solution. Taking the lowest position O of the cam follower as an origin, the displacement-time equation for the vertical motion of the piston will be

$$x = \frac{h}{2}\left(1 - \cos\frac{2\pi v_0 t}{l}\right) \tag{h}$$

When v_0t is equal to zero or l, representing the beginning and end of the sweep of the cam, this expression gives $x = 0$ as it should, while for $v_0t = l/2$, representing the halfway mark, we get $x = h$, as it should be.

Differentiating expression (h) successively with respect to time, we obtain

$$\dot{x} = \frac{h\pi v_0}{l} \sin\frac{2\pi v_0 t}{l} \tag{i}$$

for the velocity-time equation, and

$$\ddot{x} = \frac{2h\pi^2 v_0^2}{l^2} \cos \frac{2\pi v_0 t}{l} \tag{j}$$

for the acceleration-time equation. Expressions (h), (i), and (j), of course, hold only within the time interval $0 < t < l/v_0$.

Denoting by P the vertical component of the pressure between cam and follower, the equation of motion [Eq. (34)] for the rise and fall of the piston will be

$$\frac{W}{g}\ddot{x} = P - W$$

from which

$$P = W\left(1 + \frac{\ddot{x}}{g}\right) \tag{k}$$

We see from this expression that for P to be always positive, we must have

$$\ddot{x} \geqq -g \tag{l}$$

Now from Eq. (j), the maximum negative value of $\ddot{x}$ is $-2h\pi^2 v_0^2/l^2$, when

$$v_0 t = \frac{l}{2}$$

that is, when the cam is at the halfway point and the piston is in its highest position. Substituting this value into condition (l) and transposing, we obtain

$$\frac{2h\pi^2 v_0^2}{l^2} \leqq g$$

from which

$$v_0 \leqq \sqrt{\frac{gl^2}{2\pi^2 h}} \tag{m}$$

Taking, for example, $l = 20$ in. and $h = 4$ in., we obtain $v_0 \leqq 44.2$ ips.

PROBLEM SET 6.3

1. An elevator of gross weight $W = 1{,}000$ lb starts to move upward with constant acceleration and acquires a velocity $v = 6$ fps, after traveling a distance $s = 6$ ft. Find the tensile force S in the cable during this accelerated motion. Neglect friction. *Ans.* $S = 1{,}093$ lb.

2. The elevator of Prob. 1, when stopping, moves with constant deceleration and from the constant velocity $v = 6$ fps comes to rest in 2 sec. Determine the pressure P transmitted during stopping to the floor of the elevator by the feet of a man weighing 170 lb. *Ans.* $P = 154$ lb.

3. A train weighing 200 tons without the locomotive starts to move with constant acceleration along a straight horizontal track and in the first 60 sec acquires a velocity of 35 mph. Determine the tension S in the draw-bar between the locomotive and train if the total resistance to motion due to friction and air resistance is constant and equal to 0.005 times the weight of the train. *Ans.* $S = 12{,}620$ lb.

4. The driver of an automobile, traveling along a straight level highway, suddenly applies the brakes so that the car slides for 2 sec, covering a distance of 32.2 ft before coming to a stop. Assuming that during this time the car moved with constant deceleration, find the coefficient of friction between the tires and the pavement. *Ans.* $\mu = 0.5$.

5. A mine cage of weight $W = 1$ ton starts from rest and moves downward with constant acceleration, traveling a distance $s = 100$ ft in 10 sec. Find the tensile force in the cable during this time. *Ans.* $S = 1{,}876$ lb.

6. A particle of weight W is dropped vertically into a medium that offers a resistance proportional to the square of the velocity of the particle. The buoyancy of the medium is negligible, and the resisting force is f when the velocity is 1 fps. What uniform velocity will the particle finally attain? *Ans.* $v = \sqrt{W/f}$.

7. A weight $W = 1{,}000$ lb is supported in a vertical plane by a string and pulleys arranged as shown in Fig. A. If the free end A of the string is pulled vertically downward with constant acceleration $a = 6$ ft/sec², find the tension S in the string. Neglect friction in the pulleys. *Ans.* $S = 546.6$ lb.

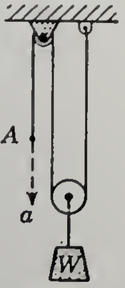

FIG. A

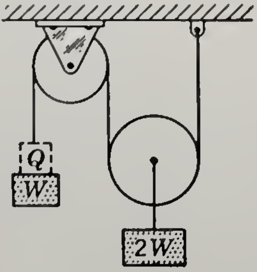

FIG. B

8. Weights W and $2W$ are supported in a vertical plane by a string and pulleys arranged as shown in Fig. B. Find the magnitude of an additional weight Q applied on the left which will give a downward acceleration $a = 0.1g$ to the weight W. Neglect friction and inertia of pulleys. *Ans.* $Q = W/6$.

9. A weight W attached to the end of a small flexible rope of diameter $d = \frac{1}{4}$ in. is raised vertically by winding the rope on a reel as shown in Fig. C. If the reel is turned uniformly at the rate of 2 rps, what will be the tension S in the rope? Neglect inertia of the rope and slight lateral motion of the suspended weight W. *Ans.* $S = 1.016W$.

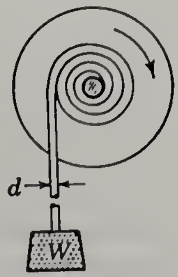

FIG. C

***10.** In Fig. D a weight W is raised vertically by means of a cam which moves horizontally from right to left with constant speed v_0. Counting time t from the instant when the cam is in the position shown by dotted lines and the weight W is in its lowest position, find the compressive force S in the vertical rod CD as a function of time. The contour ACB of the cam face is a parabola with vertical axis and vertex at A. Neglect friction.

*11. Solve Prob. 10 if the contour ACB of the cam face in Fig. D is a quarter sine wave.

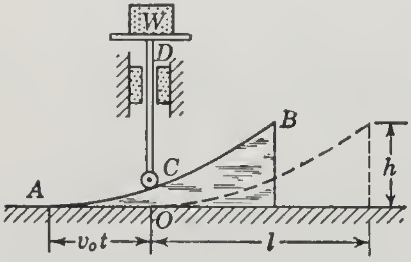

FIG. D

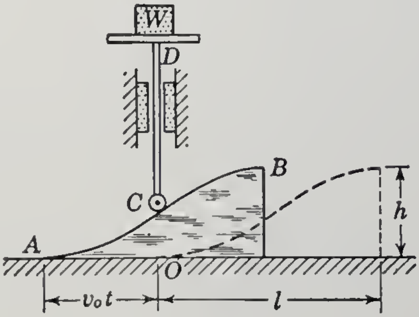

FIG. E

*12. In Fig. E the cam AB moves horizontally to the left with constant speed v_0, while the cam follower C is constrained to move vertically. Find the maximum compressive force S induced in rod CD during the sweep of the cam, if its face ACB is a half cosine wave. Neglect friction.

6.4. Motion of a particle acted upon by a constant force. We come now to the second kind of dynamics problem, in which the acting force is given and the resulting motion of the particle is required. We begin with the simplest case of a particle acted upon by a *constant force*, the direction of which remains unchanged. Under the action of such a force, the particle moves rectilinearly in the direction of the force and with constant acceleration. Referring to Fig. 204, let the line of motion be taken as the x axis and let X denote the magnitude of the force. Then if the particle starts from A with *initial displacement* x_0 and *initial velocity* $\dot{x}_0$ when $t = 0$, our problem is to find its velocity and displacement at any other time t.

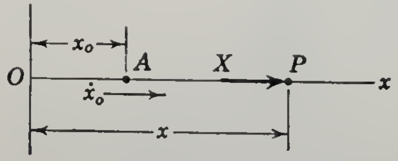

FIG. 204

We begin with the equation of motion [Eq. (34)], which we write in the form

$$\ddot{x} = g\frac{X}{W} = a \tag{35}$$

From this expression we see at once that when the magnitude of the constant force X is given, we need only to divide by the mass W/g of the particle to obtain the corresponding constant acceleration a of the motion produced. To find $\dot{x}$ and x as functions of time, we must integrate this differential equation. To do this, we write

$$d(\dot{x}) = a\,dt \tag{a}$$

Then remembering that a is a constant, we obtain by integration

$$\dot{x} = at + C_1 \tag{b}$$

where C_1 is a constant of integration. To find the meaning of this constant, let us consider the conditions of motion of the particle at the initial moment when $t = 0$. Since $\dot{x}_0$ is the velocity of the particle at this instant, we find from Eq. (b) that $C_1 = \dot{x}_0$, that is, that it represents the initial velocity of the particle. Putting $\dot{x}_0$ for C_1 in Eq. (b), we obtain

$$\dot{x} = \dot{x}_0 + at \tag{36}$$

This is the general velocity-time equation for the rectilinear motion of a particle under the action of a constant force X which produces constant acceleration $a = Xg/W$.

Writing Eq. (36) in the form

$$dx = \dot{x}_0\,dt + at\,dt$$

and integrating again, we obtain

$$x = \dot{x}_0 t + \tfrac{1}{2}at^2 + C_2 \tag{c}$$

in which C_2 is again a constant of integration. Considering conditions at the initial moment ($t = 0$) and observing that the body has at this instant an initial displacement x_0 from the origin, we find from Eq. (c) that $C_2 = x_0$. Substituting this value in Eq. (c), we obtain

$$x = x_0 + \dot{x}_0 t + \tfrac{1}{2}at^2 \tag{37}$$

This is the general displacement-time equation for the rectilinear motion of a particle under the action of a constant force X producing constant acceleration $a = Xg/W$.

Proceeding as above with Eq. (35), we always obtain two arbitrary constants of integration for the determination of which we must have two given conditions such, for example, as the initial displacement and the initial velocity of the particle. When, in addition to the differential equation of motion, we have two such conditions, the motion can always be completely defined. It is important to keep in mind that initial conditions influence the motion of a particle quite as much as does the acting force. We might say that initial conditions represent the heredity of the motion, while the acting force represents its environment.

Equations (35) to (37), can be used to study any rectilinear motion of a particle under the action of any constant force. Having the magnitude of the force X given, we begin by calculating the acceleration a from Eq. (35). Then using this value of a together with given initial displacement x_0 and initial velocity $\dot{x}_0$ in Eqs. (36) and (37), we find velocity and displacement of the particle for any desired instant of time t.

In the particular case of a freely falling body, the acting force $X = W$ and Eq. (35) gives $a = g$. Thus for this case Eqs. (36) and (37) become

$$\dot{x} = \dot{x}_0 + gt \tag{36a}$$
$$x = x_0 + \dot{x}_0 t + \tfrac{1}{2}gt^2 \tag{37a}$$

When the body starts to fall from rest and the origin is chosen to coincide with its initial position, we have $x_0 = 0$ and $\dot{x}_0 = 0$ and Eqs. (36a) and (37a) reduce to the well-known expressions

$$\dot{x} = gt \tag{36b}$$
$$x = \tfrac{1}{2}gt^2 \tag{37b}$$

for a freely falling body, as established by Galileo on the basis of his experiments.

EXAMPLES

1. At a given instant several particles start to move, under the action of their gravity forces, from point O along the various inclined planes OA, OB, OC, and OD which make the angles α, β, γ, and $\pi/2$, respectively, with the horizontal (Fig. 205). Neglecting friction, prove that at any instant t these particles will all be on the same circle $OABCD$ constructed with the distance OD, that the freely falling particle has traveled, as a diameter.

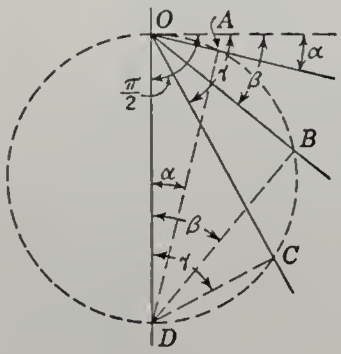

FIG. 205

Solution. Taking the line of motion in each case as the x axis and the starting point O as the origin, we have neither initial displacement nor initial velocity to consider. Then from Eq. (37b), the distance traveled by the freely falling particle is

$$OD = \tfrac{1}{2}gt^2 \tag{d}$$

For the case of a particle moving along any inclined plane which makes the angle θ with the horizontal, and neglecting friction, the active force in the direction of motion will be[1] $X = W \sin \theta$, and from Eq. (35) we obtain $a = g \sin \theta$. Substituting this value of a into Eq. (37b), we obtain

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[1] The component of the gravity force perpendicular to the inclined plane is balanced by the reaction of the plane and should not be considered.

$$x = \tfrac{1}{2}gt^2 \sin\theta \tag{e}$$

Comparing Eqs. (d) and (e), we conclude that the distances traveled by the other particles are as follows:

$$x_1 = OD \sin\alpha \qquad x_2 = OD \sin\beta \qquad x_3 = OD \sin\gamma$$

which, as we see from the geometry of the figure, proves the statement of the problem.

2. From the top of a tower of height $h = 120$ ft a ball is dropped at the same instant that another is projected vertically upward from the ground with an initial velocity $v_0 = 60$ fps. How far from the top do they pass and with what relative velocity?

Solution. For the ball that is dropped we choose the top of the tower as the origin and consider downward displacement as positive. Then this ball will have neither initial displacement nor initial velocity and from Eq. (37b) its displacement at any instant will be

$$x_1 = \tfrac{1}{2}gt^2 \tag{f}$$

For the other ball we choose the ground as the origin and consider upward displacement as positive. Then from Eq. (37a) its displacement at any instant will be

$$x_2 = v_0t - \tfrac{1}{2}gt^2 \tag{g}$$

When the balls pass, we must have

$$x_1 + x_2 = h \tag{h}$$

Substituting the values of x_1 and x_2 from Eqs. (f) and (g) into Eq. (h), we obtain

$$\tfrac{1}{2}gt^2 + v_0t - \tfrac{1}{2}gt^2 = h$$

from which, using the given numerical data, $t = 2$ sec. Using this value of t in Eq. (f), we find

$$(x_1)_{t=2} = 64.4 \text{ ft}$$

Differentiating Eqs. (f) and (g) once each with respect to time, we find that at the instant $t = 2$ sec, the two balls are moving downward with velocities of 64.4 and 4.4 fps, respectively. Hence the balls pass 64.4 ft below the top of the tower with a relative velocity of 60 fps 2 sec after starting.

3. A particle projected vertically upward is at a height h after t_1 sec and again after t_2 sec. Find this height h and also the initial velocity v_0 with which the particle was projected.

Solution. Neglecting air resistance, the particle at all times is moving under the action of its own gravity force W which is always directed vertically downward. We take the x axis along the vertical line of motion, the origin at the starting point, and consider upward displacement as positive. Then, from

Eq. (37a), we have for the instant $t = t_1$

$$h = v_0 t_1 - \tfrac{1}{2} g t_1^2 \tag{i}$$

Likewise, for the instant $t = t_2$

$$h = v_0 t_2 - \tfrac{1}{2} g t_2^2 \tag{j}$$

The elimination of v_0 between Eqs. (i) and (j) gives

$$h = \tfrac{1}{2} g t_1 t_2 \tag{k}$$

while the elimination of h gives

$$v_0 = \tfrac{1}{2} g (t_1 + t_2) \tag{l}$$

4. A small block of weight W is placed on an inclined plane as shown in Fig. 206a. What time interval t will be required for the block to traverse the distance AB if it is released from rest at A and the coefficient of kinetic friction on the plane is $\mu = 0.3$? What is the velocity at B?

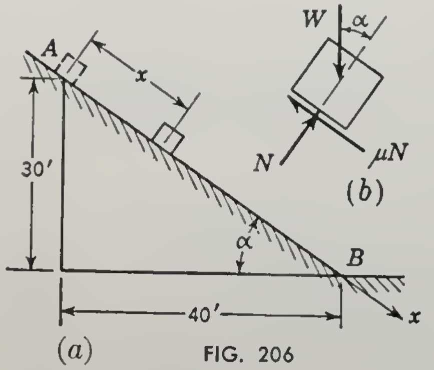

FIG. 206

Solution. From the free-body diagram of the block (Fig. 206b), we see that the resultant force acts directly down the plane and has the constant magnitude

$$X = W(\sin \alpha - \mu \cos \alpha)$$

Then from Eq. (35) we find for the acceleration

$$a = g(\sin \alpha - \mu \cos \alpha) \tag{m}$$

We note that to attain sliding of the block, this expression must be positive, which requires $\tan \alpha \geqq \mu$. Substituting the given numerical data into expression (m) we find

$$a = \tfrac{9}{25} g = 11.58 \text{ ft/sec}^2$$

Using this value of a in Eq. (37), together with the given initial conditions $x_0 = \dot{x}_0 = 0$, we obtain

$$x = \tfrac{1}{2} a t^2 = 50 \text{ ft}$$

from which $t = 2.94$ sec. Substituting this value of t into Eq. (36), we find for the velocity of the block at B, $\dot{x}_b = 34.1$ fps.

PROBLEM SET 6.4

1. A body starts to move vertically upward under the influence of gravity with an initial velocity $\dot{x}_0 = 20$ fps. Find (*a*) the maximum height to which it will rise and (*b*) the time required for it to return to its initial position. Take the starting point as the origin so that $x_0 = 0$ and neglect air resistance. *Ans.* (*a*) $x_{\max} = 6.2$ ft; (*b*) $t = 1.24$ sec.

2. A train is moving down a slope of 0.008 with a velocity of 30 mph. At a certain instant the engineer applies the brakes and produces a total resistance to motion equal to one-tenth of the weight of the train. What distance x will the train travel before stopping? *Ans.* $x = 327$ ft.

3. An elevator weighing 1,000 lb is moving upward with a uniform velocity of 12 fps. In what distance x will it stop after the power is shut off if the friction force opposing motion is 20 lb? *Ans.* $x = 2.19$ ft.

4. To determine experimentally the coefficient of friction between two materials, a small block of weight $W = 10$ lb is projected with initial velocity $v_0 = 30$ fps along a horizontal plane covered with the same material. If the block travels a total distance $x = 45$ ft before coming to rest, what is the coefficient of friction? *Ans.* $\mu = 0.31$.

5. Referring to Fig. A, find the acceleration a of the falling weight P if the coefficient of friction between the block Q and the horizontal plane on which it slides is μ. Neglect inertia of the pulley and friction on its axle. The following numerical data are given: $P = 10$ lb, $Q = 12$ lb, $\mu = \frac{1}{3}$. *Ans.* $a = 3g/11$.

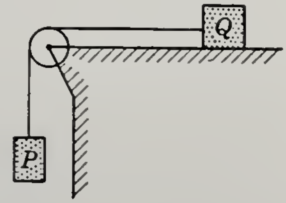

FIG. A

6. A train moves with a uniform speed of 36 mph along a straight level track. At a certain instant the engineer moves the throttle so as to increase the traction by 20 per cent. What distance x will the train cover before acquiring a speed of 45 mph if the resistance to motion is constant and equal to $\frac{1}{200}$ of the weight of the train? *Ans.* 4.61 miles.

7. A police investigation of tire marks shows that a car traveling along a straight level street had skidded for a total distance of 145 ft after the brakes were applied. The coefficient of friction between tires and pavement is estimated to be $\mu = 0.6$. What was the probable speed of the car when the brakes were applied? *Ans.* 51 mph.

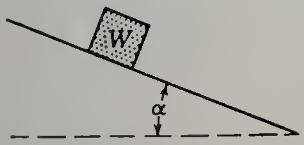

FIG. B

8. A small block of weight W rests on an adjustable inclined plane as shown in Fig. B. Friction is such that sliding of the block impends when $\alpha = 30°$. What acceleration will the block have when $\alpha = 45°$? Neglect any difference between static and kinetic friction. *Ans.* $a = 0.3g$.

9. Two small cars of weights $W_1 = 200$ lb and $W_2 = 100$ lb are connected by a flexible but inextensible string overrunning a pulley C and are free to roll on an inclined plane as shown in Fig. C. If the cars are released from rest in the positions shown, find the time t required for them to exchange positions. Neglect rolling resistance and friction in the pulley. *Ans.* $t = 6.1$ sec.

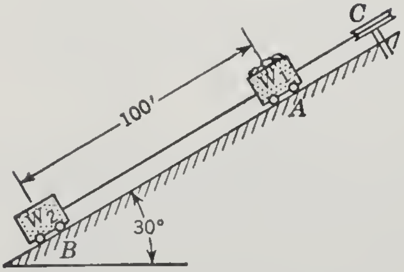

FIG. C

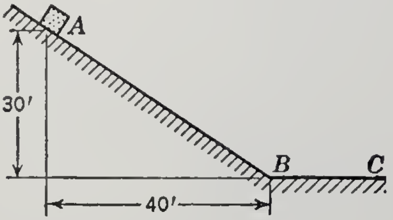

FIG. D

10. A small block starts from rest at point A and slides down the inclined plane AB in Fig. D. What distance s along the horizontal plane BC will it travel before coming to rest? The coefficient of kinetic friction between the block and either plane is $\mu = 0.3$. Assume that the initial velocity with which it starts to move along BC is of the same magnitude as that gained in sliding from A to B. *Ans.* $s = 60$ ft.

6.5. Force as a function of time. Sometimes we have to investigate the rectilinear motion of a particle under the action of a force the magnitude of which changes with time according to some given law. We shall discuss here two methods of integrating the differential equation of motion in such case.

First Method. If the acting force $X = F(t)$ can be expressed analytically, we substitute it directly into the equation of motion [Eq. (34)] and then proceed to separate the variables and integrate in the same manner as was done in Art. 6.4. In so doing, we always introduce two constants of integration, and to obtain a definite solution, two conditions of motion, such as initial displacement and initial velocity, must also be known.

Suppose, for example, that a particle of weight W starts from rest at the origin ($x_0 = 0$ and $\dot{x}_0 = 0$) and moves rectilinearly under the action of a force $X = Pe^{-kt}$, where P is the initial magnitude of the force and k is a constant. Then the equation of motion [Eq. (34)] becomes

$$\frac{W}{g}\ddot{x} = Pe^{-kt} \tag{a}$$

To separate the variables, we write this equation in the form

$$\frac{W}{g}\,d\dot{x} = Pe^{-kt}\,dt \tag{b}$$

which expresses the differential relation between velocity $\dot{x}$ and time t.

Integrating both sides of Eq. (b) we obtain

$$\frac{W}{g}\dot{x} + C_1 = -\frac{P}{k}e^{-kt} \tag{c}$$

where C_1 is the constant of integration. To evaluate this constant, we substitute the initial condition $\dot{x} = 0$ when $t = 0$, which gives $C_1 = -P/k$. Putting this back in Eq. (c), the velocity-time equation becomes

$$\dot{x} = \frac{Pg}{Wk}(1 - e^{-kt}) \tag{d}$$

This velocity-time relationship is represented graphically in Fig. 207*a* and we see that the velocity approaches asymptotically to the value Pg/Wk. Thus the motion tends to become uniform, and after kt becomes large (say $kt > 5$) it may be so considered without serious error.

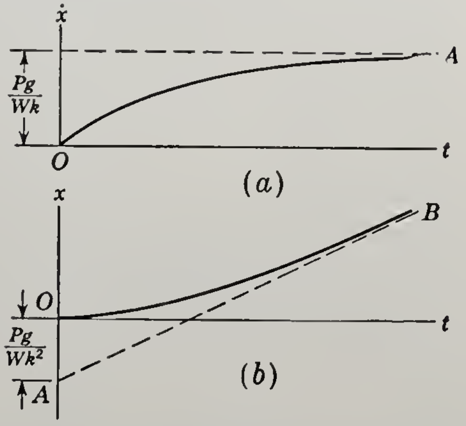

FIG. 207

Writing Eq. (d) in the form

$$dx = \frac{Pg}{Wk}(1 - e^{-kt})\,dt,$$

and integrating again, we obtain

$$x + C_2 = \frac{Pgt}{Wk} + \frac{Pg}{Wk^2}e^{-kt} \tag{e}$$

where C_2 is again a constant of integration. To evaluate this constant, we use the initial condition $x = 0$ when $t = 0$ and find

$$C_2 = +\frac{Pg}{Wk^2}$$

Putting this value back in Eq. (e) and rearranging the terms, the displacement-time equation becomes

$$x = \frac{Pg}{Wk^2}(kt + e^{-kt} - 1) \tag{f}$$

The corresponding displacement-time diagram is shown in Fig. 207*b*. For large values of t, the middle term in the parentheses of Eq. (f) approaches zero, and we obtain for the equation of the asymptote, shown by the dotted line in Fig. 207*b*,

$$x = \frac{Pg}{Wk^2}(kt - 1) \tag{g}$$

Again we may consider that for values of $kt > 5$, Eq. (g) represents the motion with sufficient accuracy for practical calculations. The motion represented by Eq. (f) is that experienced by fine particles of mist or dust falling through the air. It also holds for the settlement of particles of silt in a quiet reservoir of water.

Proceeding as above, the displacement-time equation of a particle moving under the action of any force $X = F(t)$ can be found if the initial displacement x_0 and the initial velocity $\dot{x}_0$ are given.

Second Method. Sometimes the acting force $X = F(t)$ cannot be expressed by an analytic function but is represented graphically by a curve called a *force-time diagram,* as shown in Fig. 208. In such case, we begin with the equation of motion [Eq. (34)] in the form

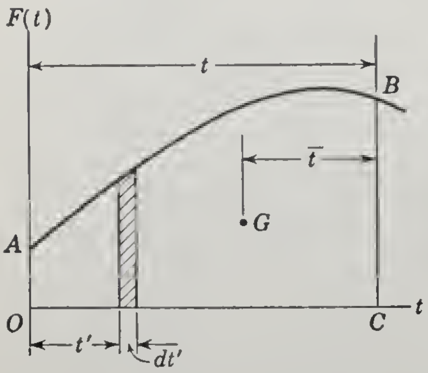

FIG. 208

$$m\ddot{x} = F(t')$$

where t' instead of t is used as a temporary symbol for time.[1] Dividing both sides of the equation of motion by the mass m of the particle and making a first separation of variables, we have

$$d\dot{x} = \frac{1}{m} F(t')\, dt' \tag{h}$$

This equation expresses the differential relation between velocity $\dot{x}$ and time t'. We see that the increment of velocity imparted to the particle during the time interval dt' is proportional to the area $F(t')\, dt'$ of the shaded strip of the force-time diagram in Fig. 208. Summing up all such increments of velocity from the initial moment $t' = 0$ to any time $t' = t$, we have

$$\int_{\dot{x}_0}^{\dot{x}} d\dot{x} = \frac{1}{m} \int_0^t F(t')\, dt'$$

where $\dot{x}_0$ is the velocity corresponding to $t' = 0$ and $\dot{x}$ is that corresponding to $t' = t$. Performing the indicated integration on the left side of this equation, we obtain the velocity-time equation in the following form:

$$\dot{x} = \dot{x}_0 + \frac{1}{m} \int_0^t F(t')\, dt' \tag{38a}$$

---

[1] We do this so that the symbol t can be used as a limit of integration.

The first term on the right side of this equation is the initial velocity. The second term represents the additional velocity imparted to the particle by the acting force $F(t')$. We see that the integral in this term simply represents the area of the force-time diagram up to the ordinate t. Hence the velocity-time equation can be expressed in the equivalent form

$$\dot{x} = \dot{x}_0 + \frac{1}{m}\Big[A_F\Big]_0^t \tag{38b}$$

where $\Big[A_F\Big]_0^t$ denotes the area of the force-time diagram between the ordinates 0 and t.

To express the displacement $x = f(t)$ also in terms of the area of the force-time diagram, we begin with Eq. (h) which states that the increment of velocity imparted to the particle at the instant t' is equal to $F(t')\,dt'/m$. As a result of this one increment of velocity, the displacement of the particle at some later time t will be

$$\frac{1}{m}F(t')\,dt'(t - t') \tag{i}$$

Now since each elemental strip of the force-time diagram by itself produces a like effect, we conclude that the displacement of the particle at the time t, due to the continuous action of the force $X = F(t')$ from $t' = 0$ to $t' = t$, will be

$$\frac{1}{m}\int_0^t F(t')(t - t')\,dt' \tag{j}$$

Finally, taking account of initial displacement and initial velocity, the complete displacement-time equation will be

$$x = x_0 + \dot{x}_0 t + \frac{1}{m}\int_0^t F(t')(t - t')\,dt' \tag{39a}$$

Considering the last term of this expression as represented by the integral (j), we see that $F(t')\,dt'(t - t')$ represents the statical moment of the shaded strip in Fig. 208 with respect to the ordinate t as an axis. Thus the sum of such statical moments as represented by the integral (j) can be expressed as the product of the total area A_F of the force-time diagram up to the ordinate t and the moment arm $\bar{t}$ to its centroid G, as shown in Fig. 208. Using this graphical interpretation of the integral (j), the displacement-time equation (39a) can be written in the equivalent form

$$x = x_0 + \dot{x}_0 t + \frac{1}{m}\Big[A_F\Big]_0^t\,\bar{t} \tag{39b}$$

where the symbol $\Big[A_F\Big]_0^t$ stands for the area $OABC$ of the force-time diagram as before and $\bar{t}$ is its moment arm measured from the ordinate t to the centroid G, as shown in Fig. 208.

This second method of handling the problem of rectilinear motion of a particle under the action of a force $X = F(t)$ can be used in all cases. If the function $F(t)$ can be expressed analytically, it can be substituted directly into Eqs. (38*a*) and (39*a*) after which the definite integrals can be evaluated. In doing this, however, it is important to write the force function as $F(t')$ so as not to confuse the variable t' with t. If the acting force $F(t')$ cannot be expressed analytically but is represented by a force-time diagram as in Fig. 208, we use Eqs. (38*b*) and (39*b*). In using these equations, of course, we must choose a definite value of time t at which to evaluate the velocity and displacement and we must be able to locate the centroid of the corresponding portion of the force-time diagram. Several examples will serve to illustrate the procedure in each particular case.

EXAMPLES

1. A particle of weight W moves rectilinearly under the action of a force $X = P \cos \omega t$. Develop the velocity-time and displacement-time equations if $x_0 = 0$ and $\dot{x}_0 = 0$.

Solution. In this case, we shall use the first method of integrating the differential equation of motion which is

$$\frac{W}{g}\ddot{x} = P \cos \omega t$$

Writing this in the form

$$\frac{W}{g}\,d\dot{x} = P \cos \omega t \, dt$$

and integrating, we obtain

$$\frac{W}{g}\dot{x} + C_1 = \frac{P}{\omega} \sin \omega t$$

Substituting the initial condition $\dot{x} = 0$ when $t = 0$, we find $C_1 = 0$ and the velocity-time equation becomes

$$\dot{x} = \frac{Pg}{W\omega} \sin \omega t \tag{k}$$

Writing Eq. (k) in the form

$$\frac{W}{g}\,dx = \frac{P}{\omega} \sin \omega t \, dt$$

and integrating again, we obtain

$$\frac{W}{g}x + C_2 = -\frac{P}{\omega^2}\cos\omega t$$

Using the initial condition $x = 0$ when $t = 0$, we find $C_2 = -P/\omega^2$ and the displacement-time equation becomes

$$x = \frac{Pg}{W\omega^2}(1 - \cos\omega t) \tag{l}$$

We see that the particle performs simple harmonic motion centered about the point $x = Pg/W\omega^2$ (see p. 251).

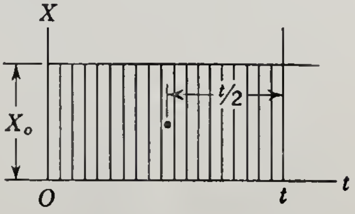

FIG. 209

2. Using Eqs. (38*b*) and (39*b*), find the velocity and displacement at any time t for a particle of weight W moving rectilinearly under the action of a constant force X_0. Assume $x_0 = \dot{x}_0 = 0$.

Solution. For the case of a constant force X_0 the force-time diagram will be as shown in Fig. 209. The area of this diagram up to the ordinate t is X_0t, and consequently the velocity of the particle at this instant is, by Eq. (38*b*),

$$\dot{x} = \frac{X_0t}{m} = at \tag{m}$$

which checks the value given by Eq. (36), page 267.

The statical moment with respect to the ordinate at t of the same area is $X_0t(t/2)$, and consequently from Eq. (39*b*) the displacement at any time t is

$$x = \frac{X_0}{m}\frac{t^2}{2} = \tfrac{1}{2}at^2, \tag{n}$$

which checks the value given by Eq. (37).

3. A particle of mass m is acted upon by a force that has the initial magnitude X_0 when $t = 0$ and decreases at a uniform rate until, when $t = t_1$, its magnitude is zero. Find the velocity and displacement of the particle when $t = t_2$, assuming that $t_2 > t_1$. Assume $x_0 = 0$ and $\dot{x}_0 = 0$.

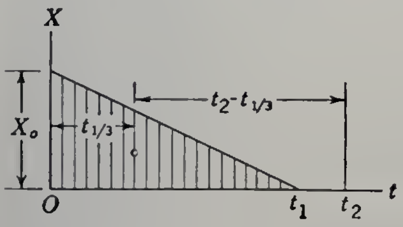

FIG. 210

Solution. The force-time diagram is shown in Fig. 210. The total area of this diagram up to the time t_2 is $X_0t_1/2$ and hence, by Eq. (38*b*), the velocity is

$$(\dot{x})_{t_2} = \frac{X_0}{m}\frac{t_1}{2} \tag{o}$$

From Eq. (o) we see that after the time t_1 the velocity of the particle remains constant which is to be expected, since no forces act upon it after this time.

The centroid of the area has the abscissa $t_1/3$, and hence $\bar{t} = t_2 - t_1/3$ as shown in the figure. Then from Eq. (39*b*) we find for the displacement of the particle at the moment t_2

$$(x)_{t_2} = \frac{X_0}{m}\frac{t_1}{2}\left(t_2 - \frac{t_1}{3}\right) \tag{p}$$

PROBLEM SET 6.5

1. Forces act upon a particle during short intervals of time as shown by the force-time diagram given in Fig. A. Find the velocity and displacement of the particle at any time $t > t_3$. Assume $x_0 = 0$ and $\dot{x}_0 = 0$ when $t = 0$.

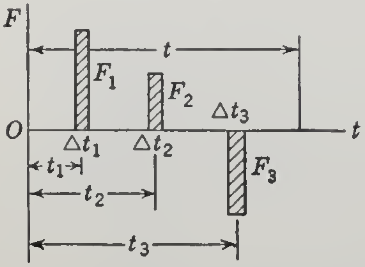

FIG. A

2. A particle initially at rest is submitted to the action of a force $X = kt$. Prove that the ratio $x/\dot{x}$ increases as a linear function of time. *Ans.* $x/\dot{x} = t/3$.

3. The magnitude of a force acting upon a body of mass W/g is initially zero and increases uniformly with time, being equal to W at the end of the first second. Find the velocity and displacement of the body after 6 sec, assuming $x_0 = 0$ and $\dot{x}_0 = 1$ fps. *Ans.* $(\dot{x})_{t=6} = 581$ fps; $(x)_{t=6} = 1{,}165$ ft.

4. A particle of mass m moves rectilinearly under the action of a force $X = F(t)$ as represented by the force-time diagram OCB in Fig. B. If this curve is a parabola, find the displacement at time t_1. *Ans.* $x = Pt_1^2/3m$.

5. A particle of weight W moves rectilinearly under the action of a force

$$X = P \sin \omega t$$

Derive the general displacement-time equation, assuming $x_0 = 0$ and $\dot{x}_0 = 0$. *Ans.* $x = (Pg/W\omega^2)(\omega t - \sin \omega t)$.

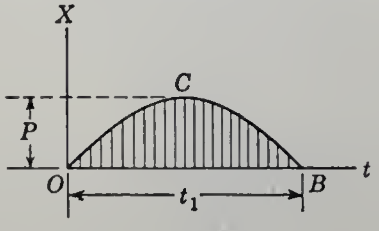

FIG. B

6. The magnitude of the force acting on a body of weight $W = 3.14$ lb is given by the equation $X = X_0 \sin \omega t$, in which $\omega = 8$ radians/sec. If after one complete force cycle the displacement of the body is 10 ft, find the maximum value X_0 of the acting force. Assume $x_0 = 0$ and $\dot{x}_0 = 0$. *Ans.* $X_0 = 9.94$ lb.

7. A body is acted upon for 5 sec by a constant force $X_1 = 10$ lb, immediately after which it is acted upon by a force in the opposite direction of constant magnitude $X_2 = 3$ lb. What time t_1 must this second force act to bring the body to rest? What time t_2 must it act to return the body to its starting point? Assume $\dot{x}_0 = 0$. *Ans.* $t_1 = 16.7$ sec; $t_2 = 35.7$ sec.

8. Under the action of a force $X = X_0 - kt$ a particle starts from rest at the origin and moves along the x axis. At what instant t will it again be at the origin? Assume $X_0 = 12$ lb and $k = 2$ lb/sec. *Ans.* $t = 18$ sec.

6.6. Force proportional to displacement—free vibrations. Very often we encounter the case of a particle moving rectilinearly under the action of a force that is some function of the displacement of the particle. The simplest as well as one of the most important instances in this case is that in which the acting force is proportional to the displacement. As an example, let us consider a body of weight W suspended on a helical spring and constrained to move only in a vertical direction, as shown in Fig. 211. Under the static action of the weight W the spring will be elongated a certain amount δ_{st} and because of the tension thus induced will exert a vertical upward reaction on the suspended body that just balances the gravity force W. Thus the resultant force acting on the body is zero, and it remains in equilibrium. If, now, the body is displaced downward from its position of static equilibrium, the tension in the spring will be increased and upon release the body will begin to move upward under the action of the unbalanced force. From experiments we know that the tension in the spring, so long as its elastic limit is not exceeded, is proportional to its elongation. Thus during motion of the body we conclude that, for any displacement x from the position of static equilibrium, the total tension in the spring is

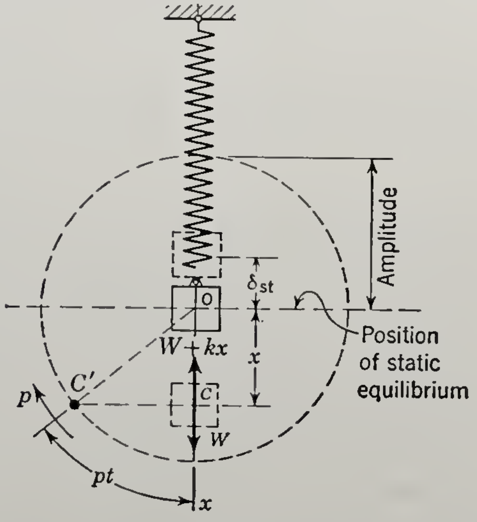

FIG. 211

$$S = W + kx \tag{a}$$

where k is the factor of proportionality between the elongation and the corresponding tension in the spring and is called the *spring constant.* It represents the tension to which the spring must be subjected in order to produce in it an elongation equal to unity and is usually expressed in units of pounds per inch (lb/in.).

Using expression (a) for the spring tension, the differential equation of motion [Eq. (34)] becomes[1]

$$\frac{W}{g}\ddot{x} = W - (W + kx)$$

which reduces to

$$\frac{W}{g}\ddot{x} = -kx \tag{b}$$

From Eq. (b) we see that for positive displacement we have negative acceleration, and vice versa; i.e., the body is always being accelerated toward its position of static equilibrium with a force that is proportional to the displacement from that position. To simplify the writing of Eq. (b), we divide both sides by W/g and introduce the notation

$$\frac{kg}{W} = p^2 \tag{c}$$

after which Eq. (b) may be written

$$\ddot{x} + p^2x = 0 \tag{40}$$

This is the differential equation for *free vibrations* of a particle.

We seek now the solution of Eq. (40); i.e., we must find for x a function of time that will satisfy this equation. A study of the equation shows that we need for $x = f(t)$ such a function that after twice differentiating it with respect to time we obtain the same function with which we started, multiplied by $-p^2$. In this way only can Eq. (40) be satisfied. The functions cos pt and sin pt satisfy this requirement. In fact we can multiply each of these functions by an arbitrary constant, add them together, and still have a function that satisfies the above requirement. Doing this we obtain

$$x = C_1 \cos pt + C_2 \sin pt \tag{41}$$

in which C_1 and C_2 are the two arbitrary constants. This is the *general solution* of Eq. (40), since by an appropriate choice of the constants C_1 and C_2 we can adapt this solution to any initial conditions of motion of the weight W.

Assume, for instance, that the weight was started in motion by giving it an initial displacement x_0 from its equilibrium position and then suddenly releasing it without any initial velocity. Thus, at the initial moment when it is released, we have

$$(x)_{t=0} = x_0 \qquad (\dot{x})_{t=0} = 0 \tag{d}$$

---

[1] This equation neglects air resistance and the inertia of the spring.

Substituting the first of conditions (d) into the general solution (41), we find $C_1 = x_0$. Differentiating expression (41) once with respect to time, we obtain the general velocity-time equation

$$\dot{x} = -pC_1 \sin pt + pC_2 \cos pt \tag{e}$$

Substituting the second of conditions (d) into Eq. (e), we find $C_2 = 0$. Replacing C_1 and C_2 by their values just determined, the solution for the assumed initial conditions of motion becomes

$$x = x_0 \cos pt \tag{41a}$$

Comparing Eq. (41a) with Eq. (f), page 251 we see that the weight W has a *simple harmonic motion.* The displacement-time, velocity-time, and acceleration-time diagrams representing this motion are shown in Fig. 212. From the displacement-time curve we see that the maximum numerical value of the displacement is x_0 and this is called the *amplitude of vibration.* We also see that the time required for the moving weight to make one complete cycle along its path is $2\pi/p$ sec. This time is called the *period of vibration* and will be designated by τ. We may write

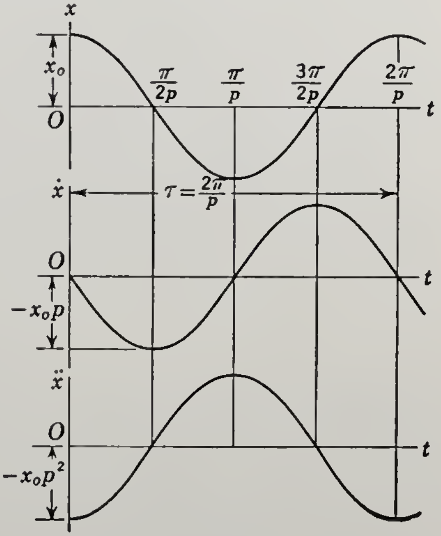

FIG. 212

$$\tau = \frac{2\pi}{p} = 2\pi\sqrt{\frac{W}{kg}} = 2\pi\sqrt{\frac{\delta_{st}}{g}} \tag{42}$$

From the first form of Eq. (42) we see that the period of vibration is inversely proportional to the factor p. A special significance can be attached to this factor in the following manner: From Eq. (c) where it was first introduced, we see that it has the dimension of a number divided by seconds (sec^{-1}), which is the same as the dimension of ω defining angle of rotation in radians per unit of time. Referring now to Fig. 211, let us construct a circle, the center of which coincides with the position of static equilibrium of the moving weight W and the radius of which is equal to the amplitude of vibration. We see then that, as the weight moves along its vertical path, the horizontal projection of its center of gravity on this circle travels around the circle at the uniform rate of p radians/sec, making one complete turn around the circle for each complete cycle of the oscillation of the weight. This circle will be called the *circle of reference*. Thus p represents the number of radians per second that the projection of the moving weight travels around the circle of reference as the weight itself oscillates along the vertical diameter of this circle.

From the second form of Eq. (42), where p has been replaced by its value from Eq. (c), we see that the period of vibration depends only upon the magnitude of the weight W and that of the spring constant k, being directly proportional to $\sqrt{W}$ and inversely proportional to $\sqrt{k}$. Thus we can change the period of vibration only by varying one or the other or both of these two quantities. A small weight on a stiff spring will have a short period of vibration, while a large weight on a very flexible spring will have a long period.

From the definition of the spring constant k, we conclude that the term W/k represents the elongation produced in the spring by the static action of the weight W. Using the notation

$$\delta_{st} = \frac{W}{k} \tag{f}$$

we obtain the third form of Eq. (42), from which we see that to calculate the period of vibration we need to know only the static elongation of the spring under the action of the load W.[1] This elongation can often be conveniently obtained by direct experiment, and such an experimentally determined value when substituted into Eq. (42) will give the period τ.

Having the period of vibration, i.e., the time required for one complete cycle, we can easily obtain the number of vibrations per second. This number is called the *frequency of vibration* and will be denoted by f. Then

$$f = \frac{1}{\tau} = \frac{p}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{kg}{W}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}} \tag{43}$$

In the preceding discussion we have assumed that at the initial moment the body W was given an initial displacement x_0 and suddenly released without initial velocity. Let us consider now another special case where vibration of the weight is started by giving it, by means of a vertical impact, an initial velocity in its position of static equilibrium. Then at the initial moment we have

$$(x)_{t=0} = 0 \qquad (\dot{x})_{t=0} = \dot{x}_0 \tag{g}$$

---

[1] It is seen that the period of vibration of the suspended weight is the same as that of a mathematical pendulum, the length of which is equal to the static elongation produced in the spring by the action of the weight W.

Substituting the first of these conditions in Eq. (41) and the second in Eq. (e), we find $C_1 = 0$ and $C_2 = \dot{x}_0/p$, and Eq. (41) becomes

$$x = \frac{\dot{x}_0}{p} \sin pt \tag{41b}$$

The displacement-time, velocity-time, and acceleration-time curves for this case are shown in Fig. 213. This is still a simple harmonic motion. It is seen that the vibration has the same period $\tau = 2\pi/p$ as before, and we observe that the period of vibration does not depend upon either initial displacement or initial velocity but only on the spring constant k and the weight W. Only the amplitude is altered, being in this case equal to $\dot{x}_0/p$.

In a more general case we may start vibration of the weight by displacing it from its equilibrium position by an amount x_0 and then giving to it in this displaced position some initial velocity $\dot{x}_0$ In this case we have at the initial moment

$$(x)_{t=0} = x_0 \qquad (\dot{x})_{t=0} = \dot{x}_0 \tag{h}$$

and the general solution (41) becomes

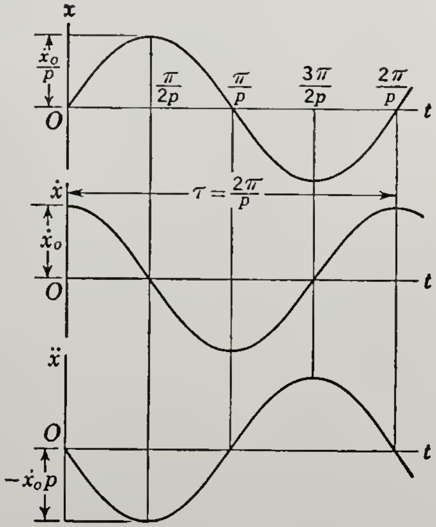

FIG. 213

$$x = x_0 \cos pt + \frac{\dot{x}_0}{p} \sin pt \tag{41c}$$

In order to get the displacement-time diagram for this case, we need only to sum up the ordinates of the displacement-time diagrams for the two preceding cases. The result of this summation again is a simple harmonic motion.

To find the amplitude in this general case, we proceed as follows: In Fig. 214, we lay out from the origin O a vector $\overline{OB} = x_0$ and making the angle pt with the x axis. Then the projection of this vector on the x axis represents the first term in Eq. (41c). Now from B, we lay out at right angles to $\overline{OB}$ the vector $\overline{BC} = \dot{x}_0/p$. Then the projection of $\overline{BC}$ on the x axis represents the second term in Eq. (41c), as can be seen from the figure. Since the sum of the projections of $\overline{OB}$ and $\overline{BC}$ on the x axis is the same as the projection of their resultant $\overline{OC}$, we conclude that Eq. (41c) can be written in the equivalent form

$$x = A \cos pt' \tag{41d}$$

where A is the amplitude of vibration and t' is a new time variable measured from the instant when the vibrating mass first reaches an extreme position after being released, i.e., when the rotating vector $\overline{OC}$ coincides with the x axis. From the right triangle OBC in Fig. 214, we see that the amplitude of vibration in the general case is given by the formula

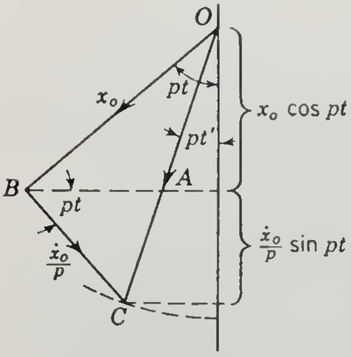

FIG. 214

$$A = \sqrt{(x_0)^2 + \left(\frac{\dot{x}_0}{p}\right)^2} \tag{44}$$

From this discussion, we conclude that in the general case where the particle has both initial displacement x_0 and initial velocity $\dot{x}_0$, the ensuing motion is the same as if we were to give the particle an initial displacement A as given by formula (44) and no initial velocity.

EXAMPLES

1. A weight W when attached to the end of a rubber band produces in it a static elongation $\delta_{st} = 5$ in. as shown in Fig. 215. If the weight is raised until the tension in the band is zero and then released without initial velocity, what maximum elongation will be produced in the band due to this sudden application of load and with what frequency will the suspended weight W oscillate?

Solution. Taking the position of static equilibrium of the suspended weight as the origin and considering downward displacement as positive, we conclude that at the moment of release the weight has the initial displacement

$$x_0 = -\delta_{st} \tag{i}$$

and that the initial velocity is zero. Hence from Eq. (41a) the displacement from the position of static equilibrium at any instant t is

$$x = -\delta_{st} \cos pt \tag{j}$$

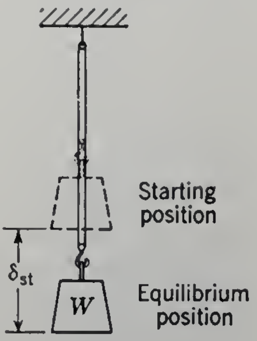

FIG. 215

When the angle $pt = \pi$, $\cos pt = -1$ and the displacement x has its maximum positive value

$$x_{\max} = +\delta_{st} \tag{k}$$

Comparing expressions (i) and (k), we conclude that the total elongation produced in the band by the sudden application of the load W is just twice that produced by the same load when gradually applied.

From Eq. (43) the frequency of vibration is

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}} = \frac{1}{2\pi}\sqrt{\frac{386}{5}} = 1.4 \text{ oscillations/sec}$$

2. A ship floats with an average depth of immersion equal to h, and the area of the water line section is A (Fig. 216). Neglecting friction and the inertia of any water that may be set in motion, prove that, if the ship is displaced downward slightly from its floating position and then released, it will oscillate up and down with a simple harmonic motion. Find also the period τ of this oscillation.

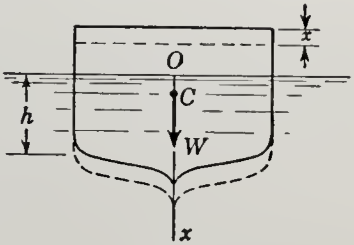

FIG. 216

Solution. Denoting the weight per unit volume of water by w, we conclude that the weight of the ship is Ahw, since it displaces its own weight in water.

Considering now any small vertical displacement x of the ship as shown, the increase in buoyancy is evidently Axw, and since for a downward displacement x this unbalanced force is upward, the differential equation of motion [Eq. (34)] becomes

$$\frac{Ahw}{g}\ddot{x} = -Axw$$

which reduces to

$$\ddot{x} + \frac{g}{h}x = 0 \tag{1}$$

Comparing Eq. (1) with Eq. (40), we conclude that in this case the constant factor g/h corresponds to p^2 in the general equation and hence the ship oscillates with a simple harmonic motion, the period of which is

$$\tau = \frac{2\pi}{p} = 2\pi\sqrt{\frac{h}{g}}$$

Owing to neglecting the inertia of the water, however, this result is not in very good agreement with experiments.

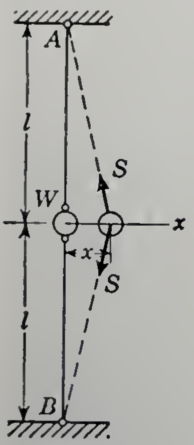

FIG. 217

3. A small ball of weight W is attached to the middle of a tightly stretched perfectly flexible wire AB of length $2l$ (Fig. 217). Prove that for small lateral displacements and high initial tension in the wire the ball will have a simple harmonic motion, and calculate the period.

Solution. If the wire is initially subjected to a high tensile force S, then any small change in this tension due to lateral displacement of the ball can be neglected. When the ball has any small displacement x as shown in the figure, the wires will be inclined slightly to the vertical and together will give an unbalanced horizontal reaction on the ball which will be

$$X = -2S\frac{x}{l}$$

Then Eq. (34) becomes

$$\frac{W}{g}\ddot{x} = -2S\frac{x}{l}$$

or

$$\ddot{x} + \frac{2S}{W}\frac{g}{l}x = 0 \tag{m}$$

Again we see that this is the equation of simple harmonic motion where the constant factor $2Sg/Wl$ corresponds to p^2 in Eq. (40). We have then for the period of vibration.

$$\tau = \frac{2\pi}{p} = 2\pi\sqrt{\frac{Wl}{2Sg}}$$

We see that for a given ball and wire the period is inversely proportional to the square root of the tension to which the wire is subjected. Thus within limits the period of vibration can be adjusted to any desired time by a suitable device for changing the tension in the wire.

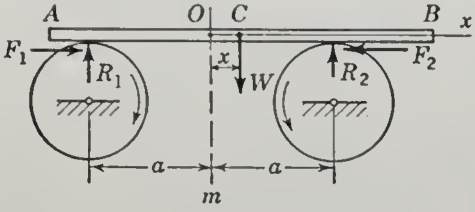

FIG. 218

4. A prismatic bar AB of weight W is placed horizontally on the top edges of two identical disks which rotate with equal angular speeds in opposite directions (Fig. 218). Show that, if the center of gravity C of the bar is displaced from the middle plane Om and then the bar is released, it will perform simple harmonic motion, the period of which depends on the distance a and the coefficient of kinetic friction μ (assumed constant) between the bar and the disks.

Solution. We take the origin O of coordinates midway between the disks and the x axis positive to the right along the line of motion of the bar. The forces acting on the bar for any displacement x of its center of gravity from the middle plane are shown in the figure. From statics we find the vertical reactions R_1 and R_2 to be

$$R_1 = \frac{W}{2a}(a - x) \qquad R_2 = \frac{W}{2a}(a + x)$$

The corresponding friction forces acting parallel to the axis of the bar will be

$$F_1 = \mu R_1 \qquad F_2 = \mu R_2$$

Then the differential equation of motion becomes

$$\frac{W}{g}\ddot{x} = -F_2 + F_1$$

and since, for the position of the bar shown in the figure, $F_2 > F_1$, we see that there is an unbalanced force acting on the bar tending to return it to its middle position. Substituting the values of F_2 and F_1, we obtain

$$\frac{W}{g}\ddot{x} = -\frac{\mu W}{a}x$$

or

$$\ddot{x} + \frac{\mu g}{a}x = 0$$

This is the equation of simple harmonic motion where $\mu g/a = p^2$ and we find the period to be

$$\tau = \frac{2\pi}{p} = 2\pi\sqrt{\frac{a}{\mu g}}$$

This device can be used for experimentally determining the coefficient of kinetic friction. Solving the above equation for μ gives

$$\mu = \frac{4\pi^2 a}{g\tau^2} \tag{n}$$

which can be used for calculating the coefficient of friction if the period τ is observed by direct test.

PROBLEM SET 6.6

1. A 10-lb weight is suspended by a helical spring having a constant $k = 5$ lb/in. Neglecting the mass of the spring, find the period τ for small amplitudes of vertical vibration. *Ans.* $\tau = 0.452$ sec.

2. If 10-lb tension produces an elongation of 1 in. in a given spring, find the frequency of vibration of a 1-lb weight suspended from the end of the spring. *Ans.* $f = 9.90\ \text{sec}^{-1}$.

3. A 7-lb weight produces a static elongation of 1.2 in. in a given spring. Determine the period of vibration of a weight $W = 10$ lb suspended by the same spring. *Ans.* $\tau = 0.418$ sec.

4. A 10-lb weight suspended vertically by a spring vibrates with an amplitude of 3 in. and a frequency of 60 oscillations/min. Find (*a*) the spring constant k, (*b*) the maximum tension induced in the spring, and (*c*) the maximum velocity of the weight. *Ans.* $k = 1.02$ lb/in.; $S_{\max} = 13.06$ lb; $\dot{x}_{\max} = 6\pi$ ips.

5. The body of a freight car weighs 50,000 lb when empty and is observed to settle 3 in. during the loading of 60,000 lb of cargo. What period of vertical vibration will the car have on its springs: (*a*) when loaded; (*b*) when empty? *Ans.* $\tau_1 = 0.75$ sec; $\tau_2 = 0.505$ sec.

6. The two springs shown in Fig. A each have a spring constant $k = 1$ lb/in., and the attached ball has the weight $W = 1$ lb. If

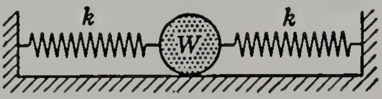

FIG. A

the ball is initially displaced 1 in. to the right, find the period of oscillation of the ball and the velocity with which it passes through its middle position. Neglect friction. *Ans.* $\tau = 0.226$ sec; $\dot{x}_{\max} = 27.8$ ips.

7. How will the results of Prob. 6 be changed if there is an initial 5 lb-tension in each spring?

8. A small pan suspended from a helical spring as shown in Fig. B when empty has an observed period of vibration τ_0. When carrying a known weight P, the observed period of vibration is τ_1, and when carrying an unknown weight W (without P), the observed period is τ_2. Find the magnitude of the weight W. *Ans.* $W = P(\tau_2^2 - \tau_0^2)/(\tau_1^2 - \tau_0^2)$.

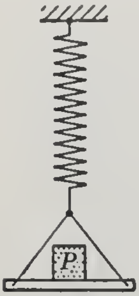

FIG. B

9. If in the case of the arrangement shown in Fig. B the observed period of vibration with a known weight P on the pan is τ_1 and the observed period of vibration with a known weight Q on the pan is τ_2, find the spring constant k. *Ans.* $k = 4\pi^2(P - Q)/g(\tau_1^2 - \tau_2^2)$.

10. A flat wood bar is placed on rotating steel disks as discussed in Example 4 and is observed to oscillate with a frequency of 40 oscillations/min. Find the coefficient of kinetic friction between the wood and steel if the distance $a = 10$ in. *Ans.* $\mu = 0.45$.

11. To keep the bar in Prob. 10 from falling to one side, the rims of the disks are grooved as shown in Fig. C. How will the period of oscillation of the bar be affected by such grooves? *Ans.* $\tau = 2\pi\sqrt{a \sin \alpha/\mu g}$.

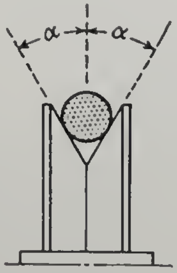

FIG. C

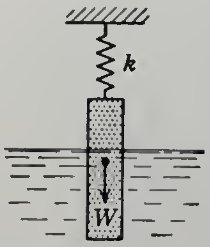

FIG. D

12. A solid right circular cylinder of weight $W = 10$ lb and cross-sectional area $A = 18$ in.^{2} is suspended by a spring of constant $k = 1$ lb/in. and hangs partially submerged in water ($w = 62.4$ lb/ft^3) as shown in Fig. D. Calculate the period τ for small vertical oscillations. Neglect inertia of the water. *Ans.* $\tau = 0.786$ sec.

6.7. D'Alembert's principle. The differential equation of rectilinear motion of a particle [Eq. (34)] can be written in the form

$$X - m\ddot{x} = 0 \tag{a}$$

in which X denotes the resultant, in the direction of the x axis, of all applied forces and m, the mass of the particle. We see at once that this equation of motion of a particle is of the same form as an equation of *static equilibrium* and may be considered as an equation of *dynamic equilibrium*. In order to write this equation, we need only to consider, in addition to the real forces acting on the particle, a fictitious force $-m\ddot{x}$. This force, equal to the product of the mass of the particle and its acceleration and directed oppositely to the acceleration, is called the *inertia force* of the particle.

In the case of a rigid body having rectilinear translation along the x axis, all particles have the same acceleration and hence the resultant of their inertia forces is

$$-\sum m\ddot{x} = -\ddot{x}\sum m = -\frac{W}{g}\ddot{x} \tag{b}$$

where W is the total weight of the body. Further, since the inertia forces of the particles are proportional to their gravity forces, this resultant inertia force has for its point of application the center of gravity of the body. Thus, for the rectilinear translation of a rigid body we obtain the following equation of dynamic equilibrium:

$$\sum X_i + \left(-\frac{W}{g}\ddot{x}\right) = 0 \tag{45}$$

Let us consider, now, any system of particles connected between themselves and so constrained that each particle can have only a rectilinear motion. As an example of such a system, we take the case of two weights W_1 and W_2, attached to the ends of a flexible but inextensible string overhanging a pulley as shown in Fig. 219. We assume that the inertia of the pulley and friction on its axle are negligible. Assuming motion of the system in the direction shown by the arrow on the pulley, we obtain upward acceleration $\ddot{x}$ of the weight W_2 and downward acceleration $\ddot{x}$ of the weight W_1. The corresponding inertia forces act as shown in the figure. By adding these inertia forces to the real forces (such as W_1 and W_2 and the string reactions S), we obtain, for each particle, a system of forces in equilibrium. Hence, we conclude that the entire system of forces is in equilibrium. Therefore, instead of writing a separate equation of equilibrium for each particle, we can write one equation of equilibrium for the entire system. This may be done either by equating to zero the algebraic sum of moments of all forces (including inertia forces) with respect to the axis

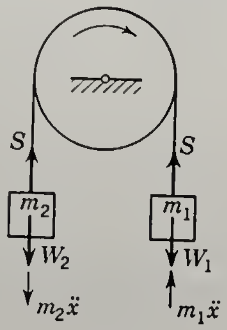

FIG. 219

of the pulley or by using the principle of virtual work (see page 218). In either case we need not consider the internal forces S of the system and obtain directly

$$W_2 + m_2\ddot{x} = W_1 - m_1\ddot{x}$$

from which

$$\ddot{x} = \frac{W_1 - W_2}{W_1 + W_2}g \tag{c}$$

The foregoing procedure may be used in more complicated systems, and instead of writing as many equations of motion as there are particles, we need to write only one equation of dynamic equilibrium, assuming, of course, that we are dealing with a system having one degree of freedom. In this way, we avoid consideration of all internal forces as well as reactions exerted by ideal constraints and a substantial simplification is realized.

D'Alembert was the first to point out that equations of motion could be written as equilibrium equations simply by introducing inertia forces in addition to the real forces acting on a system. This idea is known as *D'Alembert's principle* and is very useful in the solution of engineering problems of dynamics. Particularly when used in conjunction with the method of virtual work, it represents one of the most powerful methods of attack available.

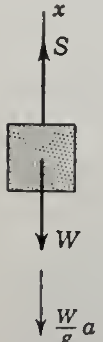

FIG. 220

EXAMPLES

1. By means of a rope, a body of weight W is moved vertically upward with a constant acceleration a. Find the tensile force in the rope.

Solution. The free-body diagram for the body is represented in Fig. 220. In order to apply Eq. (45), we must consider, besides the forces W and S, representing, respectively, the gravity force and the reaction exerted by the rope, the inertia force of magnitude $(W/g)a$ and oppositely directed to the acceleration as shown in the figure. All together then we have a system of forces in equilibrium. Taking the x axis vertical and positive upward, Eq. (45) becomes

$$S - W - \frac{W}{g}a = 0$$

from which we find for the tensile force in the rope

$$S = W\left(1 + \frac{a}{g}\right) \tag{d}$$

It is seen from Eq. (d) that the tension in the rope depends not only upon the weight W of the body but also upon the acceleration a which is given to it.

By giving a great acceleration we can produce in the rope a tension many times greater than the weight W. We are all familiar with the phenomenon of breaking a string to which a small weight is attached, simply by giving the string a quick jerk.

Equation (d) can also be used in the case of deceleration of the weight simply by changing the sign of a. In this case the tensile force becomes less than the weight of the body and we see that when the body is freely falling, that is, when $a = -g$, the tension S becomes zero.

2. A mathematical pendulum of length l and weight W is supported from the ceiling of an elevator (Fig. 221). How will its period of oscillation for small amplitudes be affected by a constant upward or downward acceleration a of the elevator?

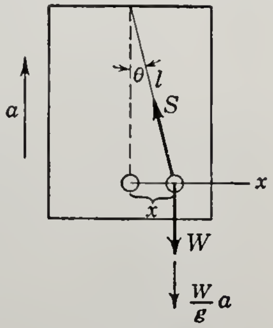

FIG. 221

Solution. Let us consider upward acceleration of the elevator as positive. Then the inertia force $-(W/g)a$ of the pendulum bob will add to the gravity force, and we have in effect an increase in the weight of the pendulum without any increase in its mass. Considering now a small displacement of the pendulum from its vertical position of equilibrium and treating the circular arc of radius l along which the weight W moves as coincident with the x axis, the projection of all forces, including the inertia force $-(W/g)\ddot{x}$, onto the x axis gives the equation of dynamic equilibrium

$$-\frac{W}{g}\ddot{x} - S\sin\theta = 0 \tag{e}$$

Projecting all forces on the direction of the string and remembering that θ is a small angle, we find

$$S = \left(W + \frac{W}{g}a\right)\cos\theta \approx W\left(1 + \frac{a}{g}\right)$$

Substituting this value of S into Eq. (e) together with the approximation $\sin\theta \approx x/l$ gives

$$\ddot{x} + \frac{g+a}{l}x = 0 \tag{f}$$

where $(g + a)/l$ corresponds to p^2 in Eq. (40) page 280 and we have

$$\tau = \frac{2\pi}{p} = 2\pi\sqrt{\frac{l}{g+a}} \tag{g}$$

We see that when $a = 0$ and the elevator is at rest or in uniform motion, Eq. (g) gives for the period of the pendulum $\tau = 2\pi\sqrt{l/g}$. In general, Eq. (g) shows that upward acceleration of the elevator decreases the period of the pendulum while downward acceleration increases its period. When $a = -g$, that is, when the elevator is freely falling, the period of the pendulum is infinite. From this we conclude that, when the elevator is freely falling, the weight W will stay in any displaced position in which it may be put and there will be no oscillations. In short, the inertia force of a freely falling body exactly balances the gravity force.

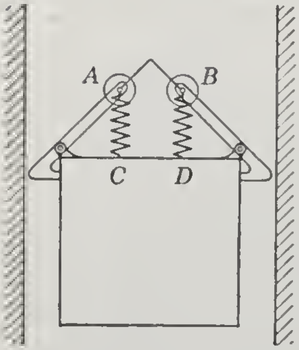

FIG. 222

Sometimes this fact must be considered in the design of elevator cages. To prevent a catastrophe in case of failure of the cable, a special toggle device (Fig. 222) is sometimes used to brake the fall of the free cage. To begin to act, the levers must tend to come into horizontal positions so that their ends will contact the sides of the shaft. It will be a mistake to use counterweights A and B for bringing the levers into action, since, as was explained above, during a free fall of the cage such counterweights will not act. To get the desired action, springs AC and BD must be used.

3. Find the maximum acceleration along a level road that the rear-wheel-drive automobile shown in Fig. 223 can attain if the coefficient of friction between tires and pavement is μ.

Solution. It is evident that the moving force F is represented by the friction developed between the rear wheels and the pavement. Since the maximum force that can be developed here depends upon the normal pressure, let us first find the reactions R_f and R_r at the front and rear wheels, respectively. When the car is at rest, these reactions are found by two moment equations of static equilibrium, as follows:

$$R_f = W\frac{b}{b+c} \qquad R_r = W\frac{c}{b+c} \tag{h}$$

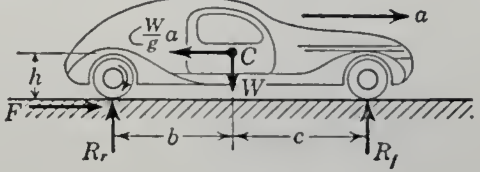

FIG. 223

When the car is in motion with acceleration a, we add the inertia force $-(W/g)a$ applied at the center of gravity of the car. Then writing two moment equations of dynamic equilibrium, we find the reactions to be

$$R_f = W\frac{b}{b+c} - \frac{W}{g}\frac{ah}{b+c} \qquad R_r = W\frac{c}{b+c} + \frac{W}{g}\frac{ah}{b+c} \tag{i}$$

Comparing Eqs. (h) with Eqs. (i), we see that owing to the acceleration of the car the reaction on the front wheels is decreased and that on the rear wheels increased, both by the same amount. From the fact that a large driving force F is dependent upon having a large reaction on the rear wheels, we conclude that a rear-wheel drive is more efficient than a front-wheel drive. By the same reasoning, front-wheel brakes will be more efficient than rear-wheel brakes.

Projecting all forces onto the horizontal and using Eq. (45), we obtain

$$F = \frac{W}{g}a \tag{j}$$

When slipping impends $F = \mu R_r$. Using this value in Eq. (j), together with the value of R_r from Eqs. (i) above, we obtain

$$a = \frac{\mu c g}{b + c - \mu h} \tag{k}$$

PROBLEM SET 6.7

1. Two weights P and Q are connected by the arrangement shown in Fig. A. Neglecting friction and the inertia of the pulleys and cord, find the acceleration a of the weight Q. Assume that $P = 40$ lb and $Q = 30$ lb. *Ans.* $a = 8.05$ ft/sec^2.

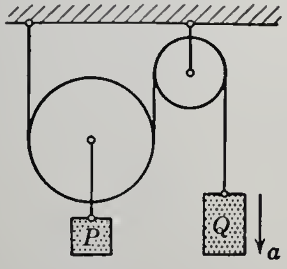

FIG. A

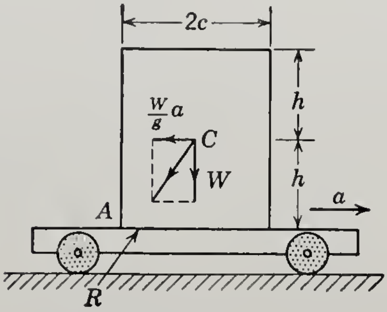

FIG. B

2. A block of weight W, height $2h$, and width $2c$ rests on a flat car which moves horizontally with constant acceleration a (Fig. B). Determine (a) the value of the acceleration a at which slipping of the block on the car will impend if the coefficient of friction is μ and (b) the value of the acceleration at which tipping of the block about the edge A will impend, assuming sufficient friction to prevent slipping. *Ans.* $a_1 = \mu g$; $a_2 = cg/h$.

3. Assuming the car in Fig. B to have a velocity of 20 fps, find the shortest distance s in which it can be stopped with constant deceleration without disturbing the block. The following data are given: $c = 2$ ft, $h = 3$ ft, $\mu = 0.5$. *Ans.* $s = 12.4$ ft.

4. Neglecting friction and the inertia of the two-step pulley shown in Fig. C, find the acceleration a of the falling weight P. Assume $P = 8$ lb, $Q = 12$ lb, and $r_2 = 2r_1$. *Ans.* $a = 2g/11$.

5. A mathematical pendulum hanging from the ceiling of a railway car inclines to the vertical by an angle α during starting of the train. What is the corresponding acceleration of the train? *Ans.* $a = g \tan \alpha$.

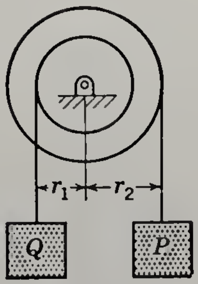

FIG. C

6. A spring-suspended mass hangs from the ceiling of an elevator cage. How will its natural period of free vertical vibration be affected by acceleration of the cage?

7. A homogeneous sphere of radius r and weight W slides along the floor under the action of a constant horizontal force P applied to a string, as shown in Fig. D. Determine the height h during this motion if the coefficient of friction between sphere and floor is μ. *Ans.* $h = r(1 - \mu W/P)$.

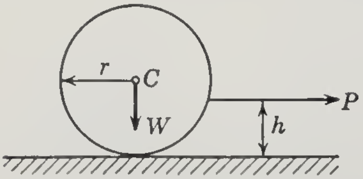

FIG. D

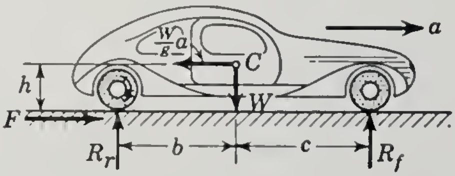

FIG. E

8. Assuming that a car of dimensions shown in Fig. E has sufficient power and that there is sufficient friction, find the maximum acceleration that it would be able to develop without tipping over backward. *Ans.* $a = bg/h$.

9. Two blocks of weights P and Q are connected by a flexible but inextensible cord and supported as shown in Fig. F. If the coefficient of friction between the block P and the horizontal surface is μ and all other friction is negligible, find (*a*) the acceleration of the system and (*b*) the tensile force S in the cord. The following numerical data are given: $P = 12$ lb, $Q = 6$ lb, $\mu = \frac{1}{3}$. *Ans.* $a = 3.58$ ft/sec²; $S = 5.33$ lb.

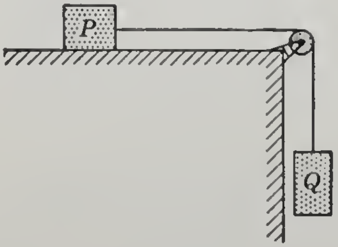

FIG. F

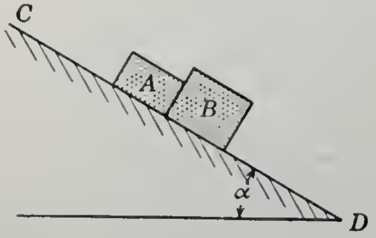

FIG. G

10. Two blocks A and B under the action of gravity slide down the inclined plane CD that makes with the horizontal the angle $\alpha = 30°$ (Fig. G). If the weights of the blocks are $W_a = 10$ lb and $W_b = 20$ lb and the coefficients of friction between them and the inclined plane are $\mu_a = 0.15$ and $\mu_b = 0.30$, find the pressure P existing between the blocks during the motion. *Ans.* $P = 0.87$ lb.

11. Neglecting friction and the inertia of the two pulleys in Fig. H find the acceleration a of the weight Q, assuming that $P = Q$. *Ans.* $a = 2g/5$.

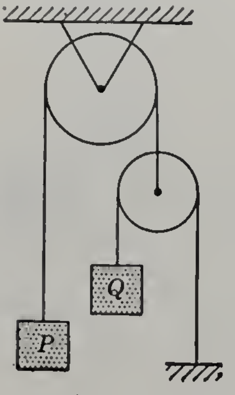

FIG. H

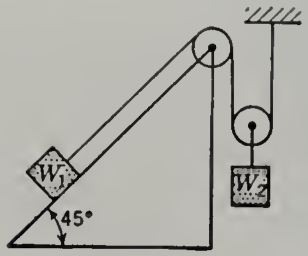

FIG. I

12. Find the tension S in the string during motion of the system shown in Fig. I if $W_1 = 200$ lb, $W_2 = 100$ lb. The system is in a vertical plane, and the coefficient of friction between the inclined plane and the block W_1 is $\mu = 0.2$. Assume the pulleys to be without mass. *Ans.* $S = 57$ lb.

13. A rectangular block of weight $Q = 200$ lb rests on a flatcar of weight $P = 100$ lb which may roll along the horizontal plane AB without friction (Fig. J). The car and block together are to be accelerated by the weight W arranged as shown in the figure. Assuming that there is sufficient friction between the block and the car to prevent sliding, find the maximum value of the weight W by which the car can be accelerated. What will this acceleration be? *Ans.* $W_{\max} = 100$ lb; $a = 8.05$ ft/sec².

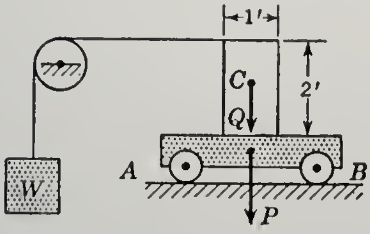

FIG. J

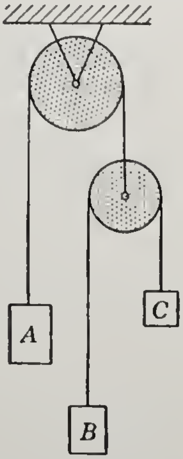

FIG. K

***14.** A system of weights and pulleys is arranged in a vertical plane as shown in Fig. K. Neglecting friction and the inertia of the pulleys, find the acceleration of each weight if their magnitudes are in the ratio $W_a : W_b : W_c = 3:2:1$. *Ans.* $\ddot{x}_a = g/17$; $\ddot{x}_b = 5g/17$; $\ddot{x}_c = -7g/17$.

6.8. Momentum and impulse. The differential equation of rectilinear motion of a particle may be written in the form

$$\frac{W}{g}\frac{d\dot{x}}{dt} = X$$

or

$$d\left(\frac{W}{g}\dot{x}\right) = X\,dt \qquad \text{(a)}$$

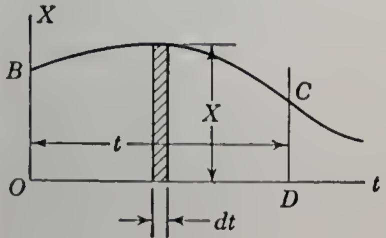

FIG. 224

In our further discussion, we shall assume that the force X is known as a function of time and is given by a force-time diagram as shown in Fig. 224. The right side of Eq. (a) then is represented by the area of the shaded elemental strip of height X and width dt in the force-time diagram. This quantity is called the *impulse* of the force X in the time dt. The expression $(W/g)\dot{x}$ on the left side of Eq. (a) is called the *momentum* of the particle. Thus Eq. (a) states that the differential change in momentum of the particle during the element of time dt is equal to the impulse of the acting force during the same time. We see that both impulse and momentum have the dimension force × time. They are usually expressed in units of pound-seconds (lb-sec).

Integrating Eq. (a), we obtain

$$\frac{W}{g}\dot{x} + C = \int_0^t X\,dt \tag{b}$$

in which the constant of integration C can be evaluated from the initial condition of the motion. Assuming that at the initial moment, $t = 0$, the particle has the velocity $\dot{x}_0$ directed along the x axis, we find from Eq. (b) that

$$C = -\frac{W}{g}\dot{x}_0$$

and the equation becomes

$$\frac{W}{g}\dot{x} - \frac{W}{g}\dot{x}_0 = \int_0^t X\,dt \tag{46}$$

Thus the total change in momentum of a particle during a finite interval of time is equal to the impulse of the acting force during the same interval. We see that this impulse is represented by the area $OBCD$ of the force-time diagram (Fig. 224).

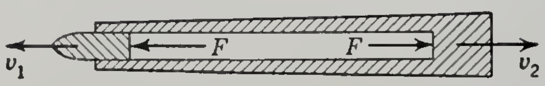

FIG. 225

The equation of momentum and impulse is particularly useful when we are dealing with a system of particles, since in such case calculation of the impulse can often be eliminated. As a specific example, let us consider the case of a gun and shell as shown in Fig. 225, which may be considered as a system of two particles. During the extremely short interval of explosion, the forces F acting on the shell and gun and representing the gas pressure in the barrel are varying in an unknown manner and a calculation of the impulses of these forces would be extremely difficult. However, the relation between the velocity of the shell and the velocity of recoil of the gun can be obtained without calculation of the impulse. Since the forces F are in the nature of action and reaction between the shell and gun, they must at all times be equal and opposite, and hence their impulses for the interval of explosion are equal and opposite, since the forces act exactly the same time t. Thus if W_1 and W_2 are the weights of the shell and gun, respectively, we find, assuming the initial velocities to be zero and neglecting all external forces, that

$$\frac{W_1}{g} v_1 = \int F\,dt \qquad \frac{W_2}{g} v_2 = \int F\,dt$$

Then for the entire system

$$\frac{W_1}{g} v_1 = \frac{W_2}{g} v_2$$

from which we obtain

$$\frac{v_1}{v_2} = \frac{W_2}{W_1}$$

We see that the velocities of the shell and gun after discharge are in opposite directions and inversely proportional to the corresponding weights.

We obtain a great simplification in the above example owing to the fact that no external forces act on the system but only internal forces in the nature of action and reaction. Internal forces in a system of particles always appear as pairs of equal and opposite forces and need not be considered when applying the equation of momentum and impulse. Thus we may state that, in the case of any system of particles to which no external forces are applied, the momentum of the system remains unchanged, since the total impulse is zero. This is sometimes called the principle of *conservation of momentum.*

EXAMPLES

1. A flat car can roll without resistance along a horizontal track as shown in Fig. 226. Initially, the car together with a man of weight w is moving to the right with speed v_0. What increment of velocity Δv will the car obtain if the man runs with speed u relative to the floor of the car and jumps off at the left?

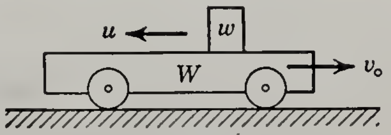

FIG. 226

Solution. Considering velocities to the right as positive, the initial momentum of the system is

$$\frac{W + w}{g} v_0 \tag{c}$$

Likewise, the final momentum of the car is

$$\frac{W}{g} (v_0 + \Delta v) \tag{d}$$

while that of the man is

$$\frac{w}{g} (v_0 + \Delta v - u) \tag{e}$$

Since no external forces act on the system, the law of conservation of momentum gives then

$$\frac{W + w}{g} v_0 = \frac{W}{g}(v_0 + \Delta v) + \frac{w}{g}(v_0 + \Delta v - u)$$

which reduces to

$$W\,\Delta v - wu + w\,\Delta v = 0$$

and we find for the increment of velocity given to the car

$$\Delta v = \frac{wu}{W + w} \tag{f}$$

We note that this increase in the velocity of the car is independent of the initial velocity v_0 of the system.

2. Assuming that the car in Fig. 226 is initially at rest and holds n men, each of weight w, prove that, if the men run with relative velocity u and jump in succession, they can impart to the car a greater velocity than if they all run and jump simultaneously.

Proof. When the men run together, the effect is the same as for one man of weight nw. Hence, by Eq. (f) of Example 2, we have for the total velocity imparted to the car in this case

$$v_1 = \frac{nwu}{W + nw} \tag{g}$$

When the men run and jump in succession, we again use Eq. (f) above and write

$$v_2 = \sum_{i=1}^{i=n} \frac{wu}{W + iw} \tag{h}$$

Since there are n terms in this series, each of which is as great as or greater than $wu/(W + nw)$, we conclude that $v_2 > v_1$.

PROBLEM SET 6.8

1. A man of weight W holds one end of a rope that passes over a frictionless pulley above his head and carries an inert weight W at its other end. Thus when the man puts his full weight on the rope, he will be just balanced by the weight at the other end. Discuss what will happen if the man tries to climb upward along the rope.

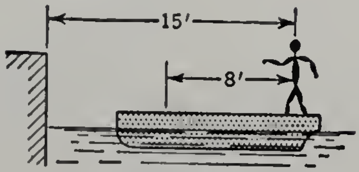

FIG. A

2. A man weighing 160 lb stands in a boat so that he is 15 ft from a pier on the shore (Fig. A). He walks 8 ft in the boat toward the pier and then stops. How far from the pier will he be at the end of this time? The boat weighs 200 lb, and there is assumed to be no friction between it and the water. *Ans.* $10\frac{5}{9}$ ft.

3. A locomotive weighing 60 tons has a velocity of 10 mph and backs into a freight car weighing 10 tons that is at rest on a level track. After coupling is made, with what velocity v will the entire system continue to move? Neglect all friction. *Ans.* $v = 8.57$ mph.

4. A 150-lb man sits in a 75-lb canoe and fires a 1-oz rifle bullet horizontally directly over the bow of the canoe. Neglecting friction of the water, find the velocity v with which the canoe will move after the shot if the rifle has a muzzle velocity of 2,200 fps. *Ans.* $v = 0.611$ fps.

5. A wood block weighing 5 lb rests on a smooth horizontal surface. A revolver bullet weighing $\frac{1}{2}$ oz is shot horizontally into the side of the block. If the block attains a velocity of 10 fps what was the muzzle velocity v of the bullet? *Ans.* $v = 1{,}610$ fps.

6. Five men lined up at one end of a floating raft, initially at rest, run in succession with velocity $u = 10$ fps. relative to the raft and dive off at the far end. Neglecting resistance of the water to horizontal motion of the raft, find its velocity after the last man dives. Each man weighs 170 lb, and the raft weighs 1,000 lb. *Ans.* $v = 5.78$ fps.

6.9. Work and energy. Writing the differential equation of rectilinear motion of a particle in the form

$$\frac{W}{g}\frac{d\dot{x}}{dt} = X$$

and multiplying both sides by dx, we obtain

$$\frac{W}{g}\frac{dx}{dt}\,d\dot{x} = X\,dx$$

or

$$d\left(\frac{W}{g}\frac{\dot{x}^2}{2}\right) = X\,dx \tag{a}$$

Assuming in our further discussion that the force X is known as a function of the displacement x of the particle as represented by the *force-displacement diagram* in Fig. 227, we see that the right side of Eq. (a) is represented by the area of the elemental strip of height X and width dx of this diagram. This quantity represents the *work* done by the force X on the infinitesimal displacement dx, and the expression in the parentheses on the left side of Eq. (a) is called the *kinetic energy* of the particle. Thus Eq. (a) states that the differential change in the kinetic energy of the moving particle is equal to the work done by the acting force on the corresponding infinitesimal displacement dx. We

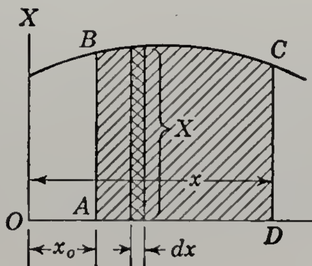

FIG. 227

see that both work and kinetic energy have the dimension of force times length. They are usually expressed in units of foot-pounds (ft-lb) or inch-pounds (in.-lb).

Integrating Eq. (a), we find

$$\frac{W}{g}\frac{\dot{x}^2}{2} + C = \int_0^x X\,dx \tag{b}$$

in which the constant of integration C can be evaluated from the initial conditions of the motion. Assuming that, when the displacement $x = x_0$, the particle has the velocity $\dot{x} = \dot{x}_0$ and substituting these simultaneous conditions into Eq. (b), we find

$$C = -\frac{W}{g}\frac{\dot{x}_0^2}{2} + \int_0^{x_0} X\,dx$$

and the equation becomes

$$\frac{W}{g}\frac{\dot{x}^2}{2} - \frac{W}{g}\frac{\dot{x}_0^2}{2} = \int_{x_0}^{x} X\,dx \tag{47}$$

We see at once that the definite integral on the right side of this equation is represented by the area $ABCD$ of the force-displacement diagram (Fig. 227). This is the total work of the force X on the finite displacement of the particle from x_0 to x. The work of a force is considered positive if the force acts in the direction of the displacement and negative if it acts in the opposite direction. Analogous to Eq. (a), Eq. (47) states that the total change in kinetic energy of a particle during a displacement $x - x_0$ is equal to the work of the acting force on this displacement.

The equation of work and energy is especially useful in cases where the acting force is a function of displacement and where we are particularly interested in the velocity of the particle as a function of displacement. Imagine, for example, that we desire the velocity with which a weight W falling from a height h strikes the ground. In this case the acting force $X = W$ and the total work is Wh. Thus if the body starts from rest, the initial velocity $\dot{x}_0 = 0$ and Eq. (47) becomes

$$\frac{W}{g}\frac{v^2}{2} = Wh$$

from which

$$v = \sqrt{2gh} \tag{c}$$

If the same body slides without friction along the inclined plane AB (Fig. 228), starting from an elevation h above point B, we can again use the equation of work and energy to find the velocity that it will have at B. In this case only the component $W \sin \alpha$ of the gravity force does work on the displacement; the component perpendicular to the inclined plane is at all times balanced by the reaction of the plane. In short, the resultant of all forces acting on the body is $X = W \sin \alpha$ in the direction of motion, and this force acts through the distance $h/\sin \alpha$. Hence its work is

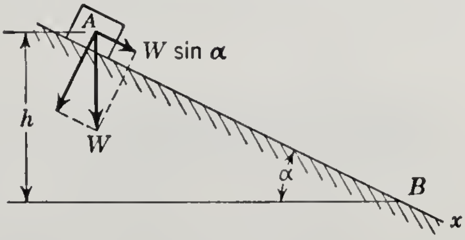

FIG. 228

$$W \sin \alpha \frac{h}{\sin \alpha} = Wh$$

and Eq. (47) gives again

$$v = \sqrt{2gh} \tag{d}$$

Comparing Eq. (d) with Eq. (c), we see that this velocity is the same as that gained in a free fall through the height h.

If the coefficient of friction between the block and the inclined plane in Fig. 228 is μ, it will be necessary in using Eq. (47) to consider also the work of friction. In such case, the resultant acting force in the direction of motion is

$$X = W \sin \alpha - \mu W \cos \alpha$$

Then through the displacement $h/\sin \alpha$ between A and B the work done is

$$Wh - \mu Wh \cot \alpha$$

and Eq. (47) gives

$$v = \sqrt{2gh(1 - \mu \cot \alpha)} \tag{e}$$

When $\alpha = \pi/2$, this expression agrees with Eq. (c) for a freely falling body, and when $\mu = 0$, it agrees with Eq. (d). We note also that to obtain a real value for v from Eq. (e), we must have $\mu < \tan \alpha$, since otherwise the block does not slide.

EXAMPLES

1. For throwing a ball of weight W the device shown in Fig. 229a is used. If the spring has a constant k and is initially compressed an amount δ_0, find the velocity with which the ball will leave the gun. Friction should be neglected.

Solution. Denoting by X the force exerted by the spring on the ball after release and during expansion of the spring, we have $X = kx$, where x is measured from the position of the ball corresponding to the unstressed length of the spring. The corresponding force-displacement diagram is shown in Fig. 229*b*. We see that the initial force on the ball when $x = \delta_0$ is $k\delta_0$ and that it decreases uniformly to zero when $x = 0$. The corresponding work of the force X is represented by the area *oab* of the diagram and is $k\delta_0\,\delta_0/2 = k\delta_0^2/2$. Equation (47) then becomes

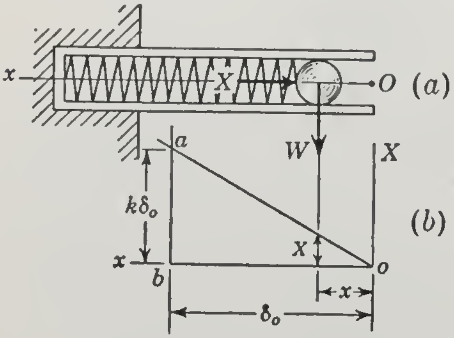

FIG. 229

$$\frac{W}{g}\frac{\dot{x}^2}{2} = \frac{k\delta_0^2}{2}$$

from which

$$\dot{x} = \delta_0\sqrt{\frac{kg}{W}} \tag{f}$$

As would be expected, this is the same result obtained in Art. 6.6 for the maximum velocity of a weight W attached to a spring of constant k and set in vibration by an initial displacement δ_0.

2. A weight W that can slide freely up and down a prismatic steel bar BC of length l and cross-sectional area A is allowed a free fall through the distance h (Fig. 230). Assuming the mass of the bar to be negligible compared with that of the falling weight, find the resulting dynamic elongation δ of the bar.

Solution. The weight starts from rest, travels a total distance $h + \delta$, and again comes to rest after having stretched the bar an amount δ. Hence the net change in kinetic energy is zero, and we conclude from Eq. (47) that the net work of all forces acting through this displacement must also be zero. During the free part h of the fall the only acting force is the constant gravity force W, but during the latter part δ, while the bar is being stretched, we have, besides W, the reaction of the bar proportional to its elongation and opposing motion. Using the notation $k = AE/l$ to represent the spring constant for the bar, this opposing force, which varies linearly from zero when $x = h$ to $k\delta$ when $x = h + \delta$, produces negative work represented by the area *oab* (Fig. 230) of the amount $-k\delta^2/2$. Equating to zero the total work of all forces, we obtain

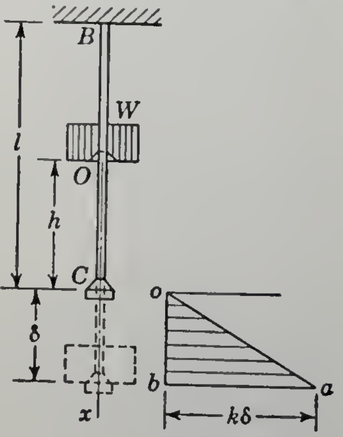

FIG. 230

$$W(h + \delta) - \frac{k\delta^2}{2} = 0 \tag{g}$$

which gives, for determining δ, the following quadratic equation:

$$\frac{k}{2}\delta^2 - W\delta - Wh = 0 \tag{h}$$

Solving Eq. (h) for δ, and using the notation $W/k = \delta_{st}$, we obtain

$$\delta = \delta_{st} \pm \sqrt{\delta_{st}^2 + 2h\delta_{st}} \tag{i}$$

Only the plus sign of Eq. (i) should be used for calculating the maximum dynamical elongation of the bar.

We note that even when $h = 0$, that is, when the weight W is suddenly applied to the bar without initial velocity, the resulting deflection is twice that which the same weight produces when gradually applied. This is a conclusion previously reached in Example 1, page 284.

3. Neglecting air resistance, find the initial velocity v_0 that would have to be given to a projectile at the surface of the earth to have it rise to an infinite height.

Solution. Let us consider the projectile at any distance x from the center of the earth (Fig. 231). Neglecting air resistance, the only force acting on it is the gravitational attraction. This is inversely proportional to the square of the distance of the particle from the center of the earth and can be represented by the equation

$$X = -W\frac{r^2}{x^2} \tag{j}$$

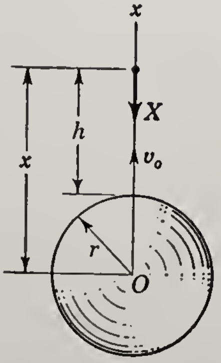

FIG. 231

This equation satisfies the known condition that, when $x = r$, the force $X = -W$. For any height h of the projectile the work done by the acting force X will be

$$\int_r^x X\,dx = -Wr^2\int_r^x \frac{dx}{x^2} = Wr^2\left[\frac{1}{x}\right]_r^x = Wr^2\left(\frac{1}{x} - \frac{1}{r}\right) \tag{k}$$

Setting $x = \infty$, and equating change in kinetic energy to work, we obtain

$$-\frac{W}{g}\frac{v_0^2}{2} = -Wr$$

from which

$$v_0 = \sqrt{2gr} \tag{l}$$

This is sometimes called the *velocity of escape*. Taking $r = 3{,}960$ miles, we obtain $v_0 = 6.95$ mps.

PROBLEM SET 6.9

1. A block of weight W is given an initial velocity v_0 along a rough horizontal plane and is brought to rest by friction in a distance x. Determine the coefficient of friction, assuming that it is independent of velocity. *Ans.* $\mu = v_0^2/2gx$.

2. When a ball of weight W rests on a spring of constant k (Fig. A), it produces a static deflection of 1 in. How much will the same ball compress the spring if it is dropped from a height $h = 1$ ft? Neglect the mass of the spring. *Ans.* $\delta = 6$ in.

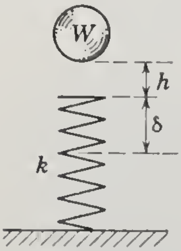

FIG. A

3. Determine the dynamical deflection δ that will be produced at the center of a simply supported beam by allowing a 4,000-lb weight to drop onto it from a height of 4 in. When gradually applied, the same load produces a static deflection of 0.1 in. Neglect the mass of the beam. *Ans.* $\delta = 1.00$ in.

4. An arrow weighing 0.0322 lb is shot from a 35-lb draw bow at full draw $d = 16$ in. Assuming a linear relation between draw and force, calculate the velocity v with which the arrow leaves the bow. *Ans.* $v = 216$ fps.

5. A gun weighing 150,000 lb fires a 1,000-lb projectile with a muzzle velocity of 3,600 fps. The gun is nested in springs having a total spring constant $k = 150{,}000$ lb/in. Assuming that the explosion is over before the gun has a chance to move perceptibly, how far will it recoil after the explosion? *Ans.* 1.22 ft.

6. A particle of mass m moves rectilinearly along the x axis under the action of a force $X = kx$, where k is a constant. Find the velocity v as a function of displacement x if the initial conditions of motion are $x_0 = 0$ and $\dot{x}_0 = v_0$. *Ans.* $v = \sqrt{v_0^2 + kx^2/m}$.

7. The driver of an automobile moving with constant speed $v_0 = 40$ mph along a straight level road steps on the accelerator so as to increase the power by 20 per cent. How far will the car travel before attaining a speed $v = 50$ mph? Assume that the resistance to motion remains constant and equal to 5 per cent of the weight of the car. *Ans.* $x = 0.567$ miles.

8. A small block of weight $W = 10$ lb is given an initial velocity $v_0 = 10$ fps down the inclined plane shown in Fig. B. If the coefficient of friction between the plane and the block is $\mu = 0.3$, find the velocity v of the block at B after it has traveled a distance $x = 50$ ft. *Ans.* $v = 29.5$ fps.

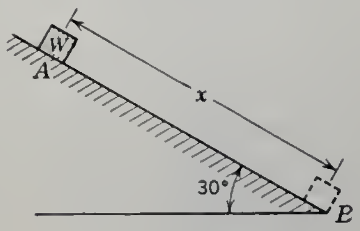

FIG. B

9. Find the velocity of escape for a rocket fired from the surface of the moon. Assume that the moon has the same density as the earth and that its total mass is $\frac{1}{75}$ that of the earth. *Ans.* $v_0 = 1.65$ mps.

6.10. Ideal systems: conservation of energy. The method of work and energy discussed in Art. 6.9 for a single particle can be extended to apply to a system of connected particles like that shown in Fig. 232. In so doing, we limit our attention to *ideal systems* with *one degree of freedom.*[1] That is, we assume that the system has frictionless constraints and inextensible connections and that its configuration can be completely specified by one coordinate such as x_1 in Fig. 232. In this case, for example, we assume a smooth inclined plane, frictionless bearings, inextensible strings, and neglect entirely the rotational inertia of the pulleys. Then the system may be regarded simply as three particles m_1, m_2, m_3, each of which performs a rectilinear motion. Furthermore, from kinematics, the displacements and velocities of all three masses can be expressed in terms of one variable, say the coordinate x_1 of the particle m_1.

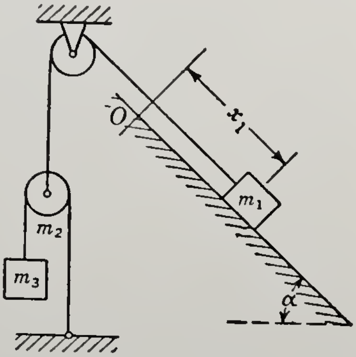

FIG. 232

We begin with a consideration of the kinetic energy of the system which is simply the sum of the kinetic energies of its several individual particles. Denoting the mass of each particle by m_i and its velocity by $\dot{x}_i$ and introducing the symbol T for total kinetic energy, we have

$$T = \tfrac{1}{2}\Sigma(m_i\dot{x}_i^2) \qquad \text{(a)}$$

During motion of the system, we consider an infinitesimal interval of time dt during which the system changes its configuration slightly and each particle suffers a displacement dx_i along its line of motion. Then the various forces acting on and within the system produce work. Denoting by X_i the resultant force on any particle m_i and by dU the total increment of work of all forces during such displacement, we have

$$dU = \Sigma X_i\,dx_i \qquad \text{(b)}$$

Using Eq. (a) from Art. 6.9, page 299 for one particle and summing up for all particles in the system, we obtain

$$\Sigma d(\tfrac{1}{2}m_i\dot{x}_i^2) = \Sigma X_i\,dx_i$$

which, with notations (a) and (b) above, may be written

$$dT = dU \qquad \text{(c)}$$

This equation states that the differential change in total kinetic energy of the system when it changes its configuration slightly is equal to the corresponding increment of work of all forces.

Considering any two configurations of the system denoted by subscripts A and B and integrating both sides of Eq. (c) between these limits, we obtain

---

[1] Such systems are discussed at length in Chap. 5.

$$\int_{T_A}^{T_B} dT = \int_{x_A}^{x_B} dU$$

or

$$T_B - T_A = \int_{x_A}^{x_B} dU \tag{48}$$

This is the equation of work and energy for a system of particles. It states that the total change in kinetic energy of the system when it moves from configuration A to configuration B is equal to the corresponding work of all forces acting upon it. In the case of an ideal system, the reactive forces will produce no work and the work of all internal forces which occur in equal and opposite pairs will cancel one another. Thus for such systems, only the work of active external forces need be considered on the right side of Eq. (48) and a substantial simplification is realized.

Equation (48) can be written in another form by introducing the concept of potential energy. The potential energy of a system in any configuration A is defined as the work which will be done by the acting forces if the system moves from that configuration back to a certain base or reference configuration O. Thus when the system is in configuration A or B its potential energy is

$$V_A = \int_A^O dU \qquad \text{or} \qquad V_B = \int_B^O dU \tag{d}$$

To illustrate, we consider a single particle of weight w at the height x_a above a datum plane O, as shown in Fig. 233. The potential energy V_a in position A is defined as the work that the gravity force w will produce if the particle is allowed to fall to the datum plane. Clearly this work is wx_a. To obtain this result by using the definition (d), we consider the particle in any position x and note that $X = -w$. Then the increment of work during a small increase dx in the displacement is

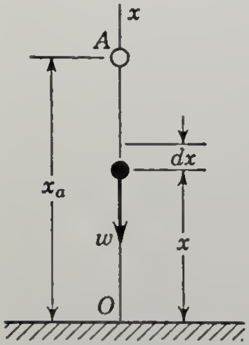

FIG. 233

$$dU = -w\,dx$$

Substituting this into the first of expressions (d), we obtain

$$V_a = -\int_{x_a}^{0} w\,dx = \int_0^{x_a} w\,dx = wx_a$$

as anticipated above.

If we have a system of particles of weights $w_1, w_2, \ldots, w_n$ at the elevations $x_1, x_2, \ldots, x_n$ above the chosen datum plane, the total potential energy of the system is

$$V = w_1x_1 + w_2x_2 + \cdots + w_nx_n = Wx_c \tag{e}$$

where W is the total weight of the system of particles and x_c is the elevation of their center of gravity.

Returning to Eq. (48), we write it in the form

$$T_B - T_A = \int_A^B dU = \int_O^B dU - \int_O^A dU$$

Then noting from expressions (d) that

$$\int_O^B dU = -\int_B^O dU = -V_B$$
$$\int_O^A dU = -\int_A^O dU = -V_A$$

we obtain

$$T_B - T_A = -V_B + V_A$$

or

$$T_B + V_B = T_A + V_A \tag{49}$$

Thus, as the system moves from one configuration to another, the total energy (kinetic + potential) remains constant. Kinetic energy may be transformed into potential energy, and vice versa, but the system as a whole can neither gain nor lose energy. This is the *law of conservation of energy* as it applies to a system of particles with ideal constraints. Such systems are sometimes called *conservative systems.*

EXAMPLES

1. If the system shown in Fig. 234, is released from rest in the configuration shown, first the velocity v of the block Q as a function of the distance x that it falls.

Solution. Neglecting the rotational inertia of the pulleys and extensibility of the string, we have a system of two connected particles. When the block Q has velocity v, the pulley P has velocity $v/2$ and the total kinetic energy is

$$T = \frac{1}{2}\frac{Q}{g}v^2 + \frac{1}{2}\frac{P}{g}\left(\frac{v}{2}\right)^2 = \frac{v^2}{2g}\left(Q + \frac{P}{4}\right)$$

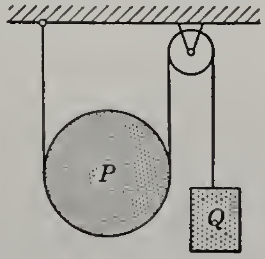

FIG. 234

Since the system is released from rest, this also represents the change in kinetic energy. Equating it to the corresponding work of the active forces P and Q, we obtain the energy equation

$$\frac{v^2}{2g}\left(Q + \frac{P}{4}\right) = Qx - P\frac{x}{2}$$

from which

$$v = \sqrt{2gx\frac{Q - P/2}{Q + P/4}}$$

2. A flexible but inextensible chain of length l and weight wl is held on a smooth table with an initial overhang a as shown in Fig. 235. Calculate the velocity v with which the chain will leave the table if released.

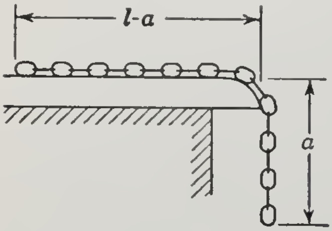

FIG. 235

Solution. Neglecting friction between the links of the chain and between the chain and table, we obtain an ideal system to which the law of conservation of energy applies. In the initial configuration, the kinetic energy is zero, and with the table top as a datum plane, the potential energy is $-wa(a/2)$ [see expression (e)]. In the final configuration (chain vertical and just leaving the table with velocity v), the kinetic energy is $(wl/g)(v^2/2)$ and the potential energy is $-wl(l/2)$. Hence Eq. (49) becomes

$$0 - \frac{wa^2}{2} = \frac{wl}{g}\frac{v^2}{2} - \frac{wl^2}{2}$$

which gives

$$v = \sqrt{\frac{g}{l}(l^2 - a^2)}$$

3. A glass U tube having a uniform bore of cross-sectional area A is open at both ends and contains a column of liquid of total length l and specific weight w as shown in Fig. 236. Using the law of conservation of energy, find the period τ of free oscillations after being disturbed from the equilibrium position as shown in the figure. Neglect friction between the fluid and the walls of the tube.

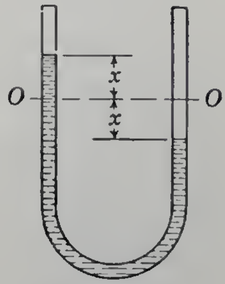

FIG. 236

Solution. When the liquid is in equilibrium, the free surfaces in the two branches of the tube will be at the same level OO, and we take this as our reference configuration. Then during oscillations, we consider any other configuration as defined by the displacements x of the two free surfaces above and below the level OO. In any such configuration, all particles of the liquid are moving with the same velocity $\dot{x}$ and the total kinetic energy of the system is

$$T = \frac{wAl}{g}\frac{\dot{x}^2}{2} \tag{f}$$

To obtain the potential energy, we must calculate the work that will be done as the liquid column is returned to the datum or middle position. In doing this, only the gravity forces acting on the liquid particles need be considered since we have an ideal system. Since the work done by the weight of the liquid in returning to the middle position is equivalent to lowering the portion above OO in the left-hand branch of the tube through a vertical distance x to fill the corresponding void in the right-hand branch of the tube, we have

$$V = wAx^2 \tag{g}$$

During motion, the total energy of the system (kinetic + potential) must remain constant. Hence the energy equation becomes

$$\frac{wAl}{g}\frac{\dot{x}^2}{2} + wAx^2 = C_1$$

To evaluate the constant C_1, we assume that at the initial moment $t = 0$, we have $x = a$ and $\dot{x} = 0$. Substituting these values in the energy equation, we find $C_1 = wAa^2$. Thus, the complete equation becomes

$$\frac{wAl}{g}\frac{\dot{x}^2}{2} + wAx^2 = wAa^2$$

which reduces to

$$\dot{x} = \sqrt{\frac{2g}{l}(a^2 - x^2)} \tag{h}$$

Introducing the notation $p = \sqrt{2g/l}$, Eq. (h) may be written in the form

$$\frac{dx}{dt} = p\sqrt{a^2 - x^2}$$

Then separating variables, we have

$$\frac{dx}{\sqrt{a^2 - x^2}} = p\,dt$$

and integrating this, we obtain

$$\arcsin\frac{x}{a} = pt + C_2 \tag{i}$$

From the initial condition $x = a$ when $t = 0$, we find $C_2 = \pi/2$ and Eq. (i) becomes

$$x = a\sin\left(pt + \frac{\pi}{2}\right)$$

or more simply

$$x = a\cos pt \tag{j}$$

We see now that the motion is simple harmonic and that its period is

$$\tau = \frac{2\pi}{p} = 2\pi\sqrt{\frac{l}{2g}}$$

Thus the liquid column oscillates with the same period as a simple pendulum of half its length. The energy method illustrated by this example is sometimes very useful in dealing with free vibrations of systems such as were discussed in Art. 6.6.

PROBLEM SET 6.10

1. For the ideal system shown in Fig. A, the weight W_1 hangs at the height $x_1 = h$ above the floor in the equilibrium configuration. Calculate the potential energy of the system with reference to this configuration if W_1 is pulled down to the floor ($x_1 = 0$). Neglect the mass of the spring and cord and rotational inertia of the pulleys. The following numerical data are given: $W_1 = W_2 = 10$ lb, $k = 2$ lb/in., $h = 4$ in. *Ans.* $V = 4$ in.-lb.

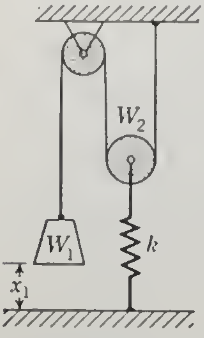

FIG. A

2. If the system in Fig. A is released from rest in the configuration defined by $x_1 = 0$, what maximum height above the floor will the block W_1 attain after release? *Ans.* $(x_1)_{\max} = 8$ in.

*3. Calculate the period of free vibration of the system in Fig. A if the weight W_1 performs small oscillations $x_1' = a \cos pt$ about its position of equilibrium. Use the same numerical data as in Probs. 1 and 2. *Ans.* $\tau = 1.60$ sec.

4. A V tube having a uniform bore of cross-sectional area A stands with its two branches inclined to the vertical by equal angles α as shown in Fig. B. Calculate the period of oscillation of a column of liquid of total length l and specific weight w if initially displaced from its equilibrium position in the tube as shown in the figure. *Ans.* $\tau = 2\pi \sqrt{(l \sec \alpha)/2g}$.

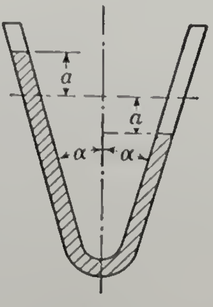

FIG. B

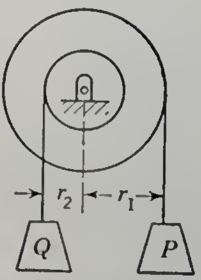

FIG. C

5. If the system in Fig. C is released from rest in the configuration shown, find the velocity v of the falling weight P as a function of its displacement x. Neglect friction and inertia of the pulley and assume the following numerical data: $P = Q = 10$ lb, $r_1 = 6$ in., $r_2 = 4$ in., $x = 10$ ft. *Ans.* $v = 12.2$ fps.

6. The two blocks in Fig. D have weights $P = 10$ lb, $Q = 5$ lb, and the coefficient of friction between the block P and the horizontal plane is $\mu = 0.25$.

If the system is released from rest and the block Q falls a vertical distance $h = 2$ ft, what velocity v will it acquire? Neglect friction in the pulley and extensibility of the string. *Ans.* $v = 4.64$ fps.

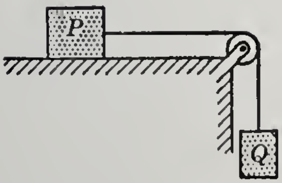

FIG. D

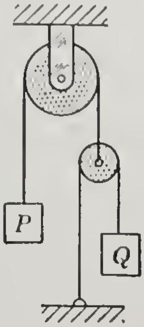

FIG. E

7. If the system in Fig. E is released from rest in the configuration shown, find the velocity v of the block Q after it falls a distance $h = 10$ ft. Neglect friction and inertia of the pulleys and assume that $P = Q = 10$ lb. *Ans.* $v = 16.05$ fps.

8. A length l of smooth straight pipe held with its axis inclined to the horizontal by an angle of 30° contains a flexible chain also of length l. Neglecting friction and assuming that, after release, the chain falls vertically as it emerges from the open end of the pipe, find the velocity v with which it leaves the pipe. *Ans.* $v = \sqrt{3gl/2}$.

9. If the system in Fig. F is released from rest in the configuration shown by solid lines, find the maximum distance h that the weight P will fall. Neglect friction and assume that the pulleys A and B are very small. *Ans.* $h = 4PQl/(4Q^2 - P^2)$.

***10.** Two equal weights Q are connected by a flexible but inextensible string overhanging a pulley as shown in Fig. G. The left-hand weight carries a loose washer of weight P, while the right-hand weight carries two such washers. The system is released from rest in the position shown, and the weights $Q + 2P$ fall a distance x_1. At this point, the weight Q passes freely through a hole in a fixed plate and the two washers are picked up by the plate. Thereafter the

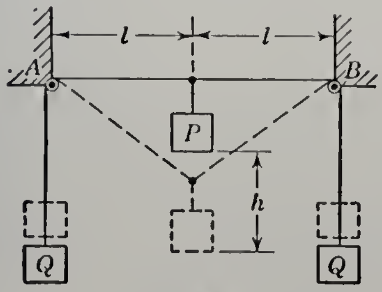

FIG. F

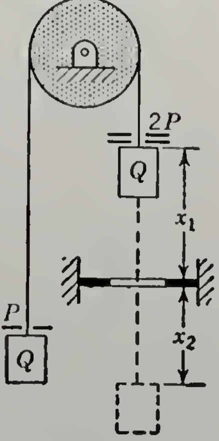

FIG. G

weight Q falls an additional distance x_2 before coming to rest. Neglecting friction and the inertia of the pulley, calculate the ratio of distances x_2/x_1. *Ans.* $x_2/x_1 = (2Q + P)/(2Q + 3P)$.

6.11. Impact. The phenomenon of collision of two moving bodies where we have active and reactive forces of very large magnitude acting during a very short interval of time is called *impact.* The magnitudes of the forces and the duration of impact depend on the shapes of the bodies, their velocities, and their elastic properties. As a simple example, we take the case of impact of two balls of weights W_1 and W_2 (Fig. 237). To measure the forces acting between the balls during impact, we cover them, at the points that come in contact, with a thin layer of soot. After impact the portions of the balls that were in contact are marked by tiny circular spots, the diameters of which can be measured by a microscope. By a static compression of the same balls, a calibration curve can be established that gives for each value of the diameter of the circle of contact the corresponding compressive force. Having such a curve, the maximum force that acts during an impact of the balls can be obtained. The duration of impact can also be measured without too much difficulty by an electrical method. Experiments with balls of brass have shown that, for a diameter of the balls of 1 in. and a velocity of impact of the order of 1 fps, the duration of impact is of the order of 15×10^{-5} sec. In this short period of time the ball that strikes completely loses its initial velocity. This indicates that there must take place a very large deceleration and forces produced at the point of contact are enormous in comparison with the weights of the balls.

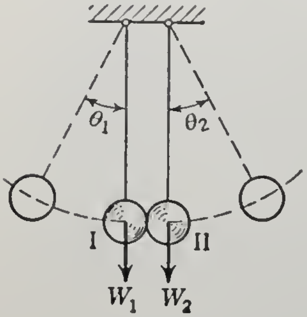

FIG. 237

The above conclusions are obtained by considering the very small local deformation of the balls at the point of contact. If we neglect this deformation and assume the bodies to be absolutely rigid, we conclude that the forces at the point of contact become infinite and the duration of impact becomes infinitesimal. In studying the action of such forces on the motion of bodies, we are not interested in the manner of variation of the forces during the infinitesimal duration t of impact but only with their summary action which depends on the time integral

$$\int_0^t F\,dt$$

i.e., on the impulse of the force as discussed in Art. 6.8. In comparison with the force of impact F, any ordinary finite forces that act during impact are negligible. Further, we can neglect any infinitesimal displacements of the bodies that occur during the infinitesimal interval of impact. With these assumptions it remains only to establish the relation between the impulses of the impact forces and the changes produced in the velocities of the bodies undergoing impact.

Let us consider first the case of impact of two spheres of weights W_1 and W_2 having, before impact, velocities v_1 and v_2, respectively (Fig. 238). We assume that these velocities are directed along the line joining the centers of the two spheres and consider them positive if they are in the positive direction of the x axis. This is called the case of *direct central impact* and is the simplest possible case. For the conditions shown in Fig. 238, impact obviously occurs only if $v_1 > v_2$. During impact, two equal and opposite forces, action and reaction, are produced at the point of contact. In accordance with the law of conservation of momentum (see Art. 6.8) such forces cannot change the momentum of the system of two balls, and if we denote by v_1' and v_2' the velocities of the balls at any instant during or after impact, we must have

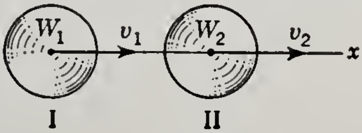

FIG. 238

$$\frac{W_1}{g} v_1 + \frac{W_2}{g} v_2 = \frac{W_1}{g} v_1' + \frac{W_2}{g} v_2' \tag{a}$$

In order to obtain the velocities v_1' and v_2' of the balls after impact this equation alone is not sufficient and we must have additional information regarding the properties of the material of the spheres.

Plastic Impact. Assume, for instance, that the material is absolutely inelastic, like putty. Then with the beginning of impact plastic deformation at the point of contact begins. The velocity of the striking ball I gradually diminishes owing to the reaction from the ball II, and at the same time the velocity of the ball II increases owing to the action of the ball I so that the relative velocity $v_1' - v_2'$ with which the plastic deformation at the surface of contact is progressing gradually diminishes. Finally, when the velocities of the balls become equal, $v_1' = v_2'$, plastic deformation ceases and, owing to the fact that the material is absolutely inelastic, there will be no tendency for the spheres to assume their original shapes. Both balls, with some permanent distortion, continue to move together with the same speed. Owing to the absence of elastic forces, there is no tendency for rebound.

Denoting by v' the common velocity of the two spheres after impact, Eq. (a) becomes

$$\frac{W_1}{g}v_1 + \frac{W_2}{g}v_2 = \left(\frac{W_1}{g} + \frac{W_2}{g}\right)v'$$

and we obtain, for the case of plastic impact,

$$v' = \frac{W_1v_1 + W_2v_2}{W_1 + W_2} \tag{50}$$

Elastic Impact. As another extreme case we assume that the material of the spheres is perfectly elastic. Hardened polished-steel or glass balls approach this condition if the local deformation during impact is not excessive. In this case of perfect elasticity there must be no loss in energy of the system and we have, in addition to the momentum equation (a), the energy equation

$$\frac{W_1}{g}\frac{v_1^2}{2} + \frac{W_2}{g}\frac{v_2^2}{2} = \frac{W_1}{g}\frac{(v_1')^2}{2} + \frac{W_2}{g}\frac{(v_2')^2}{2} \tag{b}$$

in which v_1' and v_2' are the velocities of the spheres after rebound.[1] Equations (a) and (b) together are sufficient to determine the two unknown velocities v_1' and v_2' after impact if the velocities v_1 and v_2 before impact and the weights W_1 and W_2 of the bodies are known.

For purposes of numerical calculation, Eqs. (a) and (b) can be somewhat simplified as follows. First we write them in the form

$$\begin{aligned} W_1(v_1 - v_1') &= W_2(v_2' - v_2) \\ W_1[v_1^2 - (v_1')^2] &= W_2[(v_2')^2 - v_2^2] \end{aligned}$$

Then noting that

$$\begin{aligned} v_1^2 - (v_1')^2 &= (v_1 - v_1')(v_1 + v_1') \\ v_2^2 - (v_2')^2 &= (v_2 - v_2')(v_2 + v_2') \end{aligned}$$

we divide the second equation by the first and obtain

$$v_1 + v_1' = v_2 + v_2'$$

or

$$v_1' - v_2' = -(v_1 - v_2) \tag{c}$$

This equation represents a combination of the laws of conservation of momentum and conservation of energy. It states that for an elastic impact the relative velocity after impact has the same magnitude as that before impact but with reversed sign. Using this idea in conjunction with that of conservation of momentum, we have for the case of elastic impact

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[1] A negligible part of the initial kinetic energy of the system is transformed into energy of vibrations of the spheres after rebound.

$$\begin{aligned} W_1 v_1 + W_2 v_2 &= W_1 v_1' + W_2 v_2' \\ v_1' - v_2' &= -(v_1 - v_2) \end{aligned} \tag{51}$$

To illustrate the use of Eqs. (51), let us consider several particular cases. If, in Fig. 238, we have $W_1 = W_2$, Eqs. (51) become

$$\begin{aligned} v_1 + v_2 &= v_1' + v_2' \\ v_1 - v_2 &= -v_1' + v_2' \end{aligned} \tag{d}$$

Subtracting and adding these equations, we find

$$v_1' = v_2 \qquad \text{and} \qquad v_2' = v_1$$

This shows that after an elastic impact, two equal weights simply exchange velocities.

If the weight W_2 was at rest before impact ($v_2 = 0$), Eqs. (d) give

$$v_1' = 0 \qquad \text{and} \qquad v_2' = v_1$$

In this case, the striking ball simply stops after having imparted its velocity to the other ball. This phenomenon can be observed in the case of a moving billiard ball which squarely strikes one that was at rest.

Again, if the two balls were moving toward each other with equal speeds v before impact, an exchange of velocities will simply mean that they rebound from one another with the same speed with which they collided.

As another special case, we assume that $W_2 = \infty$ while W_1 remains finite and further that $v_2 = 0$. This will represent the case of an elastic impact of a ball against a flat immovable obstruction, such as dropping a ball on a cement floor. Dividing the first of Eqs. (51) by W_2 we obtain $v_2' = 0$, as would be expected if W_2 is immovable. Then from the second equation, we find $v_1' = -v_1$. This shows that the striking ball rebounds with the same speed with which it hits the obstruction. It must be remembered that each of the examples discussed here assumes perfect elasticity so that no energy is lost during impact.

Semielastic Impact. Under actual conditions we must expect some deviation from perfect elasticity, and owing to this fact there always will be some loss in energy of the system during impact so that the relative velocity after impact is smaller than before and instead of Eq. (c) we must take

$$v_1' - v_2' = -e(v_1 - v_2) \tag{e}$$

where e is a numerical factor less than unity and is called the *coefficient of restitution* for the materials. Using this, we have for the general case of semielastic direct central impact, the following equations:

$$\begin{aligned} W_1v_1 + W_2v_2 &= W_1v_1' + W_2v_2' \\ v_1' - v_2' &= -e(v_1 - v_2) \end{aligned} \tag{52}$$

It will be noted that when $e = 0$, Eqs. (52) reduce to Eq. (50), already derived above for the case of plastic impact. Also when $e = 1$, they coincide with Eqs. (51) for the case of elastic impact. Thus with the appropriate value of the coefficient of restitution e, Eqs. (52) may be used for all cases.

EXAMPLES

1. A wooden pile of weight W_2 is driven into the ground by successive blows of a hammer of weight W_1 falling through a height h onto the head of the pile (Fig. 239). If, under the action of a single blow, the pile penetrates into the ground a distance δ, determine the total resistance R to penetration, assuming this resistance to be constant.

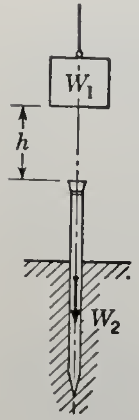

FIG. 239

Solution. Let us assume that the impact between the hammer and pile is entirely plastic, i.e., that there is no rebound of the hammer. Owing to its free fall through the height h, the hammer has, at the beginning of impact, a velocity $v_1 = \sqrt{2gh}$ and the pile has a velocity $v_2 = 0$. Now during the very short interval of impact the hammer is rapidly decelerated and the pile is accelerated until they acquire a common velocity v' which by Eq. (50) is

$$v' = \frac{W_1\sqrt{2gh}}{W_1 + W_2} \tag{f}$$

Since the duration of impact is very short, we may assume without serious error that the hammer and pile acquire this common velocity v' before any appreciable penetration of the pile has a chance to take place.

We consider now the second stage of the phenomenon in which the hammer and pile, moving together as one rigid body, are gradually brought to rest by the resisting force R acting through the displacement δ. Equating change in kinetic energy of the system to the work done by all acting forces, we obtain

$$-\frac{W_1 + W_2}{2g}\,\frac{W_1^2 2gh}{(W_1 + W_2)^2} = -R\delta + (W_1 + W_2)\delta \tag{g}$$

The sum $W_1 + W_2$ of the gravity forces is usually very small compared with the resistance R to penetration, and we can, without serious error, forget completely the second term on the right side of Eq. (g). Then we obtain for R the following expression:

$$R = \frac{W_1^2 h}{(W_1 + W_2)\delta}$$

Writing this in the form

$$R\delta = W_1 h \frac{W_1}{W_1 + W_2} \tag{h}$$

we see that what we might call the useful work $R\delta$ is less than the expended energy $W_1 h$ in the ratio $W_1/(W_1 + W_2)$. Thus we conclude that

$$1 - \frac{W_1}{W_1 + W_2}$$

or $W_2/(W_1 + W_2)$ represents the fraction of the available energy that is wasted or dissipated during impact. To have this lost energy as small a portion of the total available energy as possible, we see that the weight W_1 of the hammer should be large compared with the weight W_2 of the pile. Thus, by using a comparatively heavy mallet or sledge, a stake or post may be driven into the ground with less expended energy than if a comparatively light hammer is used.

2. Two small polished steel balls of weights W and $3W$ are suspended by fine threads so that they just touch one another in their equilibrium positions as shown in Fig. 240. Discuss what will happen if the small ball is pulled back to the position A_0 and released.

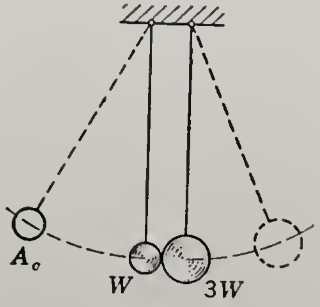

FIG. 240

Solution. Let v denote the velocity that the smaller ball will acquire in swinging downward to the point of impact. Then at the instant before impact, we have $(v_1)_1 = v$ and $(v_2)_1 = 0$. Assuming an elastic impact, Eqs. (51) become

$$\begin{aligned} v_1' + 3v_2' &= v \\ v_1' - v_2' &= -v \end{aligned} \tag{i}$$

from which we obtain $v_1' = -v/2$ and $v_2' = +v/2$. Thus as a result of the impact, the small ball rebounds with half of its striking velocity and the large one acquires this same velocity to the right.

As a result of these velocities, the two balls swing apart as two simple pendulums, and since they are pendulums of the same length, they will require the same time to return to the point of impact with velocities $(v_1)_2 = v/2$ and $(v_2)_2 = -v/2$. Thus we have a second impact for which Eqs. (51) become

$$\begin{aligned} v_1'' + 3v_2'' &= -v \\ v_1'' - v_2'' &= -v \end{aligned} \tag{j}$$

From these equations, we find for the velocities of the balls after the second impact, $v_1'' = -v$ and $v_2'' = 0$. Thus after the second impact, the small ball rebounds with velocity v and the large ball stops dead. As a result of this second rebound, the small ball swings up to the position A_0 from which it was initially released and the whole cycle begins over again. Even with slight unavoidable losses in energy of an actual system, this cyclic behavior can be observed in the case of a carefully made model.

PROBLEM SET 6.11

1. A man weighing 150 lb runs and jumps from a pier into a boat with a horizontal velocity $v_1 = 10$ fps. Assuming that the impact is entirely plastic, find the velocity with which the man and boat will move away from the pier if the boat weighs 200 lb. *Ans.* $v = 4.3$ fps.

2. Using the data of Prob. 1, compare the final kinetic energy of the boat and man together with the initial kinetic energy of the man and note that there is no conservation of energy in the case of plastic impact.

3. Several identical blocks, each of mass m, rest in a row on a perfectly smooth horizontal plane so that their centers of gravity lie on a straight line. Another block, also of mass m, is moving along this line with velocity v and squarely strikes one end of the row. Discuss what will happen if the blocks are all perfectly elastic.

4. A wood block weighing 9.95 lb rests on a rough horizontal plane, the coefficient of friction between the two being $\mu = 0.4$. If a bullet weighing 0.05 lb is fired horizontally into the block with muzzle velocity $v = 2{,}000$ fps, how far will the block be displaced from its initial position? Assume that the bullet remains inside the block. *Ans.* 3.88 ft.

5. A golf ball dropped from rest onto a cement sidewalk rebounds eight-tenths of the height through which it fell. Neglecting air resistance, determine the coefficient of restitution. *Ans.* $e = 0.9$.

6. For the two balls in Fig. A find the velocities v_1' and v_2' after an elastic impact if, before impact, $v_1 = v$, $v_2 = 0$, and $W_2 = 2W_1$. *Ans.* $v_1' = -v/3$; $v_2' = +2v/3$.

7. For the two balls in Fig. A, find the velocities v_1' and v_2' after impact if $v_1 = v$, $v_2 = 0$, $W_2 = 3W_1$, and the coefficient of restitution $e = \frac{1}{2}$. *Ans.* $v_1' = -v/8$; $v_2' = +3v/8$.

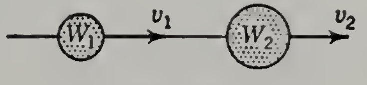

FIG A

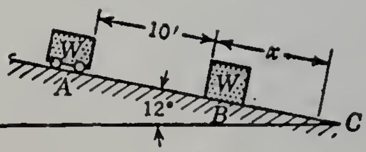

FIG. B

8. In Fig. B, a small car of weight W starts from rest at A and rolls without friction along an inclined plane to B where it strikes a block also of weight W

and initially at rest. Assuming a plastic impact at B, the car and block will move from B to C as one particle. If the coefficient of friction between the block and plane is $\mu = \frac{1}{2}$, calculate the distance x to point C where the bodies come to rest. *Ans.* $x = 14.2$ ft.

9. For the pile and pile driver shown in Fig. C, the following numerical data are given: $W_1 = 2{,}000$ lb, $W_2 = 1{,}000$ lb, $h = 10$ ft, and the coefficient of restitution $e = \frac{1}{4}$. If the resistance to penetration is constant and equal to 60,000 lb, how many blows of the hammer will be required to drive the pile 1 ft? *Ans.* Nine blows.

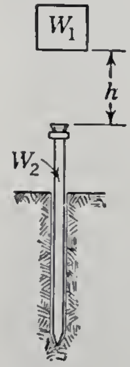

FIG. C

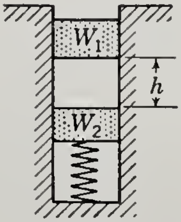

FIG. D

***10.** A weight W_1 falls through a height h onto a block of weight W_2 which is supported by a spring having a spring constant k (Fig. D). Assuming plastic impact, determine the maximum compression δ of the spring over and above that due to the static action of W_2. The following numerical data are given: $W_1 = W_2 = 10$ lb, $k = 10$ lb/in., $h = 3$ in. *Ans.* $\delta = 3$ in.

7

CURVILINEAR TRANSLATION

7.1. Kinematics of curvilinear motion. When a moving particle describes a curved path, it is said to have *curvilinear motion*. We shall now discuss the kinematics of such motion, assuming that the path of the particle is a plane curve.

Displacement. To define the position of a particle P in a plane (Fig. 241), we need two coordinates such as x and y. As the particle moves, these coordinates are changing with time and we have the displacement-time equations

$$x = f_1(t) \qquad y = f_2(t) \tag{53}$$

When these two expressions are given, the motion of the particle in its plane is completely defined.

Instead of Eqs. (53), we can also define the motion of a particle in a plane by the equations

$$y = f(x) \qquad s = f_1(t) \tag{53'}$$

where the first represents the equation of the path OA and the second gives the *displacement* s measured along the path as a function of time. Both forms (53) and (53′) are useful. Sometimes the path is predetermined as in the case of a locomotive moving along a curved track and we use Eqs. (53′). In other cases the particle may be free to choose its own path as in the case of a projectile fired from a gun and we use Eqs. (53). In this case the equation of the path can be obtained from Eqs. (53) by eliminating the parameter t.

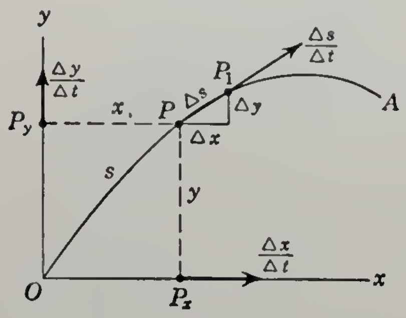

FIG. 241

Considering Eqs. (53) separately, we note that the first equation represents the rectilinear motion along the x axis of the projection P_x of the particle P moving along the curved path OA. Likewise the second equation represents the rectilinear motion along the y axis of the projection P_y of the particle P. Thus the curvilinear motion of a particle P may be considered as a compound of the two rectilinear motions of its projections P_x and P_y. This idea will be very helpful in our further discussion of velocity and acceleration.

Velocity. Let us consider now a small but finite interval of time from t to $t + \Delta t$ during which the particle in Fig. 241 moves from P to P_1 along its path. Denoting by Δs the length of the chord PP_1, we define the average velocity during this interval by the expression[1]

$$\bar{v}_{\text{av}} = \frac{\overline{\Delta s}}{\Delta t} \tag{a}$$

This average velocity is represented in Fig. 241 by a vector coinciding in direction with the chord PP_1 and its projections on the coordinate axes x and y, respectively, are

$$\begin{aligned} (v_{\text{av}})_x &= \frac{\Delta s}{\Delta t}\frac{\Delta x}{\Delta s} = \frac{\Delta x}{\Delta t} \\ (v_{\text{av}})_y &= \frac{\Delta s}{\Delta t}\frac{\Delta y}{\Delta s} = \frac{\Delta y}{\Delta t} \end{aligned} \tag{b}$$

From this we see that the projections of the average velocity of P also represent the average velocities of the projections P_x and P_y moving along the coordinate axes.

It is clear from Fig. 241 that both the magnitude and direction of the average velocity $\overline{\Delta s}/\Delta t$ of the particle will depend somewhat upon the magnitude of the chosen time interval Δt. To eliminate this ambiguity and obtain the *instantaneous velocity* $\bar{v}$ at time t, we let the time interval Δt diminish indefinitely and take the corresponding limit approached by the ratio $\overline{\Delta s}/\Delta t$. Thus the magnitude of the instantaneous velocity will be defined as follows:

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt} \tag{c}$$

and in the limit its direction will coincide with the tangent to the path at P, corresponding to the instant t. The magnitude of velocity, without regard to direction, as defined by Eq. (c) is called the *speed* of the particle.

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[1] The notations $\bar{v}$, $\overline{\Delta s}$, etc., mean that we consider the quantity as a vector. When the bar is omitted, only the magnitude need be considered.

To avoid direct consideration of the ever-changing direction of velocity in curvilinear motion, it is convenient to deal with its orthogonal projections v_x and v_y. To obtain these, we begin with expressions (b) defining the projections of the average velocity $\overline{\Delta s}/\Delta t$. Then again applying the limiting process as before, we obtain

$$\begin{aligned} v_x &= \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = \dot{x} \\ v_y &= \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt} = \dot{y} \end{aligned} \tag{54}$$

We see that the projections of the total velocity of P are identical with the velocities of the corresponding projections of the particle itself. To obtain these projections of the velocity $\bar{v}$, as functions of time, we need only to differentiate Eqs. (53) once each with respect to time. Having the velocity components $\dot{x}$ and $\dot{y}$, we obtain the magnitude and direction cosines of the total velocity $\bar{v}$ as follows:

$$v = \sqrt{\dot{x}^2 + \dot{y}^2} \qquad \cos(v,x) = \frac{\dot{x}}{v} \qquad \cos(v,y) = \frac{\dot{y}}{v} \tag{d}$$

where the symbols (v,x) and (v,y) denote the angles between the direction of the velocity vector $\bar{v}$ and the coordinate axes.

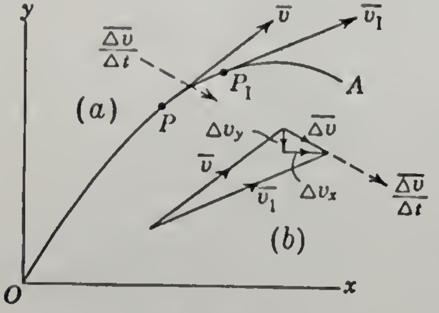


FIG. 242

Acceleration. To obtain the acceleration of a particle in curvilinear motion (Fig. 242), we again consider a small interval of time from t to $t + \Delta t$ during which the particle moves from P to P_1. Denoting by $\bar{v}$ and $\bar{v}_1$ its velocities at these two points on the path, we see from the diagram of free vectors (Fig. 242*b*) that the change in velocity in this case will be represented by a vector $\overline{\Delta v}$ directed, in general, obliquely to the path. Then the average acceleration during the interval Δt is defined as follows:

$$\bar{a}_{\text{av}} = \frac{\overline{\Delta v}}{\Delta t} \tag{e}$$

and is represented by a vector having the same direction as the velocity change $\overline{\Delta v}$, which is toward the inside of the path.

Projecting the average acceleration $\overline{\Delta v}/\Delta t$ on the coordinate axes, we obtain

$$(a_{\text{av}})_x = \frac{(\Delta v)_x}{\Delta t} = \frac{\Delta \dot{x}}{\Delta t}$$
$$(a_{\text{av}})_y = \frac{(\Delta v)_y}{\Delta t} = \frac{\Delta \dot{y}}{\Delta t} \tag{f}$$

Then again letting Δt diminish indefinitely and taking the corresponding limits approached by the ratios (f), we obtain the corresponding components of the instantaneous acceleration $\bar{a}$ as follows:

$$a_x = \lim_{\Delta t \to 0} \frac{\Delta \dot{x}}{\Delta t} = \frac{d\dot{x}}{dt} = \ddot{x}$$
$$a_y = \lim_{\Delta t \to 0} \frac{\Delta \dot{y}}{\Delta t} = \frac{d\dot{y}}{dt} = \ddot{y} \tag{55}$$

We see again, as in the case of velocity, that the projections of the resultant acceleration $\bar{a}$ are identical with the corresponding accelerations of the projections P_x and P_y of the particle itself. To obtain these components as functions of time, we need only to make a second differentiation of Eqs. (53) with respect to time. Finally, the magnitude and direction cosines of the total acceleration $\bar{a}$ will be obtained from the equations

$$a = \sqrt{\ddot{x}^2 + \ddot{y}^2} \qquad \cos(a,x) = \frac{\ddot{x}}{a} \qquad \cos(a,y) = \frac{\ddot{y}}{a} \tag{g}$$

To illustrate the use of Eqs. (53) to (55), let us consider a particular case of plane curvilinear motion represented by the equations

$$x = r\cos\omega t \qquad y = r\sin\omega t \tag{53a}$$

Squaring both equations and adding them together, we obtain, for the equation of the path,

$$x^2 + y^2 = r^2 \tag{h}$$

This shows that the particle describes a circular path of radius r and with center at the origin O as shown in Fig. 243. To obtain the velocity components, we differentiate Eqs. (53a) with respect to time and obtain

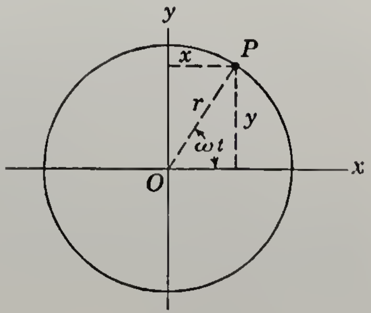


FIG. 243

$$\dot{x} = -r\omega\sin\omega t \qquad \dot{y} = r\omega\cos\omega t \tag{54a}$$

Then, from Eqs. (d), we have

$$v = \sqrt{\dot{x}^2 + \dot{y}^2} = r\omega \qquad \cos(v,x) = -\sin\omega t \qquad \cos(v,y) = \cos\omega t \tag{i}$$

This shows that the particle moves with constant speed and that for any position along the path the velocity vector $\bar{v}$ is perpendicular to OP, that is, tangent to the path at P.

Differentiating Eqs. (54a) again, we obtain for the rectangular components of acceleration

$$\ddot{x} = -r\omega^2 \cos \omega t \qquad \ddot{y} = -r\omega^2 \sin \omega t \tag{55a}$$

and Eqs. (g) give

$$a = \sqrt{\ddot{x}^2 + \ddot{y}^2} = r\omega^2 \quad \cos(a,x) = -\cos \omega t \quad \cos(a,y) = -\sin \omega t \tag{j}$$

This shows that the resultant acceleration $\bar{a}$ is constant in magnitude, normal to the path, and always directed toward the center O.

Normal and Tangential Acceleration. Let us return now to the general case of curvilinear motion of a particle P as shown in Fig. 244. In our previous discussion (Fig. 242) we resolved the total acceleration $\bar{a}$ into rectangular components a_x and a_y parallel to the fixed coordinate axes x and y. Instead of this, it is sometimes useful to resolve this acceleration at any instant t into another set of rectangular components coinciding with the normal and tangent to the path at the corresponding point P. Such components are called *normal acceleration* and *tangential acceleration*, respectively.

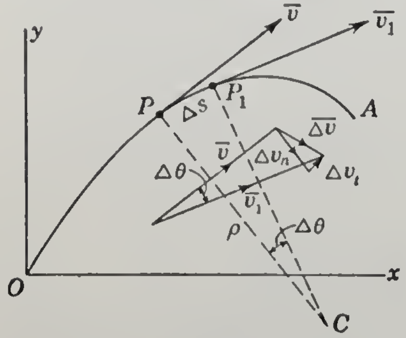


FIG. 244

To obtain these components, we begin in Fig. 244 with the total change in velocity $\overline{\Delta v}$ as the particle moves from P to P_1 and resolve this into components Δv_n and Δv_t parallel to the normal and tangent at P, respectively, as shown by the inset diagram of free vectors. Then the corresponding components of the average acceleration $\overline{\Delta v}/\Delta t$ will be

$$(a_t)_{\text{av}} = \frac{\Delta v_t}{\Delta t} \qquad (a_n)_{\text{av}} = \frac{\Delta v_n}{\Delta t} \tag{k}$$

We now note from Fig. 244 that the angle between the normals at P and P_1 is $\Delta\theta \approx \Delta s/\rho$, where ρ is the radius of curvature of the path at P. Also from the inset diagram of free vectors, we see that $\Delta v_n \approx v\,\Delta\theta$, while $\Delta v_t \approx v_1 - v = \Delta v$, where Δv without a bar represents only the change in speed of the particle as it moves from P to P_1. Using these relationships, Eqs. (k) can be written in the form

$$(a_t)_{av} = \frac{\Delta v_t}{\Delta t} \approx \frac{\Delta v}{\Delta t} \qquad (a_n)_{av} = \frac{\Delta v_n}{\Delta t} \approx \frac{v}{\rho}\frac{\Delta s}{\Delta t} \tag{k'}$$

As Δt approaches zero, the corresponding limits of these ratios give us the instantaneous tangential and normal accelerations of the particle at the particular point P on its path corresponding to the instant t. Thus we have

$$\begin{aligned} a_t &= \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} \\ a_n &= \lim_{\Delta t \to 0} \frac{v}{\rho}\frac{\Delta s}{\Delta t} = \frac{v}{\rho}\frac{ds}{dt} = \frac{v^2}{\rho} \end{aligned} \tag{56}$$

A study of Eqs. (56) shows that the tangential acceleration dv/dt depends only upon the rate of change of speed of the particle, while the normal acceleration, which is always directed toward the center of curvature of the path, depends both upon the square of the speed and the curvature of the path. If the path is a straight line, the curvature is zero and the normal acceleration vanishes. Then we have simply $a = dv/dt$, as obtained in Art. 6.1 for the case of rectilinear motion. If the speed of a particle along a curved path is constant, the tangential acceleration vanishes, and we have only normal acceleration v^2/ρ directed toward the center of curvature. We see that high speed around a sharp curve always means very large acceleration!

In summary we note that total acceleration $\bar{a}$ of a particle in curvilinear motion is a vector. This acceleration may arise as a result of change in magnitude of velocity or change of direction of velocity or both. When we resolve this total acceleration into normal and tangential components, we separate these two effects. The change in speed is accounted for by the tangential acceleration alone, while change of direction of motion is accounted for by the normal acceleration alone.

EXAMPLES

1. Small oscillations of the simple pendulum shown in Fig. 245 are represented by the displacement-time equation

$$s = s_0 \cos pt \tag{1}$$

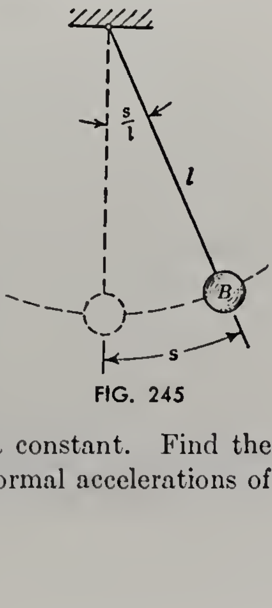


FIG. 245

where s_0 is the amplitude of swing and $p = \sqrt{g/l}$ is a constant. Find the maximum velocity and the maximum tangential and normal accelerations of the bob.

Solution. Differentiating Eq. (l) once with respect to time, we obtain, for the velocity-time equation,

$$v = \frac{ds}{dt} = -s_0 p \sin pt \tag{m}$$

The maximum value of this velocity is

$$v_{\max} = \pm s_0 p = \pm s_0 \sqrt{g/l} \tag{m$'$}$$

which occurs when $\sin pt = \pm 1$; thus $\cos pt = 0$ and the bob is at the bottom of its swing, as can be seen from Eq. (l).

To find the acceleration, we use Eqs. (56) which become

$$\begin{aligned} a_t &= \frac{dv}{dt} = -s_0 p^2 \cos pt \\ a_n &= \frac{v^2}{l} = \frac{s_0^2 p^2}{l} \sin^2 pt \end{aligned} \tag{n}$$

The tangential acceleration a_t has its maximum value when $\cos pt = \pm 1$, that is, when the bob is in either extreme position, and we obtain

$$(a_t)_{\max} = \pm s_0 p^2 = \pm \frac{s_0}{l} g \tag{n$'$}$$

The normal acceleration has its maximum value when $\sin pt = \pm 1$, that is, when the bob is in the middle position as noted before, and we obtain

$$(a_n)_{\max} = \frac{s_0^2 p^2}{l} = \left(\frac{s_0}{l}\right)^2 g \tag{n$''$}$$

2. A wheel of radius r rolls without slip along the x axis with constant speed v_0, as shown in Fig. 246. Investigate the motion of a point A on the rim of the wheel which starts from the origin O.

Solution. After a lapse of time t, the center C of the wheel will have traveled a distance $v_0 t$ as shown, and since it rolls without slip, the arc DA will also have the length $v_0 t$. Thus the angle DCA will be $v_0 t/r$. Then from the geometry of the figure, we can express the coordinates x and y of the point A as follows:

$$\begin{aligned} x &= v_0 t - r \sin \frac{v_0 t}{r} \\ y &= r - r \cos \frac{v_0 t}{r} \end{aligned} \tag{o}$$

With t as a parameter, these two equations define in rectangular coordinates the path of point A which is called a *cycloid*.

Differentiating Eqs. (o) with respect to time gives the velocity-time equations as follows:

$$\dot{x} = v_0\left(1 - \cos\frac{v_0t}{r}\right) \qquad \dot{y} = v_0 \sin\frac{v_0t}{r} \tag{p}$$

Using now the first of Eqs. (d) for the magnitude of the resultant velocity, we find

$$v = v_0\sqrt{2\left(1 - \cos\frac{v_0t}{r}\right)} = 2v_0 \sin\frac{v_0t}{2r} \tag{q}$$

From this expression we see that the maximum speed of point A is $2v_0$ when $t = \pi r/v_0$, that is, when point A is at the top A' of its path (see Fig. 246).

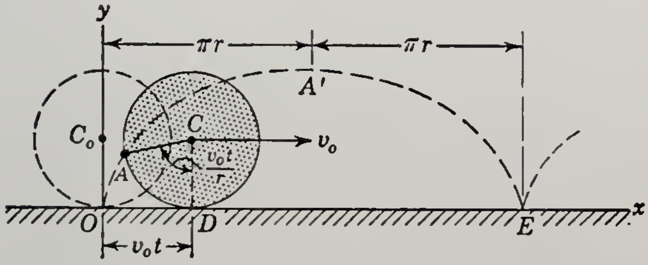


FIG. 246

Differentiating Eqs. (p) again with respect to time, we obtain the acceleration-time equations

$$\ddot{x} = \frac{v_0^2}{r}\sin\frac{v_0t}{r} \qquad \ddot{y} = \frac{v_0^2}{r}\cos\frac{v_0t}{r} \tag{r}$$

Then using the first of Eqs. (g) for the magnitude of the resultant acceleration,

$$a = \frac{v_0^2}{r} \tag{s}$$

Thus the point A has acceleration of constant magnitude always directed toward the center C of the rolling wheel, as can be established from the last two of Eqs. (g).

3. Investigate the motions of the points A, B, and C of a connecting rod (Fig. 247) if the crankpin A is moving with uniform speed v along the circle of radius r and the point B is constrained to follow the x axis.

Solution. We begin with the motion of point A, assuming that at the initial moment ($t = 0$) it has the position A_0. Then denoting by ω the angle of the arc which the point A describes in unit time, we have $\omega = v/r$, where r is the length of the crank. The angle A_0OA is then equal to ωt, and the coordinates of the point A are

$$x = r \cos \omega t \qquad y = r \sin \omega t \tag{t}$$

The coordinate x of the point B, which, of course, has rectilinear motion, is obtained by projecting on the x axis the length r of the crank and the length l of the connecting rod. Then

$$x = r \cos \omega t + l \cos \varphi \tag{u}$$

Noting from the figure that

$$r \sin \omega t = l \sin \varphi$$

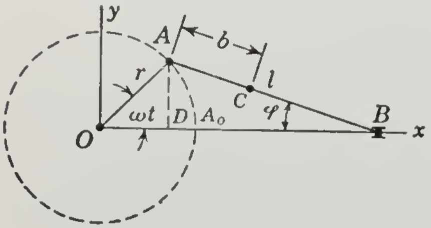


FIG. 247

we obtain the following expression for cos φ:

$$\cos \varphi = \sqrt{1 - \sin^2 \varphi} = \sqrt{1 - \frac{r^2}{l^2} \sin^2 \omega t}$$

and substituting in Eq. (u) above, we have

$$x = r \cos \omega t + l \sqrt{1 - \frac{r^2}{l^2} \sin^2 \omega t} \tag{v}$$

It is interesting to note that, while the projection D of the crankpin on the x axis performs simple harmonic motion, the motion of the point B is more complicated.

For any point C on the axis of the connecting rod at the distance b from the crankpin A, we obtain

$$x = r \cos \omega t + b \sqrt{1 - \frac{r^2}{l^2} \sin^2 \omega t} \qquad y = \frac{l - b}{l} r \sin \omega t \tag{w}$$

In the particular case where $r = l$, we have from Eq. (w)

$$x = (r + b) \cos \omega t \qquad y = (r - b) \sin \omega t \tag{x}$$

Eliminating t between Eqs. (x), we obtain for the path of C

$$\frac{x^2}{(r + b)^2} + \frac{y^2}{(r - b)^2} = 1 \tag{y}$$

Thus each point on the axis of the connecting rod describes an ellipse. This fact is utilized in a device called the ellipsograph for drawing ellipses.

PROBLEM SET 7.1

1. Construct the acceleration-time diagram for the motion of a particle that travels with constant speed $v_0 = 88$ fps along the path $ABCD$ in Fig. A, passing points A, B C, D, at the instants $t = 0$, 10, 20, 30 sec, respectively.

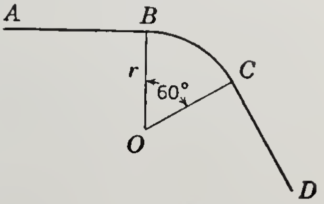


FIG. A

2. At the instant $t = 0$, a locomotive starts to move with uniformly accelerated speed along a circular curve of radius $r = 2{,}000$ ft and acquires by the end of the first 60 sec of motion a speed equal to 15 mph. Find the tangential and normal accelerations at the instant $t = 30$ sec. *Ans.* $a_t = 0.367\ \text{ft/sec}^2$; $(a_n)_{t=30} = 0.0605\ \text{ft/sec}^2$.

3. Considering only rotation of the earth, determine the resultant acceleration of a point on its surface at the latitude 40°N. Assume the radius of the earth $r = 3{,}960$ miles. *Ans.* $a = 0.085\ \text{ft/sec}^2$.

4. Prove that, if the ends A and B of a bar AB of length $2l$ (Fig. B) are constrained to move along the y and x axes, respectively, its mid-point C describes a circle of radius l with center at O while any intermediate point D describes an ellipse with major and minor semiaxes $l + b$ and $l - b$, respectively.

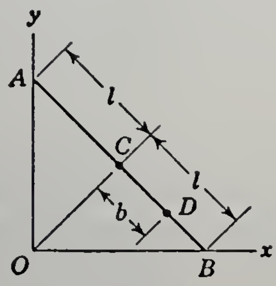


FIG. B

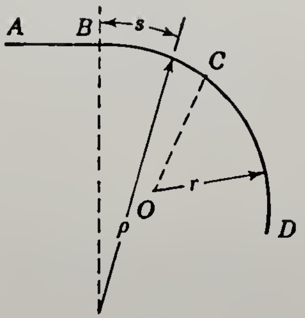


FIG. C

5. In Fig. C, the portion AB of a railroad track is straight, the portion BC is a spiral the radius of curvature of which is $\rho = r^2/s$, and the portion CD is a circle of radius r. Construct the acceleration-time diagram for a locomotive moving with constant speed $v_0 = 88$ fps from A to D if $r = 1{,}000$ ft.

6. The plane curvilinear motion of a particle is defined by the displacement-time equations $x = v_0 t$ and $y = h \cos 2\pi v_0 t/l$. Find the maximum normal acceleration of the particle if $v_0 = 10$ fps, $h = 1$ ft, and $l = 10$ ft. Find also the equation of the path $y = f(x)$. *Ans.* $(a_n)_{\max} = 4\pi^2\ \text{ft/sec}^2$.

7. Starting from rest, a particle moves along a circular path of radius r so that the distance traveled is given by the expression $s = ct^2$, where c is a constant. Find the tangential and normal components of acceleration of the particle. *Ans.* $a_t = 2c$, $a_n = 4c^2t^2/r$.

8. A particle travels with constant speed v along a parabolic path defined by the equation $y = kx^2$, where k is a constant. Find the maximum acceleration of the particle. *Ans.* $a_{\text{max}} = 2kv^2$.

7.2. Differential equations of curvilinear motion. In discussing the curvilinear motion of a particle under the action of applied forces, we return to Newton's second law of motion from which the general equation of motion [Eq. (33)] for a particle was derived. This law states that under the action of a force F a particle receives an acceleration a which is in the direction of the force and proportional to its magnitude. We recall that the law assumes no limitations regarding the motion of the particle before the force F begins to act. Thus, for the particular case where the direction of the force is constant and any initial motion of the particle is in the direction of this force, we obtain a rectilinear motion. Now if the direction of the acting force varies or if the particle has some initial motion in a direction that does not coincide with that of the acting force, we obtain a curvilinear motion of the particle. Equation (33) still holds, but since the acting force may now vary in direction as well as in magnitude, we find it advantageous to resolve both it and the acceleration a that it produces into components parallel to the rectangular coordinate axes x, y. Since each component of the force F produces an acceleration of the particle in its own direction and proportional to its magnitude, we may write

$$\frac{W}{g}\ddot{x} = X \qquad \frac{W}{g}\ddot{y} = Y \tag{57}$$

where X, Y are the components, parallel to the coordinate axes x, y of the resultant acting force F, and $\ddot{x}$, $\ddot{y}$ are the corresponding components of acceleration. These are the differential equations of motion of a particle in a plane.

Sometimes, instead of arbitrary coordinate axes x and y, we choose coordinate axes s and n, representing, respectively, tangent and normal to the path at a chosen point and then resolve both the resultant acceleration a and the resultant force F into components coinciding with these axes. Using Eqs. (56) for tangential and normal acceleration and denoting by S and N the corresponding components of the resultant force, we obtain equations of motion in the following form:

$$\frac{W}{g}\frac{dv}{dt} = S \qquad \frac{W}{g}\frac{v^2}{\rho} = N \tag{58}$$

These equations are especially useful when the path of the particle is known.

Equations (57) or (58), like Eq. (34), page 261, can be used to solve two kinds of problems: (1) The motion of the particle is given, i.e., Eqs. (53) or their equivalent are given, and it is required to find the resultant acting force that produces the given motion. (2) The active forces are given and it is required to find the resulting motion, i.e., to find, for the coordinates, such functions of time that the differential equations of motion are satisfied. The first class of problems requires only differentiation of the given displacement-time equations [Eqs. (53)], and this can usually be made without any difficulty. We shall now consider some simple problems of this kind.

EXAMPLES

1. A particle of weight W attached to a string of length l whirls in a horizontal circular path with uniform speed v. Find the tensile force T in the string.

Solution. Since the speed is uniform, there is only normal acceleration of the particle

$$a_n = \frac{v^2}{l}$$

directed toward the center of the circular path. Using the second of Eqs. (58) then, we have

$$T = N = \frac{Wv^2}{gl}$$

We see that the tensile force in the string is proportional to the square of the speed of the particle and inversely proportional to the length l of the string.

2. A simple pendulum of length l has a bob of weight W and hangs in a vertical plane as shown in Fig. 248. Find the period τ for small amplitudes of swing in the plane of the figure.

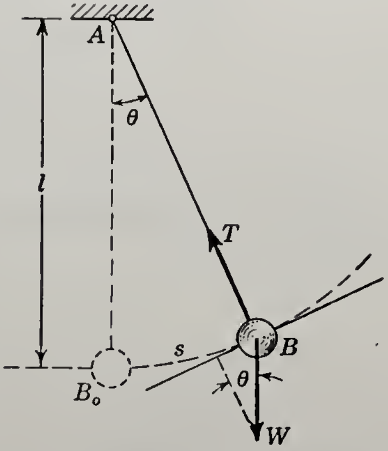


FIG. 248

Solution. During motion, we consider the bob of the pendulum in the position B defined by the distance s measured along its circular path from the equilibrium position B_0. The forces acting on the bob in this position are its weight W and the tension T in the string normal to the path. Projecting these forces onto the tangent to the path at B and using the first of Eqs. (58), we obtain

$$\frac{W}{g}\frac{dv}{dt} = -W \sin \theta$$

Then since $v = ds/dt$, while for small values of θ we have $\sin \theta \approx \theta = s/l$, this equation reduces to

$$\frac{d^2s}{dt^2} + \frac{g}{l}s = 0 \tag{a}$$

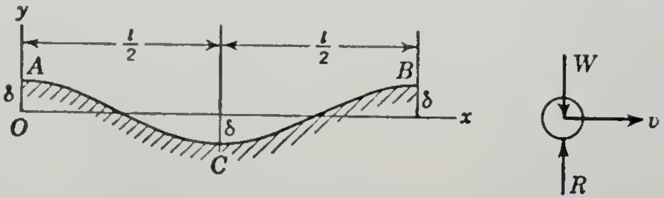
This is the differential equation of simple harmonic motion (see p. 280) and the period is

$$\tau = 2\pi\sqrt{\frac{l}{g}} \tag{b}$$

3. A particle of weight W moves with uniform speed v along the cosine curve ACB in a vertical plane (Fig. 249*a*). The curve is defined by the equation

$$y = \delta\cos\frac{2\pi x}{l} \tag{c}$$

Find the pressure R exerted by the particle on the path as it passes the lowest point C.



(*a*) (*b*)

FIG. 249

Solution. Considering the particle at C (Fig. 249*b*) and using the second of Eqs. (58), we have

$$\frac{W}{g}\frac{v^2}{\rho} = R - W$$

from which

$$R = W\left(1 + \frac{v^2}{g\rho}\right) \tag{d}$$

To find the curvature $1/\rho$, we use the known formula

$$\frac{1}{\rho} = \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{\frac{3}{2}}} \tag{e}$$

By differentiation of Eq. (c), we obtain

$$\frac{dy}{dx} = -\frac{2\pi\delta}{l}\sin\frac{2\pi x}{l}$$

$$\frac{d^2y}{dx^2} = -\frac{4\pi^2\delta}{l^2}\cos\frac{2\pi x}{l}$$

Substituting $x = l/2$ in these expressions, we find

$$\left(\frac{dy}{dx}\right)_{x=l/2} = 0 \qquad \text{and} \qquad \left(\frac{d^2y}{dx^2}\right)_{x=l/2} = \frac{4\pi^2\delta}{l^2}$$

and expression (e) becomes

$$\left(\frac{1}{\rho}\right)_{x=l/2} = \frac{4\pi^2\delta}{l^2}$$

Substituting this into Eq. (d), we obtain

$$R = W\left(1 + 4\pi^2\frac{\delta v^2}{gl^2}\right)$$

We see that the reaction R can be several times larger than the gravity force W if δv^2 is large compared with gl^2.

PROBLEM SET 7.2

1. Small oscillations of a simple pendulum of weight W and length l are defined by the equation $s = s_0 \cos pt$, where s_0 is the amplitude of swing and p is a constant equal to $\sqrt{g/l}$. Determine the maximum value of the tensile force T in the string. *Ans.* $T_{\max} = W(1 + s_0^2/l^2)$.

2. A locomotive of weight $W = 60$ tons goes around a curve of radius $r = 1{,}000$ ft at a uniform speed of 45 mph. Determine the total lateral thrust on the rails. *Ans.* 16,250 lb outward.

3. A motorcycle and rider of total weight $W = 500$ lb travel in a vertical plane with constant speed $v = 45$ mph along the circular curve AB of radius $r = 1{,}000$ ft, as shown in Fig. A. Find the reaction R exerted on the motorcycle by the track as it passes the crest C of the curve. *Ans.* $R = 432$ lb.

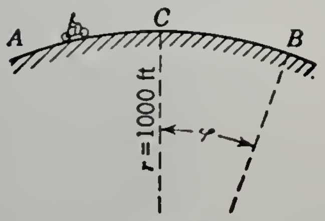


FIG. A

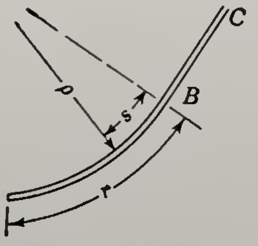


FIG. B

4. At what point B on the vertical curve in Fig. A, as defined by the angle φ, will the road reaction R become equal to zero? Assume all numerical data the same as in Prob. 3. *Ans.* $\varphi = 82°13'$.

5. In Fig. B, a portion of railroad track consists of a tangent CB and a spiral BA having length r and radius of curvature $\rho = r^2/s$, where s is measured from B toward A. A locomotive of weight $W = 60$ tons starts from rest at A and increases its speed along AB with constant tangential acceleration $dv/dt = g/10$. Find the maximum lateral thrust on the outer rail during the motion from A to B. Where does this occur? *Ans.* $N_{\max} = 3$ tons.

6. In Fig. C, a horizontal turntable rotates about a fixed vertical axis through O with constant angular speed ω so that at any instant t, the angle $A_0OA = \omega t$, as shown. As it does so, a man M of weight $W = 160$ lb, starting from O when $t = 0$, walks with constant speed $v = 3$ fps outward along the rotating line OA on the turntable. Find the total friction force F between the man's feet and the floor of the turntable at the instant $t = 2\pi/\omega$, that is, at the end of one revolution of the table. Assume $\omega = 1$ radian/sec. *Ans.* $F = 98.5$ lb.

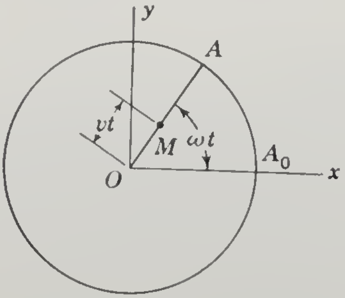


FIG. C

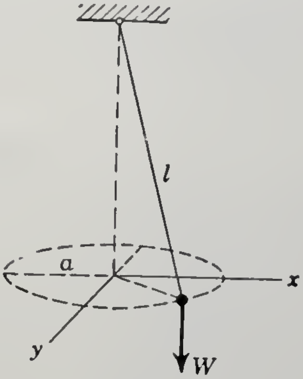


FIG. D

7. In Fig. D, the bob of a conical pendulum of length l and weight W describes a horizontal circle defined by the equations

$$x = a \cos \omega t \qquad y = a \sin \omega t$$

where a is the radius of the circular path and ω is a constant. Prove that the tension T in the string is constant during such motion and find its magnitude. *Ans.* $T = (W/g)\,\omega^2 l$.

8. The coefficient of friction between wet asphalt pavement and the tires of an automobile is found to have the value $\mu = 0.20$. At what constant speed v can the automobile travel around a curve of radius $r = 800$ ft without skidding if the road is level? *Ans.* $(v)_{\max} = 49$ mph.

9. A small block of weight W rests on a horizontal turntable at a distance r from the axis of rotation (Fig. E). If the coefficient of friction between the block and the surface of the turntable is μ, find the maximum uniform speed v_m that the block can have owing to rotation of the turntable without slipping off. *Ans.* $v_m = \sqrt{\mu g r}$.

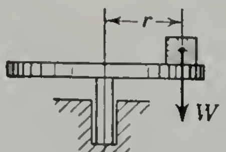


FIG. E

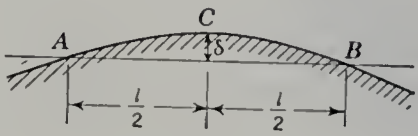


FIG F

10. An automobile of weight W travels with uniform speed v over a vertical curve ACB (Fig. F) which is parabolic. Determine the total pressure R exerted on the road by the four wheels of the car as it passes the crest C if $\delta = 4$ ft, $l = 200$ ft, $v = 60$ mph. *Ans.* $R = 0.81\ W$.

7.3. Motion of a projectile. We come now to the consideration of the integration of the differential equations of motion (57) for a particle acted upon by given forces. The simplest case we have when the resultant acting force is of constant magnitude and direction and the motion is curvilinear only by virtue of some initial velocity of the particle in a direction different from that of the acting force.

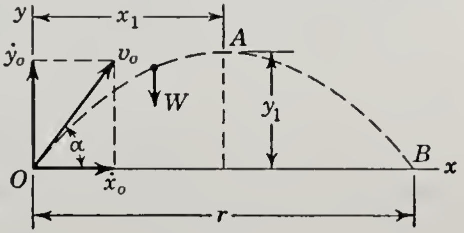


FIG. 250

As an example of the application of Eqs. (57) to such a case, let us consider the motion of a projectile fired from the origin of coordinates with an initial velocity v_0 inclined to the horizontal by the angle α (Fig. 250) and having the projections $\dot{x}_0$ and $\dot{y}_0$. Neglecting air resistance on the projectile and assuming that only the vertical gravity force W is acting, Eqs. (57) become

$$\frac{W}{g}\ddot{x} = 0, \qquad \frac{W}{g}\ddot{y} = -W$$

or

$$\ddot{x} = 0, \qquad \ddot{y} = -g \tag{a}$$

Integrating these equations once, we obtain

$$\dot{x} = C_1 \qquad \dot{y} = -gt + C_2 \tag{b}$$

Knowing that at the initial moment when $t = 0$, $\dot{x} = \dot{x}_0$ and $\dot{y} = \dot{y}_0$, we find for the constants of integration

$$C_1 = \dot{x}_0 \qquad C_2 = \dot{y}_0$$

and Eqs. (b) become

$$\dot{x} = \dot{x}_0 \qquad \dot{y} = -gt + \dot{y}_0 \tag{c}$$

These are the velocity-time equations for the projectile. We see that the particle viewed from a great height would appear to move with uniform horizontal speed $\dot{x}_0$, while viewed from some distance directly behind the gun it would appear to move as a particle projected vertically upward with the initial velocity $\dot{y}_0$.

A second integration gives

$$x = \dot{x}_0 t + D_1 \qquad y = -\tfrac{1}{2}gt^2 + \dot{y}_0 t + D_2 \tag{d}$$

Again, knowing that at the initial moment when $t = 0$ we have $x = y = 0$, we find for the constants of integration

$$D_1 = 0 \qquad D_2 = 0$$

and Eqs. (d) become

$$x = \dot{x}_0 t \qquad y = -\tfrac{1}{2}gt^2 + \dot{y}_0 t \tag{e}$$

These are the displacement-time equations for the projectile. For any value of t we can calculate from them the coordinates x and y of the moving projectile, and hence they completely define the motion.

To find the equation of the path of the projectile, we eliminate t between Eqs. (e) and obtain

$$y = -\frac{g}{2\dot{x}_0^2}x^2 + \frac{\dot{y}_0}{\dot{x}_0}x \tag{f}$$

Using $\dot{x}_0 = v_0 \cos \alpha$ and $\dot{y}_0 = v_0 \sin \alpha$, this can also be written in the form

$$y = x \tan \alpha - \frac{g \sec^2 \alpha}{2v_0^2}x^2 \tag{f'}$$

This is the equation of a *parabola* having a vertical axis. The abscissa x_1 of the vertex A of the path is obtained by observing that at this point the slope dy/dx of the curve is zero and we obtain from Eq. (f)

$$\frac{dy}{dx} = -\frac{g}{\dot{x}_0^2}x + \frac{\dot{y}_0}{\dot{x}_0} = 0$$

from which

$$x_1 = \frac{\dot{x}_0\dot{y}_0}{g} \tag{g}$$

Substituting the value (g) into Eq. (f), we obtain for the ordinate of the vertex

$$y_1 = \frac{\dot{y}_0^2}{2g} \tag{h}$$

We may obtain the same values for the coordinates of the vertex of the path in another way by using Eqs. (c) for the velocity of the projectile. Since, when the projectile is at the vertex of the path, the vertical projection $\dot{y}$ of its velocity is zero, we have

$$-gt + \dot{y}_0 = 0$$

from which

$$t = \frac{\dot{y}_0}{g}$$

Substituting this value of t into Eqs. (e), we obtain for x_1 and y_1, defining the vertex of the parabola, the same values (g) and (h) given above.

To find the point at which the path intersects the x axis, we have only to double the value of x_1 and we have

$$r = 2x_1 = \frac{2\dot{x}_0\dot{y}_0}{g} \tag{i}$$

This distance is called the *range* of the projectile. If we replace $\dot{x}_0$ and $\dot{y}_0$ by their respective values $v_0 \cos \alpha$ and $v_0 \sin \alpha$, expression (i) defining the range becomes

$$r = \frac{2v_0^2 \sin \alpha \cos \alpha}{g} = \frac{v_0^2}{g} \sin 2\alpha \tag{i'}$$

It is seen that for a given initial velocity v_0 the maximum range is obtained when $\alpha = 45°$ and this maximum range is

$$r_{\max} = \frac{v_0^2}{g} \tag{i''}$$

It must be remembered that all of the foregoing discussion of the motion of a projectile neglects the effect of air resistance, and for the speeds with which projectiles usually travel this factor is by no means negligible. Consideration of the effect of air resistance, however, greatly complicates the problem and is beyond the scope of this book.[1] In each of the following problems it will be assumed that the projectile moves without air resistance.

EXAMPLES

1. A cannon fires its projectile with such an initial velocity and at such an angle of elevation that the range is r and the maximum height to which the projectiles rises is h. Find the maximum range that can be obtained with the same initial velocity.

Solution. Denoting by v_0 and α the unknown initial velocity and angle of elevation and using Eqs. (h) and (i'), we may write

$$h = \frac{v_0^2}{g}\frac{\sin^2 \alpha}{2} \qquad r = \frac{v_0^2}{g} \sin 2\alpha \tag{j}$$

which, since the maximum range is v_0^2/g as given by Eq. (i''), may be written

$$h = r_m \frac{\sin^2 \alpha}{2} \qquad r = r_m \sin 2\alpha \tag{j'}$$

---

[1] For more detailed discussion of the problem, see the authors' "Advanced Dynamics," p. 94, McGraw-Hill, New York, 1948.

Writing the first of Eqs. (j′) in the form

$$h = \frac{r_m}{4}(1 - \cos 2\alpha)$$

we obtain from these equations

$$\cos 2\alpha = 1 - \frac{4h}{r_m} \qquad \sin 2\alpha = \frac{r}{r_m} \tag{k}$$

To eliminate the unknown angle α between Eqs. (k), we square both sides and add the equations, obtaining

$$1 = \left(1 - \frac{4h}{r_m}\right)^2 + \frac{r^2}{r_m^2}$$

from which

$$r_m = 2h + \frac{r^2}{8h}$$

2. An airplane is moving with a horizontal velocity v at a height h above a level plane (Fig. 251). If a projectile is fired from a gun at the instant when the plane is vertically above the gun, what must be the angle of elevation α and what is the minimum initial velocity v_0 of the projectile in order to hit the airplane?

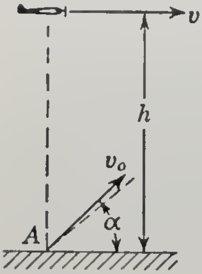


FIG. 251

Solution. It is evident that we must have

$$v_0 \cos \alpha = v \tag{l}$$

from which

$$\alpha = \arccos \frac{v}{v_0}$$

Also, if the projectile is ever to attain the height h, we must have

$$v_0 \sin \alpha \geqq \sqrt{2gh} \tag{m}$$

Eliminating α between Eqs. (l) and (m), we find

$$v_0 \geqq \sqrt{v^2 + 2gh}$$

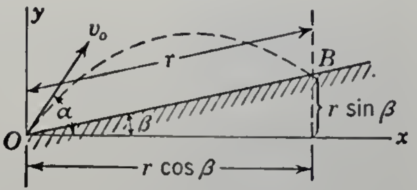


FIG. 252

3. Referring to Fig. 252, determine the range r on an inclined plane for a projectile fired from point O with initial velocity v_0 and angle of elevation α. Find also, the maximum range on the inclined plane.

Solution. Taking coordinate axes x, y, as shown, and using Eqs. (e), we write for the coordinates of point B where the projectile hits the inclined plane, the following expressions:

$$\begin{aligned} x &= v_0 t \cos \alpha = r \cos \beta \\ y &= v_0 t \sin \alpha - \tfrac{1}{2} g t^2 = r \sin \beta \end{aligned} \tag{n}$$

Eliminating t between these two equations, we find, for the range r,

$$r = \frac{2v_0^2 \cos^2 \alpha}{g \cos \beta} (\tan \alpha - \tan \beta) \tag{o}$$

If we take $\beta = 0$, this reduces to

$$(r)_{\beta=0} = \frac{v_0^2}{g} \sin 2\alpha$$

which agrees with Eq. (i′), as it should.

To find the maximum range, we set the derivative of expression (o) with respect to α equal to zero and obtain

$$\tan 2\alpha = -\cot \beta \tag{p}$$

This shows that for maximum range on the inclined plane, the initial velocity vector $\bar{v}_0$ in Fig. 252 should make the angle $\alpha = 45° + \beta/2$ with the horizontal.

PROBLEM SET 7.3

1. A mortar fires a projectile across a level field so that the range r is a maximum and equal to 1,000 yd. Find the time of flight. *Ans.* $t = 13.7$ sec.

2. In Fig. A, a projectile is fired horizontally from point A with initial velocity $v_0 = 360$ fps. Find the range r to the target B. *Ans.* $r = 2{,}840$ ft.

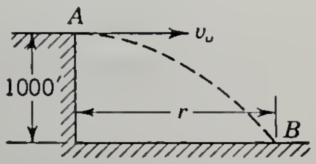


FIG. A

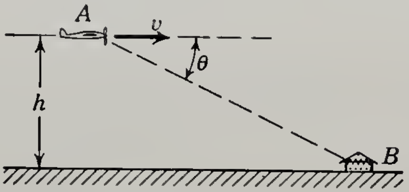


FIG. B

3. In Fig. B the pilot of an airplane flying horizontally with constant speed $v = 300$ mph at an elevation $h = 2{,}000$ ft above a level plain wishes to bomb a target B on the ground. At what angle θ below the horizontal should he see the target at the instant of releasing the bomb in order to score a hit? Neglect air resistance. *Ans.* $\theta = 22°12'$.

4. A mortar having muzzle velocity $v_0 = 707$ fps fires for maximum range across a level plain. Neglecting air resistance, calculate the time of flight of the shell. *Ans.* $t = 31.1$ sec.

5. In what proportion will the maximum range of a projectile be increased if the initial velocity is increased by 10 per cent? *Ans.* 21 per cent.

6. The maximum range of a projectile is 2,000 yd. At what angles of elevation α will the range be 1,500 yd if the initial velocity remains unchanged? *Ans.* $\alpha = 24°18'$ or $65°42'$.

7. Two adjacent guns having the same muzzle velocity $v_0 = 1{,}000$ fps fire simultaneously at angles of elevation α_1 and α_2 for the same target at range $r = 5{,}000$ yd. Calculate the time difference $t_2 - t_1$ between the two hits. *Ans.* 44.5 sec.

8. In Fig. C, a hammer of weight $W = 2$ lb starts from rest at A and slides down a roof for which the coefficient of friction is $\mu = 0.2$. Find the distance x to the point D where it hits the ground. *Ans.* $x = 14.4$ ft.

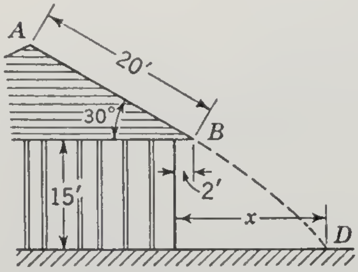


FIG. C

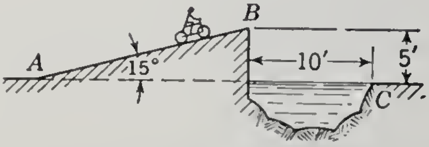


FIG. D

9. Referring to Fig. D, calculate the minimum speed v_0 with which a motorcycle stunt rider must leave the 15° ramp at B in order to clear the ditch. *Ans.* $v_0 = 15$ fps.

***10.** A boy wishes to throw a ball over a flat-roofed schoolhouse that stands 40 ft wide and 25 ft high on level ground. Assuming that the ball will leave his hand at a height of 5 ft above the ground, determine how far from the wall he should take his stand in order to make the ball clear the roof with the least effort, i.e., with the minimum initial velocity. *Ans.* 14.64 ft.

7.4. D'Alembert's principle in curvilinear motion. The equations of motion of a particle [Eqs. (57)] can be written in the form

$$X - m\ddot{x} = 0 \qquad Y - m\ddot{y} = 0 \tag{59}$$

wherein $m = W/g$ is the mass of the particle. These equations have the same form as equations of static equilibrium (page 26) and may be regarded as equations of dynamic equilibrium. In writing equations of dynamic equilibrium for a moving particle it is only necessary to consider, in addition to the real forces acting on the particle, the *inertia force*, the projections of which on the coordinate axes x and y are $-m\ddot{x}$ and $-m\ddot{y}$. The resultant inertia force balances the resultant active force, and the particle may then be considered to be in equilibrium.

In the case of any rigid body that has *curvilinear translation*, all particles of the body have the same motion and hence the same acceleration. If to each particle the inertia force is added, the resultant of these inertia forces balances the resultant of the active forces external to the body and we have again a system of forces in equilibrium. Internal forces (actions and reactions) between the various particles of the body always occur in balanced pairs and need not be considered.

Applications of D'Alembert's principle to problems of curvilinear motion will now be illustrated by several examples.

EXAMPLES

1. Determine the maximum bending moment in the side rod AB of length l of an engine if the cranks O_1A and O_2B are of length $r = 15$ in. and are making 240 rpm (Fig. 253).

Solution. Owing to uniform rotation of the cranks, each point on the axis of the side rod AB moves with constant speed v along a circle of radius r. The resultant acceleration of each element of the side rod is then directed toward the center of that circle and has the magnitude v^2/r. For a given position of the side rod the corresponding inertia forces have the parallel directions shown in the figure. If q is the weight of the rod per unit length, the intensity of the distributed inertia force is qv^2/gr. For such loading, the maximum bending moment is at the middle of the side rod and has its greatest magnitude when the rod is in the position $A'B'$, for here the inertia load and gravity load are in the same direction perpendicular to the axis of the rod. The magnitude of this moment, calculated as for a simply supported beam, is

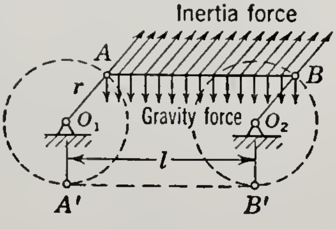


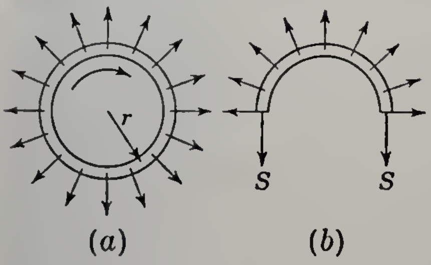
FIG. 253

$$M_{\max} = \left(q + \frac{qv^2}{gr}\right)\frac{l^2}{8} = \frac{ql^2}{8}\left(1 + \frac{v^2}{gr}\right) \tag{a}$$

For the given numerical data $v = 15 \times 8\pi$ ips, and Eq. (a) becomes

$$M_{\max} = \frac{ql^2}{8}(1 + 24.5) \tag{b}$$

This maximum bending moment in the side rod is 25.5 times that due to the weight of the bar alone. Further, from Eq. (a) we see that this moment increases with the square of the speed of operation of the engine. From this example we see that stresses due to inertia loads must play an important role in the design of high-speed machinery.



(a) (b)

FIG. 254

2. Determine the circumferential tension S produced in a uniformly rotating thin circular ring of uniform cross-sectional area A and mean radius r (Fig. 254) if the peripheral velocity of the ring is v.

Solution. If q is the weight per unit length of the ring, the intensity of the uniformly distributed inertia force due to rotation is qv^2/gr and is directed radially outward as shown in the figure. Considering the dynamic equilibrium of one-half the ring as shown in Fig. 254*b*, we obtain for the circumferential tension (see Art. 3.8, page 155)

$$S = \frac{qv^2}{gr} r = \frac{qv^2}{g} \tag{c}$$

The corresponding tensile stress is

$$s = \frac{S}{A} = \frac{qv^2}{Ag} = w\frac{v^2}{g} \tag{d}$$

where $w = q/A$ is the weight per unit volume of the material of the ring.

3. A particle of weight W attached to a fixed point O by a string of length l whirls in a horizontal circular path of radius r with uniform speed v so that the string generates a cone of height $h = \sqrt{l^2 - r^2}$ (Fig. 255*a*). Determine the relation between v, r, and h and also the tensile force S in the string during such motion.

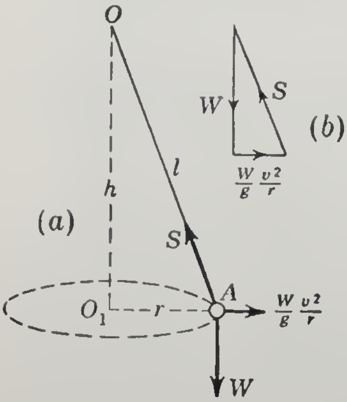


FIG. 255

Solution. Since the particle moves with uniform speed in a circular path, it has only normal acceleration v^2/r directed toward the center O_1 of the circle and the corresponding inertia force is Wv^2/gr directed outward as shown in the figure. This inertia force together with the weight W and the tensile force S are in equilibrium and must build a closed triangle of forces as shown in Fig. 255*b*. From the similarity between the triangle of forces and $\triangle OAO_1$, we may write

$$r:h = \frac{Wv^2}{gr}:W$$

from which

$$v = r\sqrt{\frac{g}{h}} \tag{e}$$

Eq. (e) expresses the required relation between v, r, and h. Also, from the triangle of forces, we obtain

$$S = W\sqrt{1 + \frac{v^4}{g^2r^2}} \tag{f}$$

Substituting, for v, the value given by Eq. (e), this may also be written in the form

$$S = W\sqrt{1 + \left(\frac{r}{h}\right)^2} \tag{f'}$$

The device is called a *conical pendulum*. Its period, i.e., the time required for one revolution, is

$$\tau = \frac{2\pi r}{v} = 2\pi\sqrt{\frac{h}{g}} \tag{g}$$

For small angles of the cone, we have $h \approx l$ and the period of the conical pendulum becomes the same as for a simple pendulum.

4. Referring to Fig. 256, determine the so-called superelevation e of the outer rail on a railroad curve of radius r so that a car traveling at speed v around the curve will exert equal pressures on the two rails. The distance between rails is b, as shown.

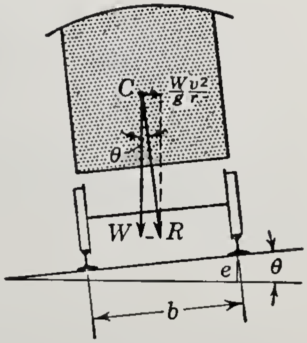


FIG. 256

Solution. Using D'Alembert's principle, we consider, in addition to the vertical gravity force W of the car, the horizontal inertia force Wv^2/gr acting at the center of gravity C and directed to the outside of the curve. Equal reactions on the two rails will be realized when the resultant R of these two forces is perpendicular to the plane of the roadbed. Thus, from the figure, we see that

$$\tan\theta = \frac{Wv^2/gr}{W} = \frac{v^2}{gr} \tag{h}$$

Knowing the value of θ from Eq. (h), we may calculate the proper superelevation e from the formula

$$e = b\sin\theta \tag{i}$$

In ordinary railroad construction, the angle θ will usually be sufficiently small so that we can take $\sin\theta \approx \tan\theta$ and in such case Eq. (i) becomes

$$e = \frac{bv^2}{gr} \tag{j}$$

This equation can also be used in computing the proper superelevation for highway curves.

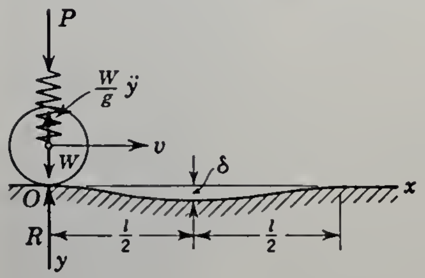


FIG. 257

5. A wheel of weight W moves with constant speed v along the horizontal x axis (Fig. 257). What effect on the pressure between the wheel and the road will a low spot in the road have if the shape of the low spot is defined by the equation

$$y = \frac{\delta}{2}\left(1 - \cos\frac{2\pi x}{l}\right)$$

Consider both the road and the rim of the wheel to be absolutely rigid. Assume also that in addition to the gravity force W the wheel is pressed against the road by a force P transmitted to it through a spring.

Solution. So long as the path of the wheel is horizontal, the pressure on the road is constant and equal to $P + W$. When the wheel is moving along the low spot, there is not only horizontal but also some vertical motion and the vertical acceleration $\ddot{y}$ corresponding to this motion must be considered. The vertical component of the velocity of the wheel for any position along the low spot is

$$\dot{y} = \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = v\frac{dy}{dx} = v\frac{\delta\pi}{l}\sin\frac{2\pi x}{l} \tag{k}$$

Then

$$\ddot{y} = \frac{d\dot{y}}{dt} = \frac{d\dot{y}}{dx}\frac{dx}{dt} = v\frac{d\dot{y}}{dx} = \frac{2\pi^2}{l^2}v^2\delta\cos\frac{2\pi x}{l} \tag{l}$$

Applying D'Alembert's principle, we find now that for any position along the low spot, defined by the displacement x, the pressure between the road and wheel is

$$R = P + W - \frac{W}{g}\ddot{y} = P + W\left(1 - \frac{2\pi^2v^2\delta}{gl^2}\cos\frac{2\pi x}{l}\right) \tag{m}$$

At the beginning of the low spot ($x = 0$), $\cos(2\pi x/l) = 1$ and the minimum pressure between the road and wheel is

$$R_{\min} = P + W\left(1 - \frac{2\pi^2v^2\delta}{gl^2}\right) \tag{m'}$$

It is seen that, when the second term in the parentheses of this expression is greater than unity, the pressure on the road is less than the force P. Thus we conclude that, if it were not for the force P, the wheel would not follow the contour of the lot spot.

At the middle of the low spot, $x = l/2$, $\cos(2\pi x/l) = -1$, and the maximum pressure between the wheel and the road is

$$R_{\max} = P + W\left(1 + \frac{2\pi^2v^2\delta}{gl^2}\right) \tag{m''}$$

Assume, for instance, that $\delta = \frac{1}{8}$ in., $l = 40$ in., and $v = 100$ fps. Then from Eq. (m''), we find

$$R_{\max} = P + W(1 + 5.75)$$

and we see that the maximum pressure can be increased very substantially owing to the effect of a slight depression in the road.

The above discussion shows the importance, in the case of vehicles, of reducing as far as possible the weight of the wheels and of having springs between the wheel axles and the body of the vehicle. By introducing elasticity in the rim of the wheel and considering elasticity of the road, further reduction in the inertia force of the wheel is obtained.

PROBLEM SET 7.4

1. A circular ring has a mean radius $r = 20$ in. and is made of steel for which $w = 0.284$ lb/in.^{3} and for which the ultimate strength in tension is 60,000 psi. Find the uniform speed of rotation about its geometric axis perpendicular to the plane of the ring at which it will burst. *Ans.* 4,300 rpm.

2. Find the proper superelevation e for a 24-ft highway curve of radius $r = 2{,}000$ ft in order that a car traveling with a speed of 50 mph will have no tendency to skid sidewise. *Ans.* $e = 2.00$ ft.

3. Racing cars travel around a circular track of 1,000-ft radius with a speed of 240 mph. What angle α should the floor of the track make with the horizontal in order to safeguard against skidding? *Ans.* $\alpha = 75°27'$.

4. A particle A of weight W is suspended in a vertical plane by two strings as shown in Fig. A. Determine the tension S in the inclined string OA (*a*) an instant before the horizontal string AB is cut and (*b*) an instant after this string is cut. Assume the string OA inextensible. *Ans.* $S_1 = W \sec \alpha$; $S_2 = W \cos \alpha$.

5. Two balls of weights $W_a = 10$ lb and $W_b = 15$ lb are connected by an elastic string and supported on a turntable as shown in Fig. B. When the turntable is at rest, the tension in the string is $S = 50$ lb and the balls exert this same force on each of the stops A and B. What forces will they exert on the stops when the turntable is rotating uniformly about the vertical axis CD at 60 rpm? *Ans.* $R_a = 39.8$ lb; $R_b = 27.0$ lb.

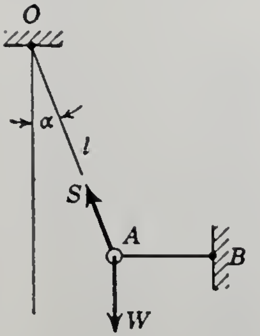


FIG. A

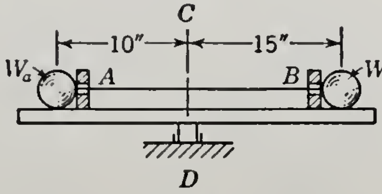


FIG. B

6. A ball of weight W is supported in a vertical plane as shown in Fig. C. Find the compressive force S in the bar BC: (*a*) just before the string AB is cut and (*b*) just after the string AB is cut. Neglect the weight of the bar BC. *Ans.* $S_a = 0.732W$; $S_b = 0.500W$.

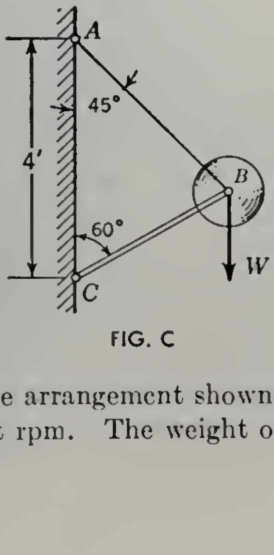


FIG. C

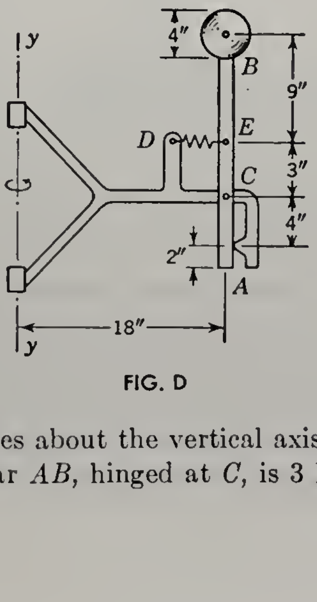


FIG. D

7. The arrangement shown in Fig. D rotates about the vertical axis yy at constant rpm. The weight of the vertical bar AB, hinged at C, is 3 lb and

the weight of the ball at the top is 6 lb. When the system is at rest, the initial tension in the spring DE is 20 lb. At what rpm will contact at A be broken? Assume the frame and bar AB to be absolutely rigid. *Ans.* 38.7 rpm.

8. At what uniform speed of rotation around the vertical axis AB will the balls C and D of equal weights W begin to lift the weight Q of the device shown in Fig. E? The following numerical data are given: $W = 10$ lb, $Q = 20$ lb, $l = 10$ in. Neglect all friction and the weights of the four hinged bars of length l. The weight Q can slide freely along the shaft AB. *Ans.* $n = 111$ rpm.

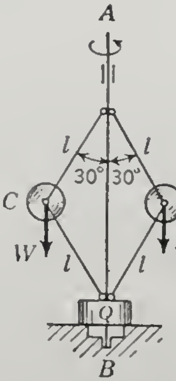


FIG. E

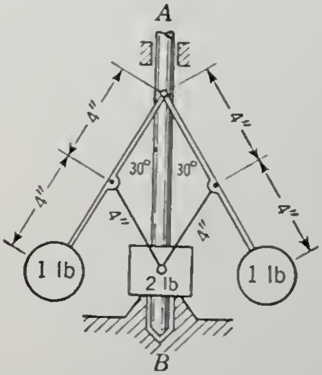


FIG. F

9. The 2-lb weight of the governor in Fig. F can slide freely on the vertical shaft AB. At what rpm about the axis AB will the 1-lb flyballs lift the sliding weight free of its support? Neglect friction and the weights of the bars. *Ans.* 101 rpm.

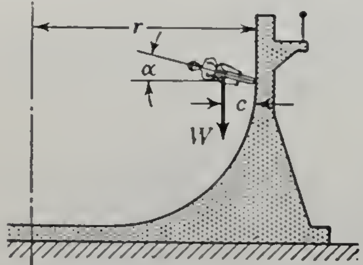


FIG. G

10. What is the minimum uniform speed that a man and motorcycle of weight W can have in going around the inside of a vertical circular drum of radius r (Fig. G) in order to prevent slipping down the wall if the coefficient of friction between the tires and the wall is μ? When the motorcycle is running at this speed, what angle α must its middle plane make with the horizontal in order to prevent tipping down? *Ans.* $v = \sqrt{(g/\mu)(r - c)}$; $\alpha = \arctan \mu$.

7.5. Moment of momentum. If a particle A of weight W moves along the curvilinear path CD in the xy plane in Fig. 258 and has at any instant the velocity v, its momentum is defined by the product $(W/g)v$. This momentum of the particle at any instant can be represented by the vector $\overline{AE}$, the direction of which coincides with that of the velocity, i.e., tangent to the path at the point defining the instantaneous position of the particle. The *moment of momentum* of the particle with respect to the origin O is defined as the product of the momentum and the perpendicular distance OB (Fig. 258). In expressing this moment of momentum of the particle, it is convenient to resolve the momentum vector $\overline{AE}$ into two rectangular components

$$\frac{W}{g}\dot{x} \qquad \text{and} \qquad \frac{W}{g}\dot{y}$$

parallel, respectively, to the coordinate axes x and y. Then, since the moment of the resultant with respect to point O is equal to the algebraic sum of the moments of its components with respect to the same point, we may express the moment of momentum of the particle with respect to the origin O as follows:

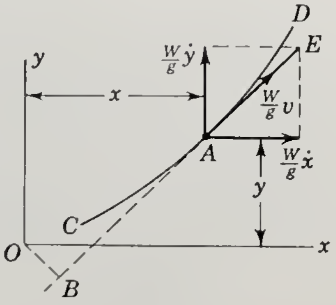


FIG. 258

$$H_0 = \frac{W}{g}(\dot{y}x - \dot{x}y) \tag{a}$$

where the symbol H will be used to denote moment of momentum and counterclockwise moment is considered as positive.

In the same manner the resultant force F acting on the particle A (Fig. 259) may be resolved into the rectangular components X and Y and the moment of this force with respect to the origin O becomes

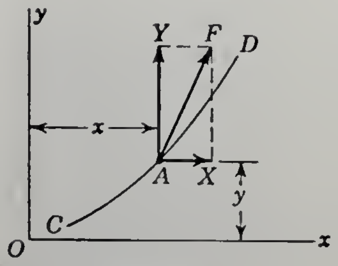


FIG. 259

$$M_0 = Yx - Xy \tag{b}$$

Let us consider now the equations of motion of a particle [Eqs. (57)]:

$$\frac{W}{g}\ddot{x} = X \qquad \frac{W}{g}\ddot{y} = Y \tag{c}$$

Multiplying the first of these equations by y and the second by x and subtracting the first from the second, we obtain

$$\frac{W}{g}(\ddot{y}x - \ddot{x}y) = Yx - Xy$$

which may be written in the form

$$\frac{d}{dt}\left[\frac{W}{g}(\dot{y}x - \dot{x}y)\right] = Yx - Xy \tag{60}$$

Using the notation H_0 for moment of momentum with respect to point

O and M_0 for moment of the resultant force with respect to the same point, Eq. (60) may be expressed in the more compact form.

$$\frac{d}{dt} H_0 = M_0 \tag{60'}$$

In either form the equation states that *the rate of change of moment of momentum of a particle with respect to any point in its plane of motion is equal to the moment of the resultant acting force with respect to the same point.*

Equation (60) is very useful when we have to deal with the plane motion of a particle under the action of a so-called *central force,* i.e., one that is always directed toward some fixed point. In such case the moment of the force with respect to this point is always zero, and hence we conclude from Eq. (60) that the rate of change of moment of momentum of the particle with respect to the same point is zero or in other words that the moment of momentum with respect to this point is constant.

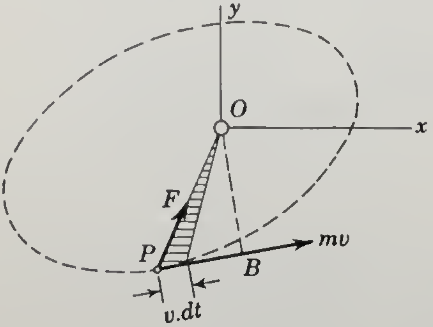


FIG. 260

As an application of Eq. (60′), let us consider briefly one aspect of the problem of *planetary motion.*[1] In Fig. 260, let point O represent the position of the sun and P that of the planet. Denoting by m the mass of the planet and by v its velocity, the momentum of the planet is mv, represented by a vector tangent to the orbit as shown. The moment of this momentum vector with respect to point O is $mv \cdot OB$. Now since the attractive force F is always directed toward O its moment about this point is zero and we conclude from Eq. (60′) that

$$mv \cdot OB = \text{const} \tag{d}$$

As the planet moves along its orbit, the radius vector OP sweeps out in time dt the small shaded area

$$dA = \frac{v\,dt}{2} OB \tag{e}$$

Hence $\frac{1}{2}v \cdot OB$ represents the rate of description of area, and we conclude from Eq. (d) that this must be constant. Thus the planet moves in its orbit in such a way that its radius vector sweeps out equal areas in equal intervals of time. This conclusion was first reached by Kepler from direct observation and was one of the established facts that helped Newton in formulating his three laws of motion.

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[1] For a more complete discussion of planetary motion, see the authors' "Advanced Dynamics," p. 87, McGraw-Hill, New York, 1948.

PROBLEM SET 7.5

1. A small ball of weight W, attached to the end of a string, is supported by a smooth horizontal plane and travels with uniform speed v_0 in a circular path of radius r (Fig. A). By pulling the string at the lower end, the radius of the path is reduced to $r/2$. Determine the new velocity of the ball and the tension S in the string. *Ans.* $v_1 = 2v_0$; $S = 8Wv_0^2/gr$.

2. A conical pendulum of length $l = 20$ in. rotates with constant speed v in a horizontal circular path of radius $r = 10$ in. as shown in Fig. B. How much string must be pulled through the pedestal to double the speed of the ball? *Ans.* 14.9 in.

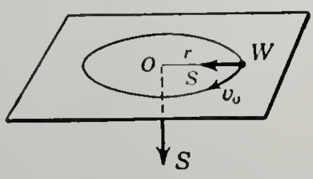

FIG. A

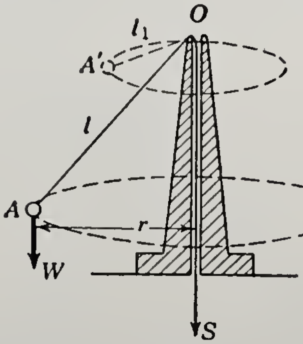

FIG. B

3. If the ball shown in Fig. A is given an initial velocity v_0 in the circular path of radius r and the coefficient of friction between it and the horizontal plane is μ, determine, by using Eq. (60), the time interval t required for the ball to come to rest. *Ans.* $t = v_0/\mu g$.

4. A heavy particle suspended vertically by a long string so that it can swing freely under the influence of gravity is allowed to describe a small horizontal elliptical path centered about the position of equilibrium O. If the major and minor semiaxes of the ellipse are a and b, respectively, find the ratio of the velocity v_x with which the particle crosses the y axis to the velocity v_y with which it crosses the x axis. *Ans.* $v_x/v_y = a/b$.

5. The motion of a particle of mass m in the xy plane is defined by the equations

$$x = a \cos pt \qquad y = b \sin pt$$

where a, b, and p, are constants. Calculate the moment of momentum of the particle with respect to the origin O. *Ans.* $H_0 = abpm$.

6. The motion of a particle of mass m in the xy plane is defined by the equations

$$x = a \cos pt \qquad y = b \sin 2pt$$

where a, b, and p are constants. Calculate the moment of momentum of the particle with respect to the origin O. *Ans.* $H_0 = 2abpm \cos^3 pt$.

7.6. Work and energy in curvilinear motion. The method of work and energy discussed in Art. 6.9 (see page 299) for the case of rectilinear motion of a particle can be used also in dealing with curvilinear motion. We begin with equations of motion (58) in the form

$$\frac{W}{g}\frac{dv}{dt} = S \qquad \frac{W}{g}\frac{v^2}{\rho} = N \tag{a}$$

where S and N are the projections of the resultant force F on the tangent and normal to the path at any point P as shown in Fig. 261.

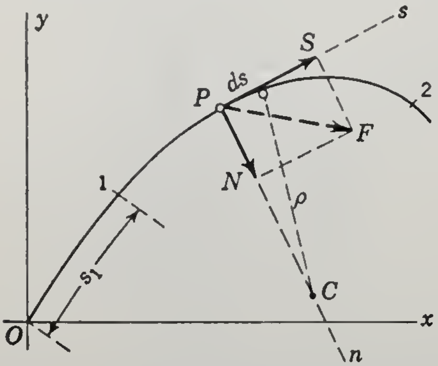

FIG. 261

Multiplying the first of Eqs. (a) by $v = ds/dt$, we obtain

$$\frac{W}{g} v \frac{dv}{dt} = S \frac{ds}{dt}$$

which may be written in the form

$$d\left(\frac{W}{g}\frac{v^2}{2}\right) = S\,ds \tag{b}$$

The left side of this expression is the differential change in *kinetic energy* of the particle during the time interval dt. The right side of Eq. (b) represents the corresponding increment of *work* done by the tangential force S acting through the displacement ds. We recall now from statics that the work of the resultant force F on the displacement ds is equal to the algebraic sum of the works of its components S and N. Furthermore, the force N is perpendicular to the direction of the displacement and does no work. Hence, the right side of Eq. (b) represents completely the work of the resultant force F on the displacement ds.

Integrating Eq. (b) between limits corresponding to any two points 1 and 2 on its path (see Fig. 261), we obtain

$$\frac{Wv_2^2}{2g} - \frac{Wv_1^2}{2g} = \int_{s_1}^{s_2} S\,ds \tag{61a}$$

This is the equation of work and energy for the case of curvilinear motion of a particle in a plane. It states that *the change in kinetic energy of the particle between any two positions on its path is equal to the work of all forces acting upon it during the motion between these two points.*

If, instead of the tangential and normal components S and N of the resultant force, we have the rectangular components X and Y, we resolve the displacement ds into components dx and dy, as shown in Fig. 262. Then since the work of the resultant is equal to the algebraic sum of the works of its components, we conclude that

$$S\,ds = X\,dx + Y\,dy \qquad \text{(c)}$$

Substituting this in Eq. (b), we obtain

$$d\left(\frac{W}{g}\frac{v^2}{2}\right) = X\,dx + Y\,dy \qquad \text{(d)}$$

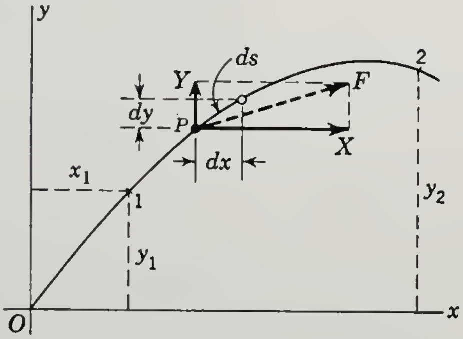

FIG. 262

Then integrating as before between limits corresponding to points 1 and 2 on the path, we obtain

$$\frac{Wv_2^2}{2g} - \frac{Wv_1^2}{2g} = \int_{x_1}^{x_2} X\,dx + \int_{y_1}^{y_2} Y\,dy \qquad (61b)$$

We see that Eq. (61*b*) differs from Eq. (61*a*) only in the manner of expressing the work done by the acting forces. Usually, the form (61*a*) will be preferable when the path of the particle is prescribed and the form (61*b*), when it is not.

The law of conservation of energy as represented by Eq. (49), page 307, for an ideal system of particles performing rectilinear motions can be used also in dealing with a system of particles that perform curvilinear motions, provided the conditions of a conservative system are satisfied.

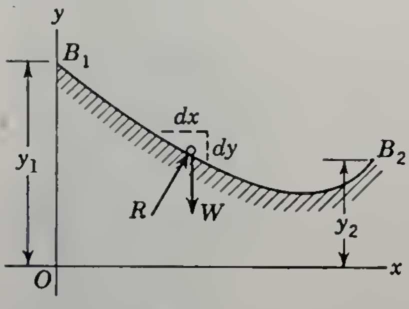

FIG. 263

EXAMPLES

1. Referring to Fig. 263, assume that a particle of weight W starts from rest at B_1 and slides under the influence of gravity along a smooth track B_1B_2 in a vertical plane. Find the velocity of the particle at B_2.

Solution. Considering the particle in any position along its path as shown, we see that the only forces acting on it are the constant vertical gravity force W and the reaction R exerted by the track. In the absence of friction, the reaction R is always normal to the direction of motion and does not produce work; hence this unknown force can be eliminated from consideration. Projecting the gravity force W onto the tangential direction, we obtain

$$S = -W \frac{dy}{ds}$$

and the energy equation (61a) becomes

$$\frac{W}{g}\frac{v_2^2}{2} = \int_{s_1}^{s_2} S\,ds = -W \int_{y_1}^{y_2} dy = W(y_1 - y_2)$$

from which

$$v_2 = \sqrt{2g(y_1 - y_2)} \tag{e}$$

We see from this example that, provided there is no friction between the particle and its track, the speed gained between B_1 and B_2 is the same as would have resulted from a free fall through the height $(y_1 - y_2)$. This conclusion will be very useful in the solution of further problems.

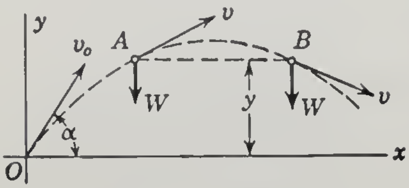

FIG. 264

2. A projectile is fired from point O with initial velocity v_0 and angle of elevation α as shown in Fig. 264. Find the velocity v of the projectile as a function of its altitude y above the ground.

Solution. Neglecting air resistance, we have to consider only the constant vertical gravity force so that $X = 0$ and $Y = -W$, and the energy equation (61b) becomes

$$\frac{W}{g}\frac{v^2}{2} - \frac{W}{g}\frac{v_0^2}{2} = \int_0^y -W\,dy = -Wy$$

from which

$$v = \sqrt{v_0^2 - 2gy} \tag{f}$$

We note that the speed of the projectile at any given elevation y is independent of the angle of elevation at which it was fired. However, in using Eq. (f), we must select only values of $y \leqq v_0^2 \sin^2\alpha/2g$, which is the maximum height to which the projectile can rise (see p. 336).

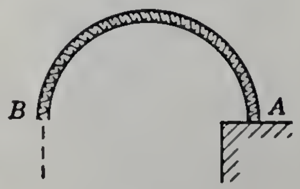

FIG. 265

3. A smooth semicircular tube AB of radius r is fixed in a vertical plane and contains a heavy flexible chain of length πr and weight $w\pi r$ as shown in Fig. 265. Assuming a slight disturbance to start the chain in motion, find the velocity v with which it will emerge from the open end B of the tube.

Solution. Neglecting friction between the links of the chain and between chain and tube, we have a system of particles with ideal constraints to which the law of conservation of energy applies. In the initial configuration, the kinetic energy of the chain is zero and its potential energy, with respect to

AB as a datum, is $w\pi r(2r/\pi)$. In the final configuration, all links of the chain have the same velocity v and its kinetic energy is $(w\pi r/g)(v^2/2)$, while the potential energy is $-w\pi r(\pi r/2)$. Thus Eq. (49), page 307, becomes

$$2wr^2 = \frac{w\pi r v^2}{2g} - \frac{w\pi^2 r^2}{2}$$

from which we find

$$v = \sqrt{2gr\left(\frac{2}{\pi} + \frac{\pi}{2}\right)} \tag{g}$$

4. A small ball of weight W starts from rest at A and moves without friction along the curved path ACB in Fig. 266. This is a parabola in a vertical plane defined by the equation

$$y = \frac{4\delta}{l^2}x^2 \tag{h}$$

Determine the reaction R exerted on the ball by the path at point C.

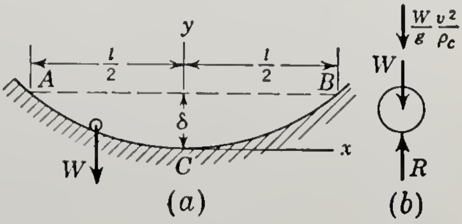

FIG. 266

Solution. Since the ball moves along AC without friction, we conclude at once that at point C it has the velocity

$$v = \sqrt{2g\delta} \tag{i}$$

directed along the horizontal tangent to the path at C.

Using D'Alembert's principle, we have the free-body diagram for the ball as shown in Fig. 266*b*, where ρ_c is the radius of curvature of the parabola at C. Writing the equation of dynamic equilibrium for the ball in this position, we obtain

$$R = W\left(1 + \frac{v^2}{g\rho_c}\right) \tag{j}$$

Substituting

$$\frac{1}{\rho_c} = \left(\frac{d^2y}{dx^2}\right)_c = \frac{8\delta}{l^2} \tag{k}$$

from Eq. (h) and $v^2 = 2g\delta$ from Eq. (i), Eq. (j) becomes

$$R = W\left(1 + \frac{16\delta^2}{l^2}\right) \tag{l}$$

It is seen that, if the ratio δ/l is large, the pressure between the ball and the support at C can be many times greater than the gravity force W.

PROBLEM SET 7.6

1. A simple pendulum of weight W and length l as shown in Fig. A is released from rest at A ($\alpha = 60°$), swings downward under the influence of

gravity, and strikes a spring of stiffness k at B. Neglecting the mass of the spring, determine the compression that it will suffer. *Ans.* $\delta = \sqrt{Wl/k}$.

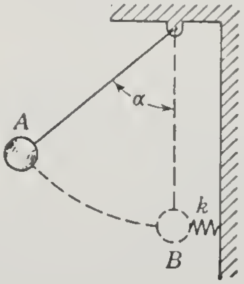

FIG. A

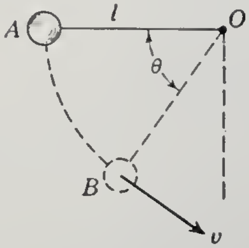

FIG. B

2. The simple pendulum in Fig. B is released from rest at A with the string horizontal and swings downward under the influence of gravity. Express the velocity v of the bob as a function of the angle θ. *Ans.* $v = \sqrt{2gl \sin \theta}$.

3. If the simple pendulum of weight W in Fig. B is released from rest in the position A, find the tension T in the string OB as a function of the angle θ. *Ans.* $T = 3W \sin \theta$.

4. If the pendulum in Fig. C is released from rest in its position of unstable equilibrium as shown, find the value of the angle φ defining the position in its downward fall at which the axial force in the rod changes from compression to tension. *Ans.* $\varphi = \arccos \frac{2}{3} = 48°11'$.

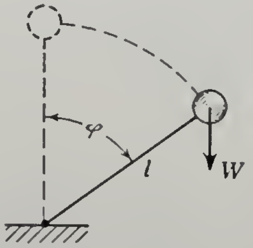

FIG. C

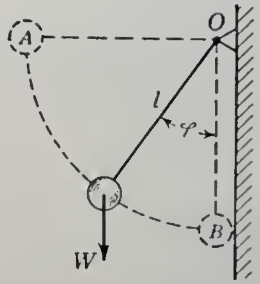

FIG. D

5. In Fig. D a simple pendulum is released from rest in the horizontal position OA and falls in a vertical plane under the influence of gravity. If it strikes a vertical wall at B and the coefficient of restitution is $e = \frac{1}{2}$, find the angle φ defining its total rebound. *Ans.* $\varphi = 41°25'$.

6. A small car of weight W starts from rest at A and rolls without friction along the loop-the-loop $ACBD$ (Fig. E). What is the least height h above the top of the loop at which the car can start without falling off the track at point B, and for such a starting position what velocity will the car have along the horizontal portion CD of the track? Neglect friction. *Ans.* $h_{\min} = r/2$; $v_c = \sqrt{5gr}$.

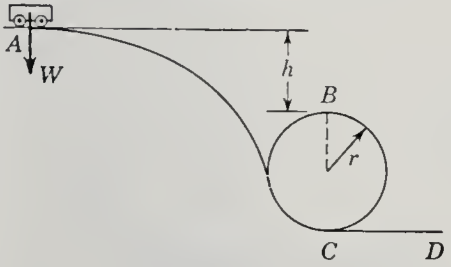

FIG. E

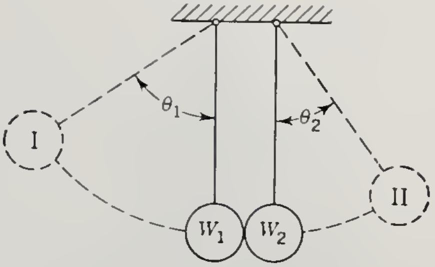

FIG. F

7. Referring to Fig. F, assume that the ball I of weight W is released from rest in the position $\theta_1 = 60°$ and swings downward to where it strikes the ball II of weight $3W$. Assuming an elastic impact, calculate the angle θ_2 through which the larger pendulum will swing after the impact. *Ans.* $\theta_2 = 28°57'$.

8. In the system shown in Fig. F, the ball I is allowed to swing downward from rest in the position defined by the angle $\theta_1 = 45°$ and to strike the ball II, which, after impact, swings upward to the position defined by the angle $\theta_2 = 30°$. If the weights of the balls are equal, find the coefficient of restitution e for the materials. *Ans.* $e = 0.35$.

9. In Fig. G a small ball of weight $W = 5$ lb starts from rest at O and rolls down the smooth track OCD under the influence of gravity. Find the reaction R exerted on the ball at C if the curve OCD is defined by the equation $y = h \sin (\pi x/l)$ and $h = l/3 = 3$ ft. *Ans.* $R = 15.95$ lb.

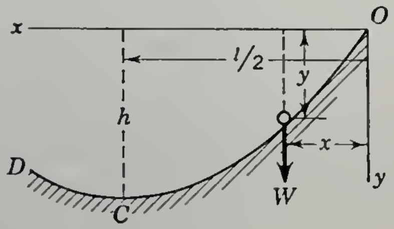

FIG. G

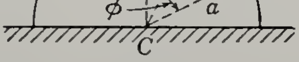

FIG. H

10. Referring to Fig. H, find the value of the angle ϕ defining the position of the point B where the particle will jump clear of the cylindrical surface after the string OA has been cut. Neglect friction. *Ans.* $\cos \phi = \frac{2}{3} \cos \alpha$.

11. A block of weight W starts from rest at A and slides in a vertical plane along the arc AB of a smooth circular cylinder of radius r (Fig. I). At point B it leaves the cylinder and travels along the dotted trajectory BC. Neglecting friction, determine the distance b defining the position of the point C at which the block strikes the horizontal plane CD. *Ans.* $b = 1.46r$.

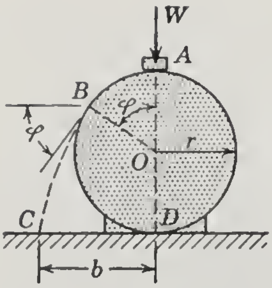

FIG. I

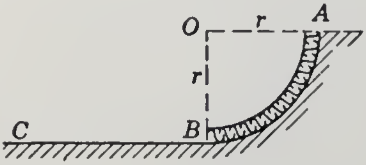

FIG. J

12. A smooth tube AB in the form of a quarter circle of mean radius r is fixed in a vertical plane and contains a flexible chain of length $\pi r/2$ and weight $w\pi r/2$ as shown in Fig. J. If released from rest in the configuration shown, find the velocity v with which the chain will move along the smooth horizontal plane BC after it emerges from the tube. *Ans.* $v = \sqrt{0.728gr}$.

13. In Fig. K, the pendulum A released from rest in the horizontal position O_1A_1 swings down and strikes the pendulum B initially at rest in the vertical position O_2B_1. If the impact is perfectly elastic, find the value of the angle φ where the axial force in OB changes from compression to tension as the ball of the lower pendulum falls along the path $B_1B_2B_3$. The weights of the bobs are $W_a = 1$ lb, $W_b = 2$ lb, and the lengths $l = 12$ in. *Ans.* $\varphi = 15°38'$.

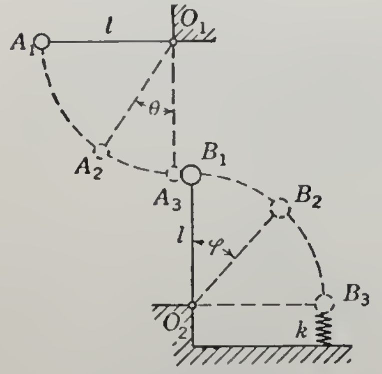

FIG. K

14. Using the data given in Prob. 13, find the angle θ defining the total rebound of the ball A after impact at A_3B_1 in Fig. K. *Ans.* $\theta = 27°16'$.

15. Using the data given in Prob. 13, find the compression δ produced in the spring at B_3 if it has stiffness $k = 6$ lb/in. and negligible mass. *Ans.* $\delta = 3.75$ in.

8

ROTATION OF A RIGID BODY ABOUT A FIXED AXIS

8.1. Kinematics of rotation. If a rigid body rotates about a fixed axis such as Oz (Fig. 267), its position at any instant can be defined by the *angle of rotation* θ, which is the angle between a certain plane OAB of the body through the axis of rotation and the immovable coordinate plane, say the xz plane through the same axis. As the body rotates, the angle of rotation θ varies with time and the rotation is completely defined if we know θ as a function of time. Thus we must have

$$\theta = f(t) \tag{62}$$

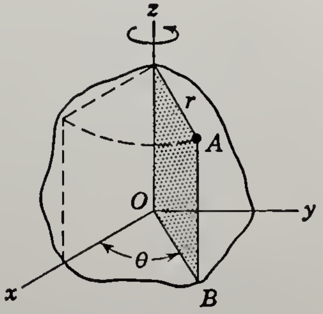

FIG. 267

The angle of rotation θ is usually expressed in units of *radians*, although it may sometimes be expressed as a number of complete revolutions, but rarely in degrees.

If the body turns so that in equal intervals of time, which can be taken as small as we like, it describes equal angles of rotation, we have the case of *uniform rotation*. The angle of rotation per second is the *angular velocity* of uniform rotation.

Generally, we have *nonuniform* rotation of a body and the angles of rotation described in equal intervals of time are not equal. If t_1 and t_2 denote two successive instants of time and θ_1 and θ_2 the corresponding angles of rotation, we define the *average angular velocity* of the rotating body during the interval of time from t_1 to t_2 by the expression

$$\omega_{av} = \frac{\theta_2 - \theta_1}{t_2 - t_1} \tag{a}$$

To define the *instantaneous angular velocity* of the body at any moment of time t, we proceed as before and consider two successive instants t and $t + \Delta t$. If $\Delta\theta$ be the angle of rotation of the body during the small interval of time Δt, the angular velocity at the instant t is

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} = \dot{\theta} \tag{63}$$

Since the angle of rotation is measured in radians and the time in seconds, angular velocity will be expressed in units of radians per second ($\sec^{-1}$). We usually consider not only magnitude but also the direction of angular velocity. The rotation indicated in Fig. 267 by an arrow and corresponding to the rotation of a right-hand-screw displacing in the positive direction of the z axis will be considered as positive, and rotation in the opposite direction as negative.

In engineering the *uniform speed* of rotating machines is usually given by the number of revolutions per minute (rpm). If n denotes this number, the angular velocity of the machine in radians per second is given by the formula

$$\omega = \frac{2\pi n}{60} = \frac{\pi n}{30} \tag{b}$$

When a body rotates about a fixed axis, each point in it, such as point A, at the distance r from the axis of rotation (Fig. 267) describes a circle of radius r. If Δs is a small displacement of the point along this circle during the interval of time Δt and if $\Delta\theta$ is the corresponding angle of rotation of the body, the velocity of point A is

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{r\,\Delta\theta}{\Delta t} = r\dot{\theta} = r\omega \tag{64}$$

The time rate of change of angular velocity of a rotating body is called the *angular acceleration* of the body. If at the instants t_1 and t_2 the body has angular velocities ω_1 and ω_2, respectively, then the *average angular acceleration* during the interval of time from t_1 to t_2 is defined by the expression

$$\alpha_{\text{av}} = \frac{\omega_2 - \omega_1}{t_2 - t_1} \tag{c}$$

To define *instantaneous angular acceleration*, we consider a small interval of time from t to $t + \Delta t$. If $\Delta\omega$ is the corresponding change in angular velocity, then the angular acceleration of the rotating body at the instant t is

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \dot{\omega} = \ddot{\theta} \tag{65}$$

Considering the circular motion of a point A of the rotating body at the distance r from the axis of rotation and proceeding as already explained in Art. 7.1, we can resolve the resultant acceleration of point A into tangential and radial components as follows:

$$a_t = \frac{dv}{dt} = r\dot{\omega} = r\ddot{\theta} \qquad a_n = \frac{v^2}{r} = \omega^2 r = \dot{\theta}^2 r \tag{66}$$

We see that, if we know the angular velocity $\dot{\theta}$ and angular acceleration $\ddot{\theta}$ of a rotating body, both components of the resultant acceleration of any point in the body at a distance r from the axis of rotation can be easily calculated from Eqs. (66).

EXAMPLES

1. The angle of rotation of a body is given as a function of time by the equation

$$\theta = \theta_0 + bt + ct^2$$

Find general expressions for the angular velocity and angular acceleration of the body. Determine also the values of the constants b and c if the initial angular velocity is 2π sec^{-1} and 1 sec later it is 4π sec^{-1}.

Solution. By using Eq. (63) we find for the angular velocity

$$\omega = \dot{\theta} = b + 2ct \tag{d}$$

Putting the simultaneous values $\omega = 2\pi$ sec^{-1} and $t = 0$ into this equation, we find $b = 2\pi$ sec^{-1}. Thus the constant b represents the initial angular velocity. Putting the simultaneous values $\omega = 4\pi$ sec^{-1} and $t = 1$ sec in Eq. (d), we find $c = \pi$ sec.$^{-2}$ The angular acceleration, from Eq. (65), is

$$\alpha = \ddot{\theta} = 2c = 2\pi \text{ sec}^{-2}$$

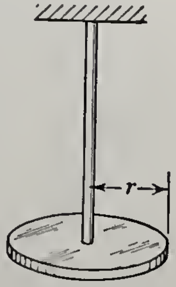

FIG. 268

We see that the given equation represents rotation of a rigid body with constant angular acceleration.

2. The circular disk suspended by a slender rod, as shown in Fig. 268, performs torsional oscillations such that its angle of rotation measured from the equilibrium position is given by the equation

$$\theta = \theta_0 \cos pt$$

Find general expressions for the angular velocity and angular acceleration of the disk and also general expressions for the radial and tangential components of the acceleration of a point on the periphery of the disk. Determine the maximum values of these components if the disk makes 2 oscillations/sec, has the radius $r = 5$ in. and maximum angular velocity 2π sec^{-1}.

Solution. Using Eqs. (63) and (65), we obtain

$$\dot{\theta} = -\theta_0 p \sin pt \tag{e}$$
$$\ddot{\theta} = -\theta_0 p^2 \cos pt \tag{f}$$

The components of acceleration of a point on the periphery of the disk, from Eqs. (66), are

$$a_t = r\ddot{\theta} = -r\theta_0 p^2 \cos pt \tag{g}$$
$$a_n = \dot{\theta}^2 r = \theta_0^2 p^2 r \sin^2 pt \tag{h}$$

From the condition that the disk has 2 oscillations/sec, we conclude that the period

$$\tau = \frac{2\pi}{p} = \frac{1}{2}\text{ sec}$$

from which $p = 4\pi \text{ sec}^{-1}$. The maximum angular velocity, from Eq. (e), then is $\theta_0 p = 2\pi \text{ sec}^{-1}$. Hence $\theta_0 = \frac{1}{2}$. Substituting the values of θ_0, p, and r into Eqs. (g) and (h), we obtain the following maximum values of the components of acceleration of a point on the periphery of the disk:

$$(a_t)_{\max} = \theta_0 r p^2 = 40\pi^2 \text{ in./sec}^2$$
$$(a_n)_{\max} = \theta_0^2 p^2 r = 20\pi^2 \text{ in./sec}^2$$

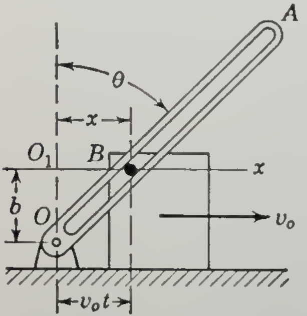

FIG. 269

3. A slotted bar OA is hinged at O and supported by a pin B which can slide freely along the slot as the block to which the pin is attached slides along a horizontal plane (Fig. 269). If at the initial moment $t = 0$, the bar OA is in a vertical position and the block moves to the right with uniform velocity v_0, as shown, find general expressions for angular velocity $\dot{\theta}$ and angular acceleration $\ddot{\theta}$ of the bar OA.

Solution. Considering the system at the instant t when the block has displacement $x = v_0 t$ and the bar has corresponding angular displacement θ, we see from the geometry of the figure that

$$\theta = \arctan \frac{v_0 t}{b} \tag{i}$$

Successive differentiations of this expression with respect to time give

$$\dot{\theta} = \frac{v_0 b}{b^2 + v_0^2 t^2} \qquad \text{and} \qquad \ddot{\theta} = -\frac{2v_0^3 bt}{(b^2 + v_0^2 t^2)^2}$$

We note from the second of these expressions that the initial angular velocity $\dot{\theta}_0 = v_0/b$ when $t = 0$ is a maximum. The student will find it instructive to make graphs of the expressions for $\dot{\theta}$ and $\ddot{\theta}$ as given above.

PROBLEM SET 8.1

1. The armature of an electric motor has angular speed $n = 1{,}800$ rpm at the instant when the power is cut off. (*a*) If it comes to rest in 6 sec, calculate the angular deceleration α assuming that it is constant. (*b*) How many complete revolutions does the armature make during this period? *Ans.* (*a*) $\alpha = 10\pi \text{ sec}^{-2}$; (*b*) 90 revolutions.

2. Referring to Fig. 268, compute the maximum tangential and normal components of acceleration of a point on the periphery of the disk if it oscillates with an amplitude $\theta_0 = 1$ radian instead of $\frac{1}{2}$ radian as in Example 2 above. *Ans.* $(a_t)_{\max} = 80\pi^2$ in./sec^2; $(a_n)_{\max} = 80\pi^2$ in./sec^2.

3. Considering the system in Fig. 269, determine the value of θ for which the negative angular acceleration $\ddot{\theta}$ of the bar OA has its maximum value. *Ans.* $\theta = 30°$.

4. A slender but rigid semicircular wire of radius r is supported in its own vertical plane by a hinge at O and a smooth peg A as shown in Fig. A. If the peg starts from O and moves with constant speed v_0 along the horizontal x axis, find the angular velocity $\dot{\theta}$ of the wire at the instant when $\theta = 60°$. *Ans.* $\dot{\theta} = v_0/r$.

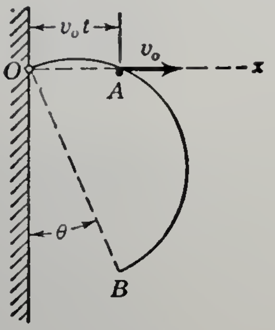

FIG. A

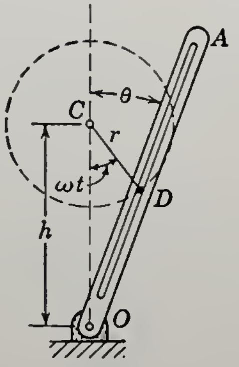

FIG. B

***5.** A slotted bar OA hinged at O is made to oscillate about its vertical position by means of a crankpin D rotating with uniform angular velocity ω as shown in Fig. B. Derive a general expression for the angular velocity $\dot{\theta}$ of the bar, counting time from the instant when the pin D is in its lowest position and plot one cycle of the curve $\dot{\theta} = f(t)$ for the particular case where $h = 2r$.

8.2. Equation of motion for a rigid body rotating about a fixed axis. In deriving the equation of motion for a rigid body rotating about a fixed axis, we shall use D'Alembert's principle and the notion of dynamic equilibrium. We consider the body as a system of rigidly connected particles and apply to each particle its inertia force. These inertia forces together with all external forces acting on the body constitute a system of forces in equilibrium. The internal forces (actions and reactions between the various particles) need not be considered, since they always occur in pairs of equal and opposite collinear forces and are of themselves balanced. Thus the algebraic sum of the moments of external forces and inertia forces, with respect to the axis of rotation, is zero.

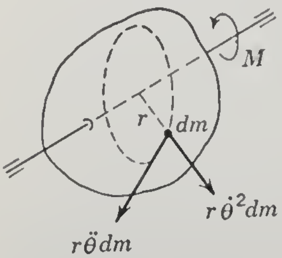

FIG. 270

In calculating the moments of inertia forces with respect to the axis of rotation, we have to consider only the tangential components of these forces, since the moments of the radial components, intersecting the axis of rotation, vanish. Denoting by *dm* the mass of a particle of the body at a distance r from the axis of rotation (Fig. 270) and using the first of Eqs. (66), we find that the tangential component of its inertia force is $-r\ddot{\theta}\,dm$ and the moment of this force with respect to the axis of rotation is $-r^2\ddot{\theta}\,dm$. Summing up the moments of these inertia forces for all particles of the body and denoting by M the resultant moment with respect to the axis of rotation of all external forces acting on the body, we may write the following moment equation of dynamic equilibrium:

$$-\int r^2\ddot{\theta}\,dm + M = 0 \tag{a}$$

where the integration must extend over the entire volume of the body. The angular acceleration $\ddot{\theta}$, being the same for all particles of the rotating body, can be taken out from under the integral sign. Then introducing the notation

$$I = \int r^2\,dm \tag{b}$$

Eq. (a) can be written in the form

$$I\ddot{\theta} = M \tag{67}$$

This is the equation of motion for a rigid body rotating about a fixed axis. The quantity I is the *moment of inertia*[1] of the body with respect to the axis of rotation, and M is the resultant moment of all external forces with respect to the same axis. We see that Eq. (67) has the same form as the equation of motion [Eq. (34)] for the rectilinear motion of a particle.

As in the case of rectilinear motion of a particle, we conclude that two kinds of problems can be solved by using Eq. (67). In the first case, the motion is known, i.e., the angle of rotation θ is given as a certain function of time t, and it is required to find the resultant moment M that produces this motion. In the second case the moment M of external forces with respect to the axis of rotation is given and it is required to find the motion that this moment produces, i.e., to find an expression for the angle of rotation θ as a function of time. Problems of the first kind are simple. It is only necessary to substitute in Eq. (67), for θ, the given function of time. Then, by differentiation, we obtain the corresponding expression for the resultant moment M. We shall now consider several examples of this kind.

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[1] For further discussion and methods of calculating moments of inertia, see Appendix II, p. A,17.

EXAMPLES

1. A right circular disk which weighs 300 lb and is 30 in. in diameter is free to rotate about its geometric axis and is constantly accelerated from rest to 300 rpm in 20 sec. Determine the constant torque M required to produce this acceleration.

Solution. The constant angular acceleration of the disk is

$$\ddot{\theta} = \frac{300 \times 2\pi}{60 \times 20} = \frac{\pi}{2} = 1.571 \text{ sec}^{-2}$$

The moment of inertia of the disk (see page A,18) is

$$I = \frac{300}{386}\,\frac{15^2}{2} = 87.5 \text{ lb-sec}^2\text{-in.}$$

Substituting these values into Eq. (67) gives

$$M = 87.5 \times 1.571 = 137.3 \text{ in.-lb} \tag{c}$$

This is the external torque required to produce the given motion.

2. A slender prismatic bar OA of weight W and length l (Fig. 271) can rotate freely about the fixed axis through O normal to the plane of the figure. By means of a horizontal bar AB and a crankshaft with crank radius r and crankpin D freely sliding in the slot DC, a simple harmonic motion is given to the end A of the bar OA. Determine the force S in the bar AB, assuming that its mass is negligible.

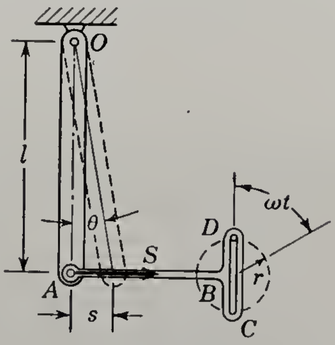

FIG. 271

Solution. Denoting by ω the uniform angular speed of the crankshaft and counting time from the instant when the crank is vertical as shown, the motion of the point A in the horizontal direction is[1]

$$s = r \sin \omega t \tag{d}$$

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[1] It is assumed that the crank radius r is small compared with the length l of the bar OA, and vertical displacement of A is neglected.

Then for small angles of oscillation of the bar OA we have

$$\theta \approx \sin\theta = \frac{s}{l} = \frac{r}{l}\sin\omega t \tag{e}$$

from which

$$\ddot{\theta} = -\frac{r}{l}\omega^2 \sin\omega t \tag{f}$$

The moment of external forces acting on the bar OA with respect to the axis of rotation consists of two parts: (1) the moment of the force S, representing the action of the bar AB on the bar OA, and (2) the moment of the gravity force W, which can be assumed applied at the center of gravity of the bar OA. Then

$$M = Sl - W\frac{l}{2}\sin\theta$$

which, by using expression (e), becomes

$$M = Sl - \frac{Wr}{2}\sin\omega t \tag{g}$$

Substituting Eqs. (f) and (g) into the general equation of motion [Eq. (67)] we obtain

$$-I_0\frac{r}{l}\omega^2\sin\omega t = Sl - \frac{Wr}{2}\sin\omega t$$

from which

$$S = \left(\frac{Wr}{2l} - I_0\frac{r\omega^2}{l^2}\right)\sin\omega t \tag{h}$$

In calculating the moment of inertia I_0 of the bar OA with respect to its axis of rotation, we assume that its cross-sectional dimensions are small compared with its length l, so that (see page A,22)

$$I_0 = \frac{W}{g}\frac{l^2}{12} + \frac{W}{g}\frac{l^2}{4} = \frac{W}{g}\frac{l^2}{3}$$

Substituting this into Eq. (h), we obtain

$$S = \frac{Wr}{l}\left(\frac{1}{2} - \frac{\omega^2 l}{3g}\right)\sin\omega t \tag{i}$$

It is seen that the force S in the bar AB depends on the angular speed ω of the crankshaft. For very small values of ω the second term in the parentheses of expression (i) can be neglected and we obtain

$$S = \frac{Wr}{2l}\sin\omega t = \frac{Ws}{2l} \tag{i'}$$

This is the magnitude of the force S in the bar AB which would be necessary to hold the bar OA in equilibrium with its lower end A displaced a distance s to the right. It is a tensile force for displacements as shown in the figure and a compressive force for displacements of A in the opposite direction.

For larger values of ω the force S as given by Eq. (i) decreases and becomes zero when $\omega^2 l/3g = \frac{1}{2}$, that is, when

$$\omega = \sqrt{\frac{3g}{2l}} \tag{j}$$

The corresponding time for one revolution of the crank is obviously

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2l}{3g}} \tag{k}$$

We shall see later that this is the period of free oscillations of the bar OA, vertically suspended at O. Thus, if the frequency of forced oscillations produced by the rotating crankshaft coincides with the natural frequency of the bar OA as a pendulum, no action from the bar AB is required to maintain these oscillations.

When ω becomes greater than the value given by Eq. (j) above, expression (i) changes sign and we have compression in the bar AB for the conditions as shown in the figure.

3. Find the pulsating torque acting in a shaft if the rotation of a flywheel of weight $W = 2{,}000$ lb at the end of the shaft is defined by the equation

$$\theta = \omega t + \alpha \sin \omega t$$

The average angular speed of the flywheel is 180 rpm, and the maximum deviation from this average angular speed is 0.01ω. The radius of gyration of the flywheel with respect to the axis of rotation is $i = 50$ in.

Solution. From the numerical data given above we have

$$\omega = 180\,\frac{2\pi}{60} = 6\pi \text{ sec}^{-1} \qquad \alpha = 0.01 \qquad I = \frac{2000}{386}\,50^2 \text{ lb-sec}^2\text{-in.}$$

Substituting these values in Eq. (67), we obtain

$$M = I\ddot{\theta} = -I\alpha\omega^2 \sin \omega t = 46{,}000 \sin \omega t \text{ in.-lb}$$

PROBLEM SET 8.2

1. A flywheel having moment of inertia $I = 600$ lb-sec^2-in. with respect to its axis of rotation and making 100 rpm, if left alone, comes to rest with constant angular deceleration in 52 sec, owing to friction in the bearings.

Determine the friction couple that produces this angular deceleration. *Ans.* $M = 120.7$ in.-lb.

2. A homogeneous sphere, of radius $a = 10$ in. and weight $W = 100$ lb, can rotate freely about a diameter. If it starts from rest and gains, with constant angular acceleration, an angular speed $n = 180$ rpm in 12 revolutions, find the acting moment M. *Ans.* $M = 24.4$ in.-lb.

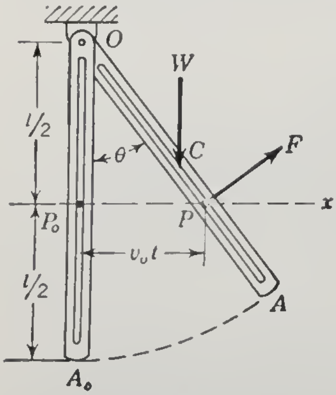


FIG. A

3. In Fig. A the pin P, sliding without friction in the slotted bar OA, moves with constant velocity v_0 along the horizontal x axis, causing the bar OA to rotate in the vertical plane about the fixed axis through O. For the particular instant when $\theta = 30°$, find the force F that the pin exerts on the bar. The following numerical data are given: $W = 5$ lb, $l = 20$ in., $v_0 = 30$ ips. *Ans.* $F = 1.29$ lb.

4. Referring again to Fig. A, assume that the pin P performs simple harmonic motion along the horizontal x axis as represented by the displacement-time equation $x = \frac{1}{2}l \sin \omega t$, where $\omega = 6 \text{ sec}^{-1}$. Under such conditions, find the magnitude of the force F exerted on the bar OA by the pin when it is in either of its extreme positions, that is, when $x = \pm l/2$. Use all other data from Prob. 3 above. *Ans.* $F = 0.30$ lb.

5. Referring again to Fig. A, find the period $\tau = 2\pi/\omega$ of the simple harmonic motion of the pin P, as described in Prob. 4, for which the force F exerted on the bar OA in an extreme position ($x = \pm l/2$) will be zero. *Ans.* $\tau = 0.978$ sec.

8.3. Rotation under the action of a constant moment.

We come now to the second kind of dynamics problems mentioned in Art. 8.2, where the active forces are known and the motion that they produce is required. That is, by integration of Eq. (67), we must determine the angle of rotation θ as a function of time. In the particular case where the moment of all external forces acting on the body with respect to the axis of rotation is constant, the necessary integration of Eq. (67) can be made in a very simple manner. Assuming that M is constant, we obtain by the first integration

$$I\dot{\theta} = Mt + C_1 \tag{a}$$

The constant of integration C_1 is found from the initial conditions of the motion. If, when $t = 0$, the angular velocity of the rotating body is $\dot{\theta}_0$, we obtain from Eq. (a), $C_1 = I\dot{\theta}_0$ and the general expression for the angular velocity becomes

$$\dot{\theta} = \frac{M}{I}t + \dot{\theta}_0 \tag{68}$$

Integrating Eq. (68), we obtain

$$\theta = \frac{Mt^2}{2I} + \dot{\theta}_0 t + C_2 \tag{b}$$

The constant of integration C_2 again must be determined from the initial conditions. If at the initial moment $t = 0$ the angle of rotation is θ_0, we obtain from Eq. (b), $C_2 = \theta_0$ and the general expression for the angle of rotation becomes

$$\theta = \frac{Mt^2}{2I} + \dot{\theta}_0 t + \theta_0 \tag{69}$$

Various simple problems of rotation of a rigid body about a fixed axis can be solved by using Eqs. (68) and (69) in the same manner as was done before in discussing the rectilinear motion of a particle under the action of a constant force (see Art. 6.3).

Writing Eq. (67), governing the rotation of a rigid body about a fixed axis, in the form

$$M - I\ddot{\theta} = 0 \tag{c}$$

we conclude that if we apply a couple $-I\ddot{\theta}$ to the rotating body in addition to the moment M of the active forces, we obtain a condition of *dynamic equilibrium.* This couple $-I\ddot{\theta}$ which equilibrates the moment M of the active forces in called the *inertia couple.* We see that this inertia couple can be used in problems of rotation in the same way that we have previously used inertia force $-m\ddot{x}$ in problems of rectilinear motion.

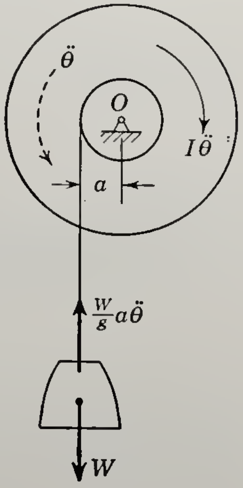


FIG. 272

Consider, for example, the circular disk in Fig. 272 which is made to rotate about its geometric axis through O by a falling weight W attached to a flexible but inextensible string wound on a shaft of radius a. Denoting by $\ddot{\theta}$ the angular acceleration of the disk, we see that the falling weight W must have vertical acceleration $a\ddot{\theta}$ if the string is inextensible. Applying the corresponding inertia force $-Wa\ddot{\theta}/g$ to the falling weight and the inertia couple $-I\ddot{\theta}$ to the disk as shown in the figure, we establish dynamic equilibrium. Then equating to zero the algebraic sum of moments, with respect to point O, of all forces (including the inertia force and inertia couple), we obtain

$$Wa - \frac{W}{g}a^2\ddot{\theta} - I\ddot{\theta} = 0$$

which may be written

$$\left(I + \frac{W}{g} a^2\right) \ddot{\theta} = Wa \tag{d}$$

where I is the moment of inertia of the disk with respect to the axis of rotation.

Equation (d) can be obtained in another way by using the *principle of virtual work* (see page 218) and equating to zero the algebraic sum of the works of all forces on any virtual displacement of the system. Such a virtual displacement, in the case of the system shown in Fig. 272, will be an infinitesimal rotation $\delta\theta$ of the pulley around the center O. Assuming this rotation to be in the counterclockwise direction, the corresponding equation of virtual work becomes

$$\left(W - \frac{W}{g} a\ddot{\theta}\right) a \,\delta\theta - I\ddot{\theta}\,\delta\theta = 0$$

from which

$$\left(I + \frac{W}{g} a^2\right) \ddot{\theta} = Wa$$

as before.

It will be noted that Eq. (d) has the same general form as Eq. (67) if we consider the quantity

$$I_s = I + \frac{W}{g} a^2 \tag{e}$$

as an *equivalent moment of inertia* of the system in Fig. 272. That is, the system behaves as a body of fictitious moment of inertia I_s rotating about a fixed axis under the action of a constant external moment Wa, and Eqs. (68) and (69) may be used directly to find angular velocity and displacement as functions of time.

EXAMPLES

1. A rotor having moment of inertia I with respect to its axis of rotation starts from rest and is brought up to an angular velocity $\dot{\theta}$ by a constant driving torque M. Develop formulas for the time t required to reach this speed and for the total angular displacement.

Solution. The constant torque M produces constant angular acceleration $\ddot{\theta} = M/I$. Using Eq. (68) and noting that $\dot{\theta}_0 = 0$, we find

$$t = \frac{I}{M}\dot{\theta} \tag{f}$$

Using this value of t in Eq. (69) and noting also that $\theta_0 = 0$, we find

$$\theta = \frac{I\dot{\theta}^2}{2M} \tag{g}$$

Equations (f) and (g) represent the required formulas.

Considering $\dot{\theta}$ as an initial angular velocity, Eqs. (f) and (g) may also be used in the case of a rotor coming to rest under the action of a constant decelerating torque M such as might be produced by constant bearing friction.

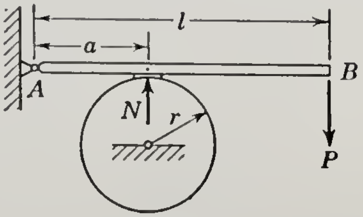


FIG. 273

2. A solid circular rotor of radius r and weight W which can rotate about its geometric axis is braked by the device shown in Fig. 273. If the coefficient of friction between rotor and brake shoe is μ and the rotor has an initial angular velocity ω before the brake is applied, how many revolutions will it make before coming to rest? Neglect friction in the bearings.

Solution. Considering the static equilibrium of the bar AB, we find that the normal force N acting between the rim of the rotor and the brake shoe is

$$N = P\frac{l}{a}$$

and hence the tangential friction force acting on the rim of the rotor is

$$\mu N = \mu P\frac{l}{a}$$

The decelerating moment of this force with respect to the axis of rotation of the rotor is

$$M = \mu Pr\frac{l}{a}$$

Then using Eq. (g) from Example 1, we find

$$\theta = \frac{W\omega^2 ra}{4\mu gPl} \tag{h}$$

The corresponding number of complete revolutions of the rotor is

$$\frac{\theta}{2\pi} = \frac{W\omega^2 ra}{8\pi\mu gPl} \tag{h'}$$

Taking, for example, $W = 386$ lb, $n = 1{,}800$ rpm, $r = 10$ in., $a = 15$ in., $l = 36$ in., $\mu = \frac{1}{3}$, and $P = 150$ lb, Eq. (h′) gives 118 revolutions required to stop the rotor.

3. Two rotors having weights W_1 and W_2 and moments of inertia I_1 and I_2 are mounted in bearings and geared together as shown in Fig. 274. A flexible cord wound around the circumference of one of the rotors carries a weight Q at its free end. Neglecting friction, find the acceleration of the weight Q if it is allowed to fall vertically downward.

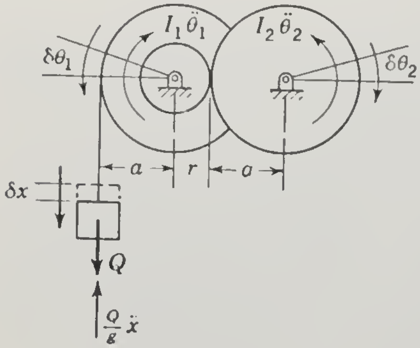


FIG. 274

Solution. Let $\ddot{x}$ be the acceleration of the falling weight; $\ddot{\theta}_1$ and $\ddot{\theta}_2$, the corresponding angular accelerations of the two rotors. Then introducing the inertia force $-(Q/g)\ddot{x}$ and the inertia couples $-I_1\ddot{\theta}_1$ and $-I_2\ddot{\theta}_2$ as shown in the figure, we put the system in dynamic equilibrium.

We now give a virtual displacement δx to the weight Q and let $\delta\theta_1$ and $\delta\theta_2$ be the corresponding angular displacements of the rotors directed as shown. Then the equation of virtual work for the system becomes

$$\left(Q - \frac{Q}{g}\ddot{x}\right)\delta x - I_1\ddot{\theta}_1\,\delta\theta_1 - I_2\ddot{\theta}_2\,\delta\theta_2 = 0 \tag{i}$$

Introducing the notations,

$$I_1 = \frac{W_1}{g} i_1^2 \qquad \text{and} \qquad I_2 = \frac{W_2}{g} i_2^2$$

and noting that

$$\ddot{\theta}_1 = \frac{\ddot{x}}{a} \qquad \ddot{\theta}_2 = \frac{\ddot{x}}{a}\frac{r}{a}$$

while

$$\delta\theta_1 = \frac{\delta x}{a} \qquad \delta\theta_2 = \frac{\delta x}{a}\frac{r}{a}$$

Eq. (i) becomes

$$\left(Q - \frac{Q}{g}\ddot{x}\right)\delta x - \frac{W_1}{g} i_1^2 \frac{\ddot{x}}{a}\frac{\delta x}{a} - \frac{W_2}{g} i_2^2 \frac{\ddot{x}}{a}\frac{r}{a}\frac{\delta x}{a}\frac{r}{a} = 0$$

Upon canceling δx, we obtain

$$\ddot{x} = g\,\frac{Q}{Q + W_1\dfrac{i_1^2}{a^2} + W_2\dfrac{i_2^2}{a^2}\dfrac{r^2}{a^2}} \tag{j}$$

Taking the following numerical data: $Q = 5$ lb, $W_1 = 20$ lb, $W_2 = 40$ lb, $i_1 = a/2$, $i_2 = a/\sqrt{2}$, and $r = a/2$, Eq. (j) gives $\ddot{x} = g/3 = 10.73$ ft/sec².

PROBLEM SET 8.3

1. A shaft of radius r rotates with constant angular speed ω in bearings for which the coefficient of friction is μ. Through what angle θ will it rotate after the driving torque is removed? *Ans.* $\theta = r\omega^2/4\mu g$.

2. The wheel of a small gyroscope is set spinning by pulling on a string wound around the shaft. Its moment of inertia is $I = 0.05$ lb-sec^2-in., and the diameter of the shaft on which the string is wound is $\frac{1}{2}$ in. If 30 in. of string is pulled off with a constant force of 12 lb, what angular velocity will be imparted to the wheel? *Ans.* $\omega = 120$ radians/sec.

3. A right circular drum of radius r and weight W rotating at 600 rpm is braked by the device shown in Fig. A. Develop a formula for the time t required to bring the drum to rest if the coefficient of friction between the drum and breaking bar is μ. The following data are given: $l = 4$ ft, $a = 2.5$ ft, $r = 15$ in., $\mu = \frac{1}{4}$, $W = 400$ lb, and $P = 100$ lb. *Ans.* $t = 12.2$ sec.

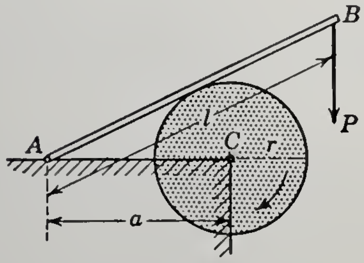


FIG. A

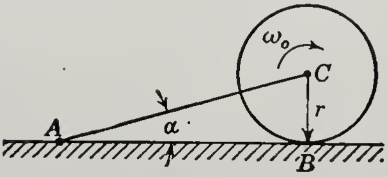


FIG. B

4. A solid right circular rotor of radius r and weight W tied to a horizontal plane by a rod AC has initial angular velocity ω_0 as shown in Fig. B. If the rotor is suddenly allowed to rest its full weight on the plane, what time t will elapse before it comes to rest? The coefficient of friction at B is μ. Numerical data are given as follows: $\omega_0 = 20\pi$ sec^{-1}, $r = 1$ ft, $\mu = \frac{1}{4}$, $\alpha = 15°$. *Ans.* $t = 3.64$ sec.

5. Solve Prob. 4 for the case where AB is a vertical wall instead of a horizontal floor. Use numerical data as above. *Ans.* $t = 13.6$ sec.

6. A solid right circular drum of radius $r = 1$ ft. and weight $W = 32.2$ lb is free to rotate about its geometric axis as shown in Fig. C. Wound around the circumference of the drum is a flexible cord carrying at its free end a weight $Q = 10$ lb. If the weight Q is released from rest, (a) find the time t required for it to fall through the height $h = 10$ ft. (b) With what velocity v will it strike the floor? *Ans.* (a) $t = 1.275$ sec; (b) $v = 15.7$ fps.

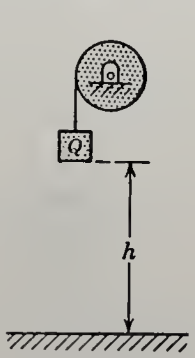


FIG. C

7. The rotor and shaft in Fig. D weigh 500 lb, and the radius of gyration with respect to the axis of rotation is 10 in. Calculate the acceleration $\ddot{x}$ of the falling weight $W = 100$ lb if the shaft radius $a = 5$ in. *Ans.* $\ddot{x} = g/21$.

8. Using the same numerical data as in Prob. 7, calculate the torque M that must be applied to the shaft of the rotor to produce an upward acceleration $g/3$ of the attached weight W. *Ans.* $M = 4{,}000$ in.-lb.

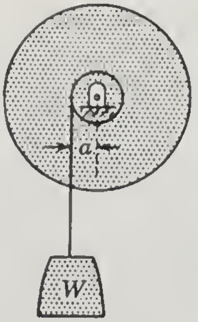


FIG. D

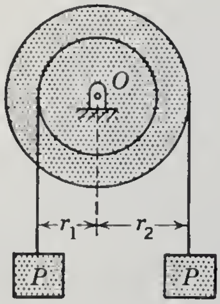


FIG. E

9. The two-step pulley in Fig. E has weight $W = 400$ lb. and radius of gyration $i_0 = 7.07$ in. Develop a formula for the downward acceleration of the falling weight P on the right if $P = 50$ lb, $r_1 = 10$ in., and $r_2 = 15$ in. *Ans.* $\ddot{x} = 39.9$ in./sec^2.

10. Figure F represents a system consisting of a block A of weight $W_a = 866$ lb, connected to a block B of weight $W_b = 644$ lb by a flexible cord which runs over a pulley of radius $r = 12$ in. and weight $W_c = 322$ lb. The block A slides along a plane inclined to the horizontal by the angle $\alpha = 30°$ and for which the coefficient of friction is $\mu = 0.1$. Friction on the axle of the pulley is assumed to supply a constant resisting torque of 10 ft-lb. Assuming that no slipping occurs between the cord and the pulley, determine (*a*) the acceleration of the blocks and (*b*) the maximum tensile force in the cord. *Ans.* (*a*) $a = 2.43$ ft/sec^2; (*b*) $S_{\max} = 595$ lb.

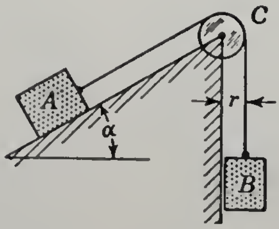


FIG. F

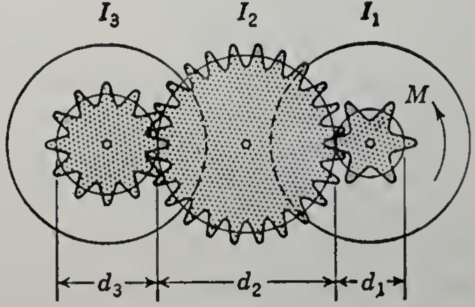


FIG. G

11. Figure G represents a system of two rotors of moments of inertia I_1 and I_3 which can rotate about their parallel geometric axes and are connected by an idling gear of moment of inertia I_2 as shown. If a constant driving torque $M = 100$ ft-lb is applied to the first rotor I_1, what angular acceleration $\ddot{\theta}_3$ of the rotor I_3 will be produced? Assume that the mechanical efficiency of the system is 1 and that the pitch diameters of the three gears are in the ratios $d_1:d_2:d_3 = 1:4:2$. The moments of inertia are $I_1 = 200$ lb-sec^2-in., $I_2 = 80$ lb-sec^2-in., $I_3 = 300$ lb-sec^2-in. *Ans.* $\ddot{\theta}_3 = 2.145$ sec^{-2}.

12. A homogeneous sphere of radius $r = 1$ ft and weight $W = 40$ lb rotates about a vertical diameter with initial angular velocity $\omega = 20\pi$ sec^{-1} and is braked by the device shown in Fig. H. Neglecting friction in the bearings, determine the time required for the brake to bring the sphere to rest if the coefficient of friction at B is $\mu = \frac{1}{3}$ and $P = 10$ lb. *Ans.* $t = 9.40$ sec.

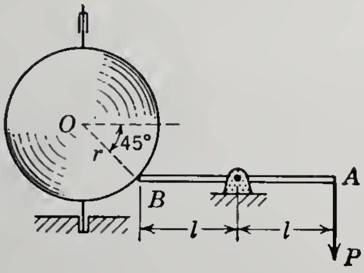


FIG. H

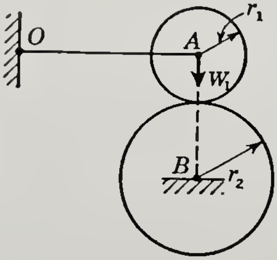


FIG. I

13. A circular rotor of weight $W_1 = 30$ lb and radius $r_1 = 10$ in. rotating with an initial angular velocity $\omega = 100$ sec^{-1} (Fig. I) is suddenly allowed to rest its full weight against another rotor of weight $W_2 = 60$ lb and radius $r_2 = 15$ in. which is free to rotate about its geometric axis parallel to that of the first, but is initially at rest. At first there will be slipping at the point of contact, but gradually the first rotor will be decelerated and the second accelerated until no further slipping occurs. Find the time t that elapses before this condition prevails. Neglect friction in the bearings, and assume that the coefficient of friction between the surfaces of rotors is $\mu = 0.25$. *Ans.* $t = 3.46$ sec.

8.4. Torsional vibration. Let us consider the device shown in Fig. 275, which consists of a circular disk attached to one end of a shaft, the axis Oz of which passes through the center of the disk and is perpendicular to its plane. When the upper end of the shaft is held rigidly as shown, the device is called a *torsional pendulum.* If by some external moment the disk be turned about the axis Oz so that the shaft is twisted, and then the external moment is removed, the disk will perform an oscillatory rotation about this axis called *torsional vibration.* In discussing these vibrations, we shall neglect the mass of the shaft and also any damping influence such as air resistance or imperfect elasticity of the shaft. Then denoting by I the moment of inertia of the disk with respect to the axis of rotation Oz and by θ the angle of rotation measured from the equilibrium position as shown in the figure, the differential equation of motion is

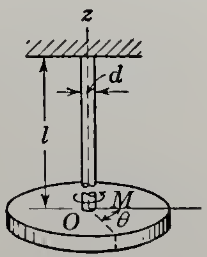


FIG. 275

$$I\ddot{\theta} = M \tag{a}$$

where M is the moment, with respect to the axis Oz, of all external forces acting on the disk. Since the gravity force coincides with this axis, it does not contribute to the moment M and it is necessary to consider only the elastic reaction exerted by the shaft on the disk. Within the elastic limit of the material of the shaft, this torque is proportional to the angle of twist θ, and we can write

$$M = -k_t\theta \tag{b}$$

where k_t is called the *torsional spring constant* of the shaft and represents the torque required to produce an angle of twist equal to 1 radian. The minus sign indicates that, when a rotation of the disk is produced in one direction as shown in the figure, the twisted shaft exerts on the disk an elastic reactive couple M in the opposite direction. In the case of a circular cylindrical shaft the relation between the torque and the angle of twist is given by the formula

$$\theta = \frac{Ml}{GJ} \tag{c}$$

in which l is the length of the shaft, G the modulus of elasticity of the material in shear, and $J = \pi d^4/32$ is the *polar moment of inertia*[1] of the circular cross section of the shaft of diameter d. Comparing expressions (b) and (c), we conclude that in the case of a shaft of circular cross section

$$k_t = \frac{GJ}{l} \tag{d}$$

Substituting the value of M from Eq. (b) into Eq. (a), we obtain

$$I\ddot{\theta} = -k_t\theta$$

or using the notation

$$p^2 = \frac{k_t}{I} = \frac{GJ}{Il} \tag{e}$$

we obtain

$$\ddot{\theta} + p^2\theta = 0 \tag{70}$$

This differential equation is seen to have the same form as Eq. (40) for the case of free vibrations of a weight on a spring as dicussed in Art. 6.5, and as before, its general solution can be written in the form

$$\theta = C_1 \cos pt + C_2 \sin pt \tag{71}$$

---

[1] Polar moment of inertia J of area (dimension, length^4) should not be confused with moment of inertia I of mass (dimension, force × time^2 × length). See Appendix I, p. A,4.

The constants of integration are found in the same way as before, from considerations of the initial conditions of motion, and we finally obtain

$$\theta = \theta_0 \cos pt + \frac{\dot{\theta}_0}{p} \sin pt \tag{71a}$$

in which θ_0 is the initial angle of rotation and $\dot{\theta}_0$, the initial angular velocity of the disk.

In the particular case where the disk is given an initial angle of rotation θ_0 and released without initial angular velocity, we obtain from Eq. (71*a*)

$$\theta = \theta_0 \cos pt \tag{71b}$$

If torsional vibration is started by some impulse that gives to the disk an initial angular velocity $\dot{\theta}_0$ while the initial angle of rotation from the position of equilibrium is zero, we obtain from Eq. (71*a*)

$$\theta = \frac{\dot{\theta}_0}{p} \sin pt \tag{71c}$$

From Eq. (71) we see that the rotation of the disk is periodical and that a complete cycle requires the time

$$\tau = \frac{2\pi}{p} = 2\pi \sqrt{\frac{Il}{GJ}} \tag{72}$$

This expression represents the *period* of torsional vibration.

If all dimensions of the torsional pendulum (Fig. 275) are increased n times, the moment of inertia I of the disk is increased in the proportion n^5 and the polar moment of inertia J, in the proportion n^4. Hence the period τ, from Eq. (72), increases in the same ratio n as the linear dimensions of the system. Thus by increasing the size of the torsional pendulum but keeping it geometrically similar, we make the period of torsional vibration larger and the vibrations slower.

The *frequency* of torsional vibration of the pendulum from Eq. (72) is

$$f = \frac{1}{\tau} = \frac{1}{2\pi} \sqrt{\frac{GJ}{Il}} \tag{73}$$

It should be noted that frequency is increased by increasing the torsional rigidity of the shaft and decreased by increasing the moment of inertia of the disk.

EXAMPLES

1. Determine the frequency of torsional vibration of a horizontal bar AB of weight $W = 4$ lb and length $a = 2$ ft (Fig. 276), which is suspended at its mid-point on a vertical steel wire of length $l = 2$ ft and diameter $d = \frac{1}{8}$ in. The modulus of elasticity in shear for the wire is $G = 12 \times 10^6$ psi. Neglect the cross-sectional dimensions of the bar and the mass of the wire.

Solution. The moment of inertia of the bar with respect to the axis of the wire is

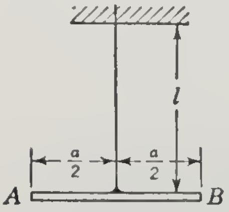


FIG. 276

$$I = \frac{W}{g}\frac{a^2}{12} = \frac{4}{386}\frac{24^2}{12} \text{ lb-sec}^2\text{-in.}$$

and the polar moment of inertia of the circular cross section of the wire is

$$J = \frac{\pi d^4}{32} = \frac{\pi}{32 \times 8^4} \text{ in.}^4$$

Substituting in Eq. (73), we find

$$f = \frac{1}{2\pi}\sqrt{\frac{12 \times 10^6 \times \pi \times 386 \times 12}{8^4 \times 32 \times 24 \times 4 \times 24^2}} = 0.781 \text{ oscillations/sec}$$

2. A steel shaft of length l and polar moment of inertia J is supported horizontally in frictionless bearings and has attached at its ends two disks having moments of inertia I_1 and I_2, as shown in Fig. 277. If the disks are turned in opposite directions, so that the shaft is twisted, and then released from rest, find the period and frequency of the torsional vibrations that will ensue.

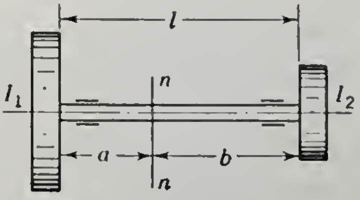


FIG. 277

Solution. Since the gravity forces and bearing reactions intersect the axis of the shaft, it is clear that during vibrations there is no external moment about this axis. Hence the system as a whole can have no angular acceleration and since both disks are at rest when $t = 0$, we conclude that after release they must always rotate in opposite directions.[1] From this it follows that there must be a cross section nn of the shaft which remains stationary during these vibrations. This cross section is called the *nodal cross section* of the shaft. In discussing the torsional vibrations of the two bodies, it may be considered as fixed and the part of the system to either side can be regarded as a simple torsional pendulum. The distances a and b defining the position of the nodal cross section can be found from the condition that the periods of oscillation for the two parts of the system must be equal. Thus using Eq. (72), we obtain

$$\tau = 2\pi\sqrt{\frac{I_1 a}{GJ}} = 2\pi\sqrt{\frac{I_2 b}{GJ}} \tag{f}$$

from which

$$\frac{a}{b} = \frac{I_2}{I_1} \tag{g}$$

---

[1] This question is also discussed in Example 1 on p. 407.

i.e., the distances of the nodal cross section from the oscillating bodies are inversely proportional to their moments of inertia. Observing also that

$$a + b = l \tag{h}$$

we find from Eqs. (g) and (h) that

$$a = \frac{I_2 l}{I_1 + I_2} \qquad b = \frac{I_1 l}{I_1 + I_2} \tag{i}$$

Substituting these values into Eq. (f), we find for the period of free torsional vibrations of the system the following expression:

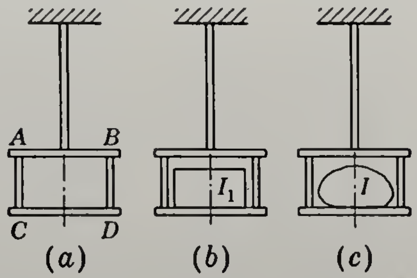
$$\tau = 2\pi \sqrt{\frac{I_1 I_2 l}{GJ(I_1 + I_2)}} \tag{j}$$

The corresponding frequency of vibration is

$$f = \frac{1}{2\pi} \sqrt{\frac{GJ(I_1 + I_2)}{I_1 I_2 l}} \tag{k}$$

We note again that frequency of torsional vibration increases with increasing torsional rigidity of the shaft and decreases with increasing inertia of the attached bodies.

Equations (j) and (k) can be used for calculating the period and frequency of free torsional vibration of a ship propeller shaft where the propeller will be represented by one of the disks and the rotor of the driving turbine, by the other. Torsional vibrations of such systems are of considerable importance in engineering.



(a) (b) (c)

FIG. 278

3. A torsional pendulum (Fig. 278*a*) consists of two parallel circular disks *AB* and *CD* connected together by rods *AC* and *BD* so that the whole acts as a rigid body attached to the vertical shaft and inside of which any body of limited size can be placed. When empty (Fig. 278*a*), the torsional pendulum has an observed period τ_0. When carrying a body of known moment of inertia I_1, which, owing to friction, oscillates with it (Fig. 278*b*), the observed period is τ_1, and when carrying a body of unknown moment of inertia I (Fig. 278*c*), the observed period of torsional oscillation is τ_2. Find the moment of inertia I of the last body with respect to the axis of rotation, i.e., the axis of the shaft.

Solution. When the pan of the pendulum is empty (Fig. 278*a*), we have, by Eq. (72),

$$\tau_0 = 2\pi \sqrt{\frac{I_0 l}{GJ}} \tag{l}$$

where I_0 denotes the moment of inertia of the pendulum itself.

When carrying the body of known moment of inertia I_1 (Fig. 278*b*), we have

$$\tau_1 = 2\pi \sqrt{\frac{(I_0 + I_1)l}{GJ}} \tag{m}$$

Finally for the condition represented in Fig. 278*c*, we have

$$\tau_2 = 2\pi \sqrt{\frac{(I_0 + I)l}{GJ}} \tag{n}$$

These three equations contain I, I_0, and GJ/l as unknowns. Eliminating the last two, we find

$$I = I_1 \frac{\tau_2^2 - \tau_0^2}{\tau_1^2 - \tau_0^2} \tag{o}$$

from which the unknown moment of inertia I can be calculated. This device is quite useful for the experimental determination of moments of inertia of irregularly shaped bodies.

PROBLEM SET 8.4

1. Calculate the period τ of the torsion pendulum shown in Fig. A. The following data are given: $W = 14$ lb, $l = 20$ in., $d = \frac{1}{8}$ in., $r = 4$ in., $h = 1$ in., $G = 12 \times 10^6$ psi. Neglect the thickness h of the disk in computing its moment of inertia. *Ans.* $\tau = 0.631$ sec.

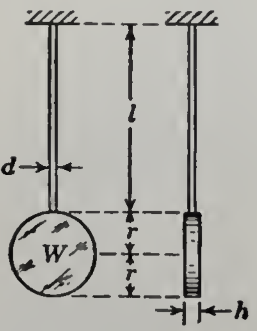


FIG. A

2. Calculate the per cent error in the period τ of the torsion pendulum in Fig. A due to neglecting the thickness h of the disk in computing its moment of inertia. *Ans.* 1 per cent.

3. The hairspring of a watch is so adjusted that to rotate the balance wheel one-quarter turn (90°) requires a torque of 0.02 oz-in. If the wheel has a period $\tau = 1$ sec in free oscillation, what is its moment of inertia I? *Ans.* $I = 20.1 \times 10^{-6}$ lb-sec²-in.

4. The circular disk shown in Fig. B*a* has a radius $r = 6$ in. and weight $W = 50$ lb. The observed frequency of torsional vibration is 1 oscillation/sec. When another body is attached to the same shaft (Fig. B*b*), the observed frequency of torsional vibration is 1.2 oscillations/sec. Find the moment of inertia of the second body with respect to the axis of the shaft. *Ans.* $I = 1.62$ lb-sec²-in.

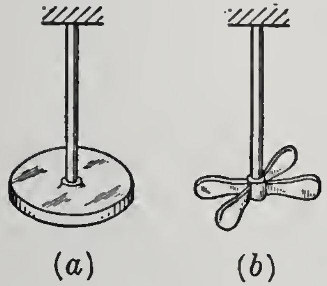


FIG. B

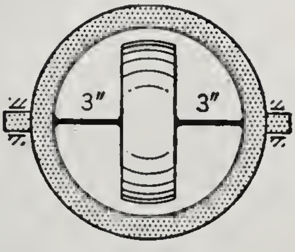


FIG. C

5. A thin ring of weight $W_1 = 3$ lb and mean radius $r_1 = 5$ in. contains a solid wheel of weight $W_2 = 5$ lb and radius $r_2 = 4$ in. as shown in Fig. C. Across a diameter of the ring is a steel rod of diameter $d = \frac{1}{8}$ in. to which both ring and wheel are rigidly attached. The whole system floats freely in bearings as shown. If the disk is initially twisted through a small angle relative to the plane of the ring and then released, calculate the natural frequency of torsional vibration of the system using $G = 12 \times 10^6$ psi. *Ans.* $f = 9.86$ oscillations/sec.

6. The propeller shaft of a ship (Fig. D) has a length $l = 12$ ft and a diameter $d = 4$ in. On one end of the shaft is a steam-turbine rotor of weight $W_1 = 2{,}000$ lb and radius of gyration $i_1 = 30$ in. On the other end is a propeller of weight $W_2 = 1{,}000$ lb and radius of gyration $i_2 = 20$ in. The shaft is of steel for which $G = 11.5 \times 10^6$ psi. Calculate the frequency of free torsional vibration of the system. Neglect friction in the bearings of the shaft. *Ans.* $f = 7.75$ oscillations/sec.

7. By what per cent will the frequency of torsional vibration of the system shown in Fig. D be increased if the diameter of the shaft is increased along half its length by 10 per cent? *Ans.* 9 per cent.

8. Using the same data as given in Prob. 6 above, calculate the frequency of vibration of the system shown in Fig. D, if the diameter of the shaft is increased along one-third of its length from 4 to 5 in. *Ans.* $f = 8.66$ oscillations/sec.

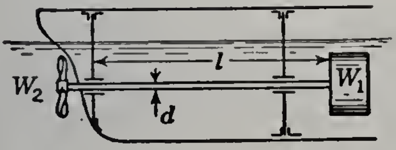


FIG. D

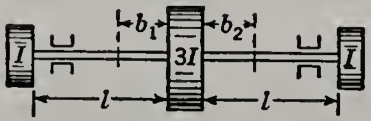


FIG. E

9. In Fig. E, a shaft of uniform diameter d carries three right circular disks symmetrically arranged with respect to the mid-point of the shaft. The moments of inertia of the end disks are each I and that of the middle disk is $3I$. If the system performs free torsional vibrations such that the middle disk always rotates opposite to the ones at the ends of the shaft, what are the

distances b_1 and b_2 to the nodal cross sections in the two portions of the shaft? *Ans.* $b_1 = 2l/5$; $b_2 = 2l/5$.

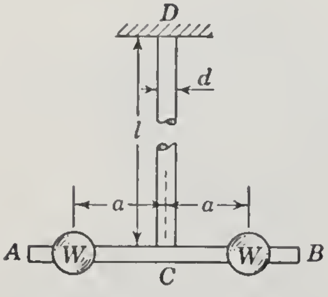


FIG. F

10. A torsion pendulum consists of a rigid bar AB attached at its mid-point C to a shaft DC as shown in Fig. F. Without the two balls, each of weight $W = 1$ lb, the observed period of free oscillation is $\tau_1 = 0.70$ sec. When the two balls are set at the distance $a = 10$ in. from the axis of the shaft as shown, the observed period of free oscillation is $\tau_2 = 1.00$ sec. If the shaft DC has length $l = 110$ in. and diameter $d = \frac{1}{4}$ in., what is the modulus of elasticity in shear for the material of the shaft? *Ans.* $G = 11.5 \times 10^6$ psi.

8.5. The compound pendulum. Any rigid body so suspended that it is free to rotate about a fixed horizontal axis through any point O and normal to the vertical xy plane as shown in Fig. 279 is called a *compound pendulum*. In its equilibrium position the pendulum hangs with its center of gravity C vertically below the point of support O, and we have the reaction at the support equal and opposite to the gravity force W. If the pendulum is rotated from this position of equilibrium and then released, it performs an oscillatory rotation about the fixed axis through point O. In investigating this motion, we denote by θ the angle of rotation of the body from its equilibrium position at any instant t and apply Eq. (67). Denoting by c the distance from the axis of rotation to the center of gravity C, the moment of all forces acting on the body, neglecting friction at the axis, is

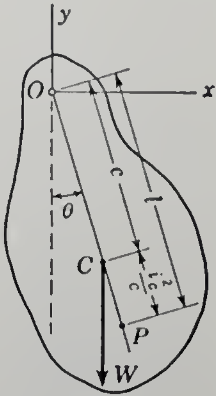


FIG. 279

$$M = -Wc \sin \theta$$

Substituting this in Eq. (67), we obtain

$$\frac{W}{g} i_0^2 \ddot{\theta} = -Wc \sin \theta \tag{a}$$

where i_0 is the radius of gyration of the pendulum with respect to the fixed axis of rotation through O.

Considering only small angles of oscillation, we have $\sin \theta \approx \theta$ and Eq. (a) becomes

$$\ddot{\theta} + \frac{cg}{i_0^2} \theta = 0 \tag{b}$$

Comparing this equation with the equation

$$\ddot{s} + \frac{g}{L} s = 0$$

for a mathematical pendulum of length L (see page 332), we conclude that the period of oscillation of a compound pendulum is equal to that of a mathematical pendulum of length

$$L = \frac{i_0^2}{c} \tag{74a}$$

This period, of course, is

$$\tau = 2\pi \sqrt{\frac{L}{g}} \tag{c}$$

The length L defined by Eq. (74*a*) is called the *equivalent length* of the compound pendulum.

The equivalent length L of a compound pendulum can be written in another form by using the relation between the radii of gyration with respect to parallel axes [see Eq. (32), Appendix II]. Denoting by i_c the radius of gyration of the body with respect to the axis through the center of gravity C and parallel to the axis of rotation, we have

$$i_0^2 = i_c^2 + c^2 \tag{d}$$

Substituting this expression for i_0^2 into Eq. (74*a*), we obtain

$$L = c + \frac{i_c^2}{c} \tag{74b}$$

It is seen that the equivalent length L of the compound pendulum, as represented in Fig. 279 by the length OP, is larger than the distance c from the axis of rotation to the center of gravity C. The point P, which is on the prolongation of the line OC, is called the *center of oscillation.*

The theory of the compound pendulum is often useful in making an experimental determination of the moment of inertia of a body with respect to a given central axis. The body is suspended on a knife-edge defining any convenient axis parallel to the central axis for which the moment of inertia is desired, and the distance c from the axis of suspension to the center of gravity of the body is carefully measured. Then the period of small oscillations is determined by direct observation, and the equivalent length of the pendulum calculated from Eq. (c). Knowing L and c, the square of the radius of gyration i_c of the body can readily be calculated from Eq. (74*b*). The desired moment of inertia is, of course, Wi_c^2/g. By making several observations for different axes of suspension and averaging the results, a very accurate value of i_c^2 can usually be found.

From Eq. (74b) we see that the equivalent length of a compound pendulum is equal to ∞ when $c = 0$, that is, when the axis of suspension passes through the center of gravity of the body, and hence the period is infinite. Again we have $L = \infty$ when $c = \infty$. This suggests that there is some value of c between zero and infinity for which the equivalent length of the pendulum is a minimum and consequently for which its frequency of oscillation is a maximum. To find this value of c, we set the first derivative with respect to c of expression (74b) equal to zero and obtain

$$c = i_c \tag{e}$$

Thus the maximum frequency of oscillation of a compound pendulum will be obtained when the center of suspension is at the distance equal to the radius of gyration i_c from the center of gravity of the body.

EXAMPLES

1. Calculate the period of small oscillations of a circular cylindrical bar OA of length l and radius r suspended from one end O (Fig. 280).

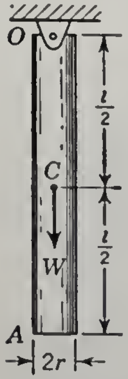


FIG. 280

Solution. From Appendix II, page A,21, we find that the moment of inertia of the bar with respect to the axis of rotation through O is

$$I_0 = \frac{W}{g}\left(\frac{r^2}{4} + \frac{l^2}{12}\right) + \frac{W}{g}\frac{l^2}{4} = \frac{W}{g}\left(\frac{r^2}{4} + \frac{l^2}{3}\right)$$

Hence

$$i_0^2 = \frac{r^2}{4} + \frac{l^2}{3}$$

and Eq. (74a) for the equivalent length becomes

$$L = \frac{i_0^2}{c} = \frac{2l}{3} + \frac{r^2}{2l} \tag{f}$$

Then, for small oscillations, the period will be

$$\tau = 2\pi\sqrt{\frac{L}{g}}$$

If the bar is slender, i.e., if the ratio r/l is small, we see that the second term in expression (f) can be neglected in comparison with the first. Thus substituting the approximate value $L \approx 2l/3$ into the formula for the period, we obtain

$$\tau = 2\pi \sqrt{\frac{2l}{3g}} \tag{g}$$

This formula which neglects the cross-sectional dimensions can be used to calculate the period of free swing of a homogeneous slender bar of any prismatic form.

2. The compound pendulum shown in Fig. 281 is so constructed that it has the same period τ for small amplitudes of swing about either the knife-edge A (Fig. 281*a*) or the knife-edge B (Fig. 281*b*). Prove that the distance l between the knife-edges is the equivalent length of the pendulum if $c \neq l/2$.

Solution. For case *a* the equivalent length by Eq. (74*b*) is

$$L_1 = c + \frac{i_c^2}{c} \tag{h}$$

while for case *b* it is

$$L_2 = (l - c) + \frac{i_c^2}{l - c} \tag{i}$$

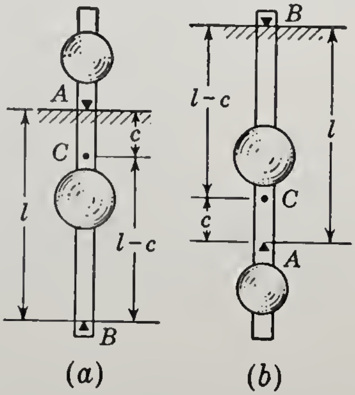


FIG. 281

and by virtue of the specified equality of periods

$$L_1 = L_2 = L$$

Eliminating i_c^2 between expressions (h) and (i), we obtain

$$(L - c)c = (L - l + c)(l - c)$$

which may be written in the form

$$l^2 - (L + 2c)l + 2cL = 0 \tag{j}$$

Solving this quadratic for l, we obtain

$$l = L \qquad \text{or} \qquad l = 2c$$

Since we excluded the possibility of $l = 2c$, we conclude that l is the equivalent length of the pendulum.

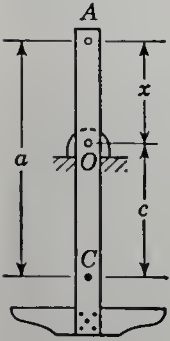


FIG. 282

3. When hanging from a nail at A, the T square in Fig. 282 has an observed period $\tau = 1.5$ sec. A simple static balance test shows also that the center of gravity C is at the distance $a = 24$ in. from the end A of the rule. At what distance x from A should a hole O be drilled in the rule so that the period of free swing about this point of suspension will be the minimum possible period?

Solution. From the previous discussion on page 382, we know that the minimum period will be obtained by making the distance

$$c = a - x = i_c$$

Also for small oscillations about A, we have, by using Eq. (74b) for equivalent length,

$$\tau = 2\pi \sqrt{\frac{a^2 + i_c^2}{ag}}$$

Substituting $a - x$ for i_c in this expression, and squaring both sides, we obtain the following quadratic in x:

$$x^2 + 2ax - \left(2a^2 - \frac{ag\tau^2}{4\pi^2}\right) = 0$$

The roots of this equation are

$$x = a\left(-1 \pm \sqrt{3 - \frac{g\tau^2}{4\pi^2 a}}\right) \tag{k}$$

Substituting the given numerical data, we obtain $x = 10.7$ in. The negative root is of no physical significance.

PROBLEM SET 8.5

1. A homogeneous circular disk of radius r and weight W hangs in a vertical plane from a pin O at its circumference. Find the period τ for small angles of swing in the plane of the disk. *Ans.* $\tau = 2\pi\sqrt{3r/2g}$.

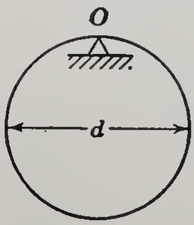


FIG. A

2. A thin circular hoop of diameter d hangs over a knife-edge at O as shown in Fig. A. Develop a formula for the period τ for small amplitudes of swing in its own vertical plane. *Ans.* $\tau = 2\pi\sqrt{d/g}$.

3. A slender wire 36 in. long is bent in the form of an equilateral triangle and hangs from a pin at O as shown in Fig. B. Determine the period τ for small amplitudes of swing in the plane of the figure. *Ans.* $\tau = 1.03$ sec.

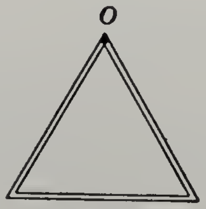


FIG. B

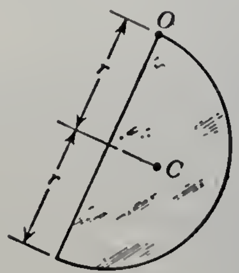


FIG. C

4. A homogeneous semicircular disk of radius r is suspended as a compound pendulum as shown in Fig. C and swings with small amplitude in its own vertical plane. Calculate the period τ. *Ans.* $\tau = 2\pi\sqrt{1.38r/g}$.

5. A homogeneous square plate with a centered circular hole is supported as a compound pendulum as shown in Fig. D. Calculate the period τ for small oscillations in the vertical plane of the plate. *Ans.* $\tau = 1.115$ sec.

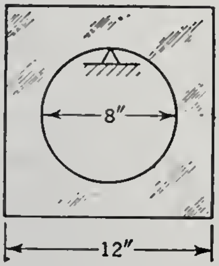


FIG. D

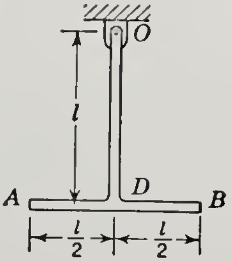


FIG. E

6. Develop a formula for the period τ for small oscillations of the compound pendulum shown in Fig. E. Treat OD and AB as identical slender bars of uniform cross section. *Ans.* $\tau = 2\pi\sqrt{17l/18g}$.

7. Determine the period of small oscillations of the compound pendulum shown in Fig. F and consisting of a disk suspended by a slender rod if the following numerical data are given: $b = 12$ in., $a = 5$ in., $W_1 = \frac{1}{2}$ lb, and $W_2 = 3$ lb. *Ans.* $\tau = 1.32$ sec.

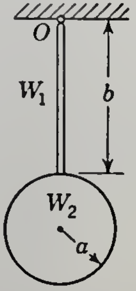


FIG. F

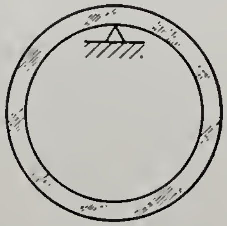


FIG. G

8. A circular ring rests with the inner face of its rim on a transverse horizontal knife-edge (Fig. G). Determine the radius of gyration i_c of the ring with respect to its center if the inner radius of the ring is 3 ft 3 in. and its period for small oscillations in the vertical plane of the ring is $\tau = 2.93$ sec. *Ans.* $i_c = 3$ ft 6 in.

9. A connecting rod of weight $W = 3.00$ lb when supported on a knife-edge as shown in Fig. H has, for small amplitudes of swing in its plane of symmetry, the observed period $\tau = 1.05$ sec. If the dimensions of the rod are as shown in the figure, calculate its moment of inertia I_0 with respect to the axis of the crank bearing. *Ans.* $I_0 = 0.353$ lb-sec²-in.

10. Calculate the period of oscillation of the ring in Fig. G if it swings in the direction normal to the plane of the figure. *Ans.* $\tau = 2.51$ sec.

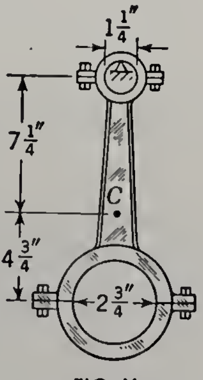


FIG. H

8.6. General case of moment proportional to angle of rotation. In many practical problems we encounter the case of a rigid body which can rotate about a fixed axis but that in so doing brings into play certain reactive forces which create a moment about that axis proportional to the angle of rotation of the body from its equilibrium position and tending always to restore it to that position. In any such case, if the body is initially disturbed from its position of equilibrium, it will perform rotational oscillations of a simple harmonic nature about the fixed axis and we usually are most interested in determining the period or frequency of these oscillations.

Consider, for example, the slender prismatic bar OA of weight W and length l which is hinged at O and supported in a horizontal position of equilibrium by a spring of constant k attached to it at A (Fig. 283). The tension in the spring for this position of equilibrium is obviously $W/2$. If the end A of the bar is given some initial displacement downward and then released, the bar will perform oscillatory rotation about the fixed axis through O normal to the plane of the figure. During these oscillations, let θ be the angle of rotation of the bar from its horizontal position of equilibrium at any instant t. Then the upward reactive force exerted by the spring on the end A of the bar is,

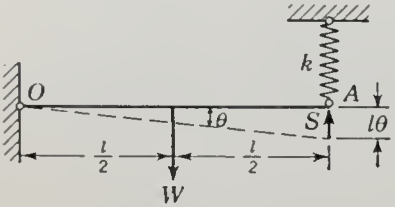


FIG. 283

$$S = \frac{W}{2} + kl\theta \tag{a}$$

and neglecting the cross-sectional dimensions of the bar OA, the equation of rotation [Eq. (67)] becomes

$$\frac{W}{g}\frac{l^2}{3}\ddot{\theta} = W\frac{l}{2} - \left(\frac{W}{2} + kl\theta\right)l$$

which reduces to

$$\ddot{\theta} + \frac{3kg}{W}\theta = 0 \tag{b}$$

Equation (b) is of the same form as Eq. (70) where $p^2 = 3kg/W$, and hence we conclude at once that we have a harmonic oscillatory rotation of the bar with the period

$$\tau = 2\pi\sqrt{\frac{W}{3kg}} \tag{c}$$

In any case of rotation of a rigid body about a fixed axis where the general differential equation of motion (67) can be brought into the form of Eq. (70), we conclude that we have a harmonic oscillatory rotation and can at once write down the expression for its period or frequency.

EXAMPLES

1. In the case of a single-phase generator (Fig. 284), the rotor produces a pulsating torque on the stator. In order to reduce the effect of this torque on the foundation, springs are introduced between the stator and the foundation as shown. The springs must be such that the natural frequency of torsional vibration of the stator is small in comparison with the frequency of the pulsating torque. It is required, then, to find the frequency of natural torsional vibration of the stator if its moment of inertia with respect to the axis O perpendicular to the plane of the figure is I. The distance between the springs is l, the total number of springs is 4, and each spring constant is k.

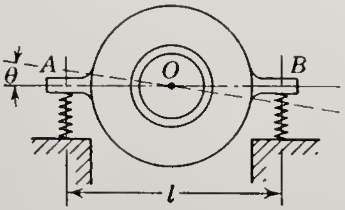


FIG. 284

Solution. If a couple in the plane of the figure is applied to the stator so that a compression of the two springs on one side and an extension of the two on the other side is produced and then this couple is suddenly removed, rotational oscillations of the stator about the axis through O will ensue. Let θ be the angle of rotation of the stator at any instant t. Then the compression and elongation of the springs for small values of θ can be taken equal to $\theta l/2$, and the corresponding reaction exerted by each spring on the stator is $k\theta l/2$. The couple formed by these elastic reactions is $k\theta l^2$, since there are two springs on each side. Hence Eq. (67) becomes

$$I\ddot{\theta} = -kl^2\theta$$

which reduces to

$$\ddot{\theta} + \frac{kl^2}{I}\theta = 0$$

and the frequency is

$$f = \frac{1}{2\pi}\sqrt{\frac{kl^2}{I}}$$

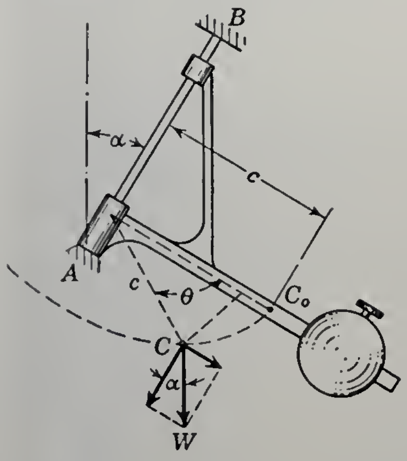


FIG. 285

2. In Fig. 285, a ball is attached to an inclined shaft AB by a bracket that can rotate without friction on the shaft. The entire weight of the system is W, the center of gravity lies at C_0, distance c from AB, and the radius of gyration with respect to AB is i_0. Find the period of oscillation for small amplitudes of swing about the equilibrium position ABC_0.

Solution. Considering a small angle of rotation θ out of the vertical plane ABC_0, we see that the gravity force W acting at C will exert a restoring moment about the axis AB of the inclined shaft. To calculate this moment, we resolve the vertical force W into components $W \cos \alpha$, parallel to AB, and $W \sin \alpha$, perpendicular to AB, as shown in the figure. Then only the latter component need be considered in calculating the moment M. The arm of this force is $c \sin \theta$ and we have

$$M = -W \sin \alpha \, c \sin \theta \tag{d}$$

Substituting this value of M into the general equation of motion $I\ddot{\theta} = M$, we obtain

$$\frac{W}{g} i_0^2 \ddot{\theta} = -Wc \sin \alpha \sin \theta$$

Considering only small values of θ such that $\sin \theta \approx \theta$, this reduces to

$$\ddot{\theta} + \left(\frac{cg}{i_0^2} \sin \alpha \right) \theta = 0 \tag{e}$$

From this we conclude that the ball performs simple harmonic motion having the period

$$\tau = 2\pi \sqrt{\frac{i_0^2}{cg} \csc \alpha} \tag{f}$$

Thus we have a compound pendulum with equivalent length

$$L = \frac{i_0^2}{c \sin \alpha}$$

By making the angle of inclination α of the shaft AB very small, we obtain a pendulum having a very large equivalent length L. This device is sometimes used in situations where a very-low-frequency oscillation is desired.

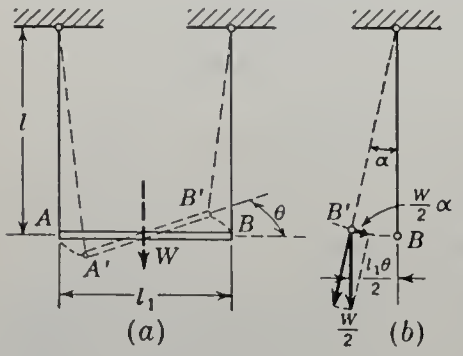


FIG. 286

3. Determine the period of small rotational oscillations of a horizontal slender prismatic bar of length l_1 and weight W suspended at its ends by two vertical and perfectly flexible strings of length l (Fig. 286*a*).

Solution. Measuring from the position of equilibrium a small angle of rotation θ of the bar AB in the horizontal plane, the corresponding angle of inclination α of each string with the vertical, as we see from Fig. 286*b*, is

$$\alpha = \frac{l_1 \theta}{2l} \tag{g}$$

Resolving the load $W/2$ transmitted to each string into two components as shown in Fig. 286b, we obtain a couple of forces $W\alpha/2$ that are approximately horizontal and tend to bring the bar back to its position of stable equilibrium. The moment of this couple is $l_1W\alpha/2$, which, using expression (g), becomes $l_1^2\theta W/4l$. Equation (67) for rotation of the bar about a vertical axis through its center of gravity then becomes

$$\frac{W}{g}\frac{l_1^2}{12}\ddot{\theta} = -\frac{l_1^2 W}{4l}\theta$$

which reduces to

$$\ddot{\theta} + \frac{3g}{l}\theta = 0$$

and we have for the period of oscillation

$$\tau = 2\pi\sqrt{\frac{l}{3g}} \tag{h}$$

i.e., the same as for a mathematical pendulum of the length $l/3$.

PROBLEM SET 8.6

1. A slender prismatic bar AB of weight $W = 5$ lb and length $l = 3$ ft is hinged at D and held in a horizontal position of equilibrium by a spring of constant $k = 5$ lb/in. (Fig. A). Find the period τ for small amplitudes of rotational oscillation of the bar in the vertical plane of the figure. The distance $a = 1$ ft. *Ans.* $\tau = 0.320$ sec.

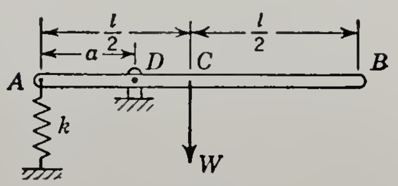


FIG. A

2. A slender prismatic bar AB of weight $W = 5$ lb and length $l = 3$ ft is hinged at A and supported in a horizontal position by a vertical spring attached to it at D (Fig. B). For small amplitudes of rotational oscillation of the bar in a vertical plane, calculate the period of oscillation if the spring constant is $k = 5$ lb/in. and $a = 1$ ft. *Ans.* $\tau = 0.555$ sec.

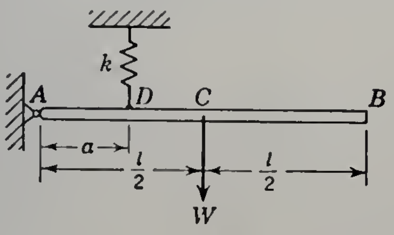


FIG. B

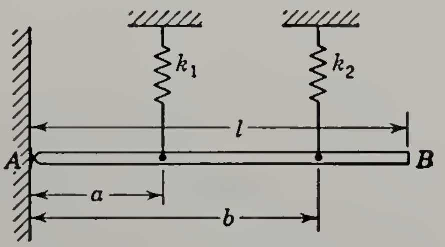


FIG. C

3. A slender prismatic bar AB of length $l = 3$ ft and weight $W = 4$ lb is hinged to a wall at A and supported in a horizontal position by two springs having constants k_1 and k_2 as shown in Fig. C. Find the period of oscillation for small amplitudes of rotation in the plane of the figure if $a = 1$ ft, $b = 2$ ft, and $k_1 = k_2 = 2$ lb/in. *Ans.* $\tau = 0.350$ sec.

4. The spring of the arrangement shown in Fig. D is such that a movement of the weight W from the position D on the bar AB to the position E produces a vertical movement of the pointer B of 0.1 in. as shown. Find the period of vibration of the weight W when in the position E. The following numerical data are given: $a = 10$ in., $b = 10$ in., $c = 15$ in., $W = 12$ lb. The bar AB is 40 in. long and weighs 4 lb. *Ans.* $\tau = 0.155$ sec.

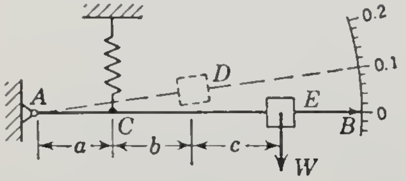


FIG. D

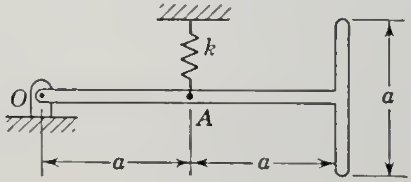


FIG. E

5. A slender T bar of uniform cross section and total weight W is supported in a vertical plane by a hinge at O and a spring of constant k at A as shown in Fig. E. Develop a formula for the period τ for small amplitudes of rotational oscillations in the plane of the figure. *Ans.* $\tau = 3\pi\sqrt{W/kg}$.

6. A slender L-shaped bar ACB is supported in a vertical plane by a pin at C and a spring at A as shown in Fig. F. The total weight of the bar is 3 lb, the spring constant $k = 1$ lb/in., and $l = 6$ in. Calculate the period τ for small amplitudes of rotational oscillation in the plane of the figure. *Ans.* $\tau = 0.479$ sec.

7. Calculate the period τ for rotational oscillations of the bar in Fig. F if the leg CB stands vertically above C instead of vertically below as shown. *Ans.* $\tau = 0.679$ sec.

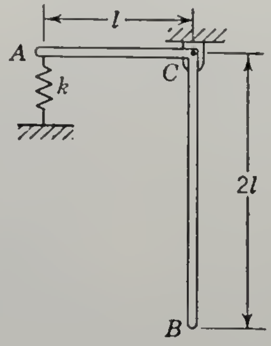


FIG. F

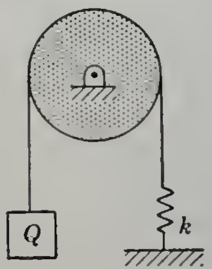


FIG. G

8. A homogeneous triangular plate of weight W, base b, and altitude h is hinged along its base and supported in a horizontal position by a spring of constant k at the vertex of the triangle. Develop a formula for the period τ for small amplitudes of rotational oscillation. *Ans.* $\tau = 2\pi\sqrt{W/6kg}$.

9. A solid circular drum of weight W and radius r can rotate without friction around its horizontal geometric axis (Fig. G). A flexible but inextensible belt overhanging the drum carries a weight Q at one end and is attached to a spring of constant k at the other end. Assuming that the belt cannot slip on the drum, find the period of oscillation of the system. The following numerical data are given: $W = 160$ lb, $r = 1$ ft, $Q = 40$ lb, $k = 60$ lb/in. *Ans.* $\tau = 0.452$ sec.

10. Find the period τ for small amplitudes of rotation of the horizontal bar AB in Fig. H about the vertical axis through its mid-point C. Neglect the thickness of the bar. *Ans.* $\tau = 3.33$ sec.

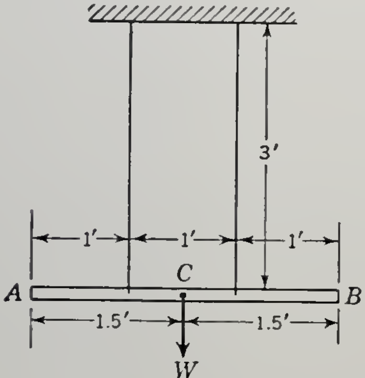


FIG. H

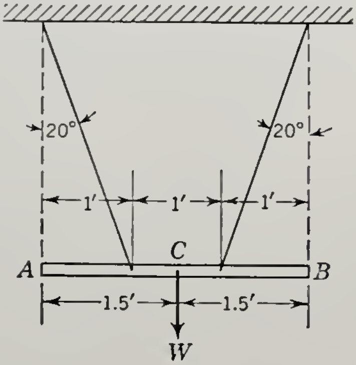


FIG. I

11. Find the period τ for small amplitudes of rotation of the horizontal slender bar AB in Fig. I about the vertical axis through the mid-point C. The inclined strings are attached to the ceiling at points vertically above A and B, as shown. *Ans.* $\tau = 1.83$ sec.

12. A homogeneous rectangular plate is free to rotate with respect to a fixed axis AB coinciding with one of its edges and inclined to the vertical by an angle $\alpha = 20°$ (Fig. J). Determine the period of small rotational oscillations if the dimensions of the plate are $a = 3$ ft and $b = 4$ ft. *Ans.* $\tau = 2.69$ sec.

13. Solve Prob. 12, assuming that the plate is an equilateral triangle with edges of length $a = 3$ ft, one of which coincides with the inclined axis AB. *Ans.* $\tau = 2.18$ sec.

14. A thin circular hoop is suspended by several uniformly spaced vertical strings of equal lengths l so that its plane is horizontal. Prove that for small amplitudes the period of rotational oscillation of the hoop in its plane is $\tau = 2\pi\sqrt{l/g}$.

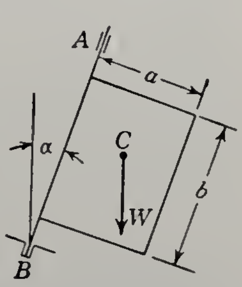


FIG. J

8.7. D'Alembert's principle in rotation. When a body rotates about a fixed axis AB as shown in Fig. 287, any mass particle dm travels in a circular path of radius r centered on the axis of rotation and normal thereto. Attaching to each particle its normal and tangential inertia forces $(-r\dot{\theta}^2\,dm)$ and $(-r\ddot{\theta}\,dm)$ in accordance with D'Alembert's principle, we equilibrate the real forces and establish dynamic equilibrium. In short, we transform the dynamics problem into a statics problem.

By so doing, we see that the inertia forces will represent some form of distributed load, as discussed in Arts. 2.5 and 3.8. In general, the distributed inertia loading on the body will produce reactions at the bearings and various stresses in the body and the shaft on which it is mounted. Such dynamic reactions and stresses are very important in the design of rotating machine parts, and D'Alembert's principle furnishes a useful method for their calculation. The examples which follow will serve to illustrate several particular cases of practical interest.

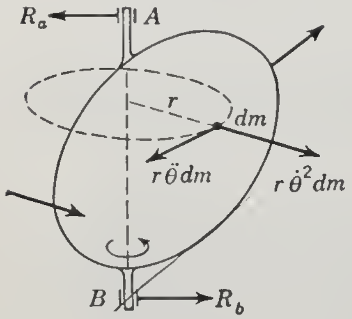


FIG. 287

EXAMPLES

1. A hub of radius r has attached to it several slender radial spokes each of length l and weight W, as shown in Fig. 288. If the hub rotates in a horizontal plane about its geometric axis with uniform angular velocity ω, prove that the radial force S exerted on it by each spoke is the same as if the entire mass of that spoke were concentrated at its center of gravity C.

Proof. Let dm be any element of mass of a spoke at the distance x from the axis of rotation. Then the inertia force for this one element is $\omega^2 x\,dm$ directed as shown in the figure. Summing up all such inertia forces, we obtain for the total radial force exerted on the hub:

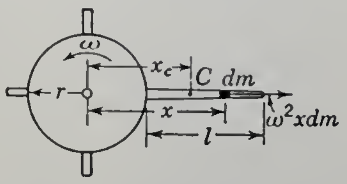


FIG. 288

$$S = \int_r^{(l+r)} \omega^2 x\,dm = \omega^2 \int_r^{(l+r)} x\,dm = \frac{W}{g}\,\omega^2 x_c \tag{a}$$

This formula can be used to calculate the tension produced at the root of a turbine blade owing to centrifugal force.

2. A slender prismatic bar of length l and weight $W = wl$ is supported horizontally by a hinge at A and a vertical string at B as shown in Fig. 289a. Under such conditions, the static reaction at A is obviously $W/2$. Find the magnitude of this reaction an instant after the string is cut but before the bar has moved perceptibly.

Solution. As soon as the string is cut the bar is free to rotate about A. At the initial instant in which we are interested, the bar has no angular velocity but only angular acceleration $\ddot{\theta}$. Hence, in this case, there will be no normal inertia forces but only tangential inertia forces perpendicular to the axis of the bar. For an element of mass $dm = w\,dx/g$ at the distance x from A, the tangential inertia force is $-x\ddot{\theta}w\,dx/g$. Dividing this by the length dx of the element, we see that the intensity of distributed load is $-\ddot{\theta}wx/g$ which varies linearly with x since $\ddot{\theta}$ is the same for all elements of the bar. Thus the load diagram representing the distributed inertia force is a triangle as shown in Fig. 289*b* and the intensity at the free end of the bar is $\ddot{\theta}wl/g = \ddot{\theta}W/g$. The resultant of this load distribution is equal to the area of the load diagram and acts through its centroid at the distance $2l/3$ from A as shown.

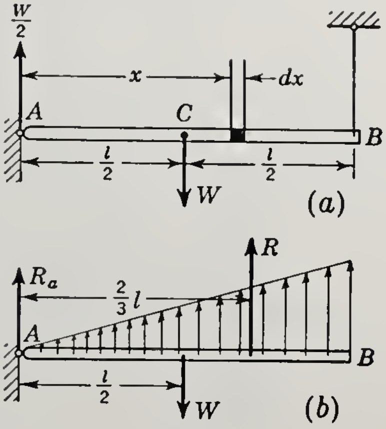


FIG. 289

We now have the bar in dynamic equilibrium under the action of three vertical forces (Fig. 289*b*). The equations of equilibrium for such a system of forces are $\Sigma F_i = 0$ and $\Sigma M_A = 0$, which become

$$R_a + R - W = 0$$
$$R\frac{2}{3}l - W\frac{l}{2} = 0 \tag{b}$$

From the second equation, we find $R = 3W/4$. Substituting this in the first equation, we obtain $R_a = W/4$. Thus, when the string is cut, there is a sudden decrease in the magnitude of the reaction at A.

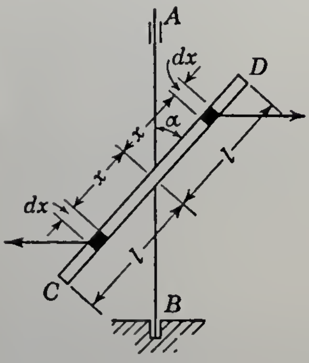


FIG. 290

3. Referring to Fig. 290, find the bending couple M exerted on the rotating shaft AB by the slender prismatic bar CD of length $2l$ and weight $W = 2ql$ which is rigidly attached to the shaft at its mid-point. The axis of the bar CD makes the angle α with the axis of the shaft AB.

Solution. Considering two infinitesimal elements of length dx, as shown in the figure, we find that the corresponding inertia forces form a couple of the magnitude

$$\frac{q\,dx}{g}\omega^2 x \sin\alpha\, 2x\cos\alpha = \frac{q\omega^2\sin 2\alpha}{g}x^2$$

where ω is the angular velocity of rotation. The resultant couple transmitted to the shaft by the inclined bar is

$$M = \int_0^l \frac{q\omega^2 \sin 2\alpha}{g} x^2\, dx = \frac{q\omega^2 l^3}{3g} \sin 2\alpha = \frac{W\omega^2 l^2}{6g} \sin 2\alpha \tag{c}$$

which is zero when $\alpha = 0$ or when $\alpha = 90°$ and maximum when $\alpha = 45°$.

4. Calculate the bending couple transmitted to the horizontal shaft AB by the hub of a flywheel the plane of which makes a small angle α with the plane normal to the shaft (Fig. 291). Consider only the mass of the rim of the flywheel and assume that this mass is uniformly distributed along the center line of the rim having a radius r.

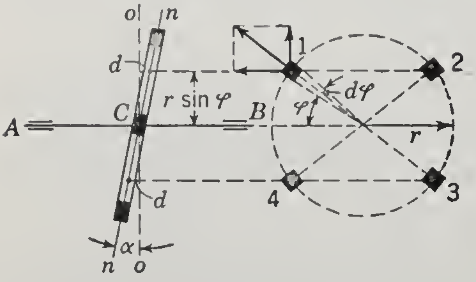


FIG. 291

Solution. Measuring an angle φ from the horizontal diameter representing the intersection of the middle plane nn of the flywheel with the plane oo normal to the shaft and considering an element of the rim corresponding to an infinitesimal angle $d\varphi$, the centrifugal force acting on this element is $(qr\, d\varphi/g)\omega^2 r$, where q denotes the weight of the rim per unit length of its center line. This force can be resolved, as shown in the figure, into two components: (1) a horizontal component equal to

$$\frac{qr^2\, d\varphi}{g} \omega^2 \cos \varphi$$

and (2) a vertical component equal to

$$\frac{qr^2\, d\varphi}{g} \omega^2 \sin \varphi$$

Considering now four equal elements 1, 2, 3, 4, symmetrically situated with respect to the horizontal and vertical diameters of the flywheel, it is evident that the horizontal components of the centrifugal forces of elements 1 and 2 and also of elements 3 and 4 balance one another. The vertical components of the centrifugal forces of elements 1 and 2 give a resultant

$$\frac{2qr^2\, d\varphi}{g} \omega^2 \sin \varphi$$

that acts at a distance $d \approx \alpha r \sin \varphi$ from the plane oo normal to the shaft. Elements 3 and 4 contribute a vertical resultant of the same magnitude but in the opposite direction and on the opposite side of the plane oo by the same distance d. Thus, all together, the inertia forces of the four elements form a couple with an arm $2d$ and acting in the vertical plane through the axis of the shaft. The magnitude of this couple is

$$\frac{2qr^2\, d\varphi}{g} \omega^2 \sin \varphi\, 2r\alpha \sin \varphi$$

Varying the angle φ from 0 to $\pi/2$ and summing up all such elemental couples, we obtain the resultant couple that represents the action of the flywheel on the shaft and is transmitted to the shaft by the hub of the flywheel. The magnitude of the moment of this couple is

$$M = \int_0^{\pi/2} \frac{2qr^2\,d\varphi}{g}\,\omega^2 \sin\varphi\; 2r\alpha \sin\varphi = \frac{W}{2g}\,\omega^2 r^2 \alpha \tag{d}$$

During rotation of the flywheel the plane of the couple also rotates, always remaining perpendicular to the diameter represented by the line of intersection of the planes *oo* and *nn*.

5. A conical pendulum consists of a slender prismatic bar AB of length l and weight ql pinned to a vertical shaft as shown in Fig. 292. If this shaft rotates with constant angular velocity ω, what will be the angle α of the cone described by the bar?

Solution. Consider one element of the bar of length dx at the distance x from the pin A. This element of mass $dm = q\ dx/g$ moves uniformly in a circular path of radius $x \sin \alpha$; it has normal acceleration $\omega^2 x \sin \alpha$. The corresponding inertia force is directed horizontally outward as shown in the figure. Considering such inertia forces for all elements of the bar together with the gravity force ql, we have a system of forces in dynamic equilibrium. Equating to zero the algebraic sum of their moments about point A, we obtain

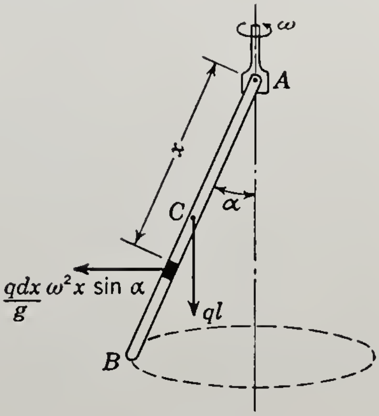


FIG. 292

$$ql\frac{l}{2}\sin\alpha - \int_0^l \frac{q\,dx}{g}\,\omega^2 x \sin\alpha\; x\cos\alpha = 0$$

which reduces to

$$\frac{l^2}{2}\sin\alpha - \frac{\omega^2}{g}\frac{l^3}{3}\cos\alpha\sin\alpha = 0$$

From this equation we obtain two solutions for α as follows:

$$\sin\alpha = 0 \tag{e}$$

or

$$\cos\alpha = \frac{3g}{2\omega^2 l} \tag{f}$$

These expressions define the angle α of the cone for all possible values of ω.

Setting $\cos\alpha = 1$ in Eq. (f), we find

$$\omega = \sqrt{\frac{3g}{2l}} \tag{g}$$

For angular velocities below this value, the solution (e) applies and we see that the bar remains vertical; that is, $\alpha = 0$. For angular velocities above this value, the solution (g) applies and we have a definite value of α for each value of ω. It is of interest to note that the angular velocity (g) at which the bar first shows a tendency to depart from the vertical position corresponds to its natural frequency of oscillation as a compound pendulum (see Example 1, page 382).

PROBLEM SET 8.7

1. Determine the tensile stress at the root of a steel turbine blade of constant cross section due to centrifugal force, if the radius of the rotor at the root of the blade is 30 in. and the length of the blade is 10 in. The speed of the rotor is $n = 1{,}800$ rpm and the weight per unit volume of steel is 0.283 lb/in.^{3} *Ans.* $s = 9{,}110$ psi.

2. Solve Prob. 1, assuming that the cross-sectional area of the blade at the end is only half that at the root and that the magnitude of the area varies along the blade following a linear law. *Ans.* $s = 6{,}730$ psi.

3. A horizontal turntable carries a slender L-shaped prismatic bar weighing 3 lb and rotates about a vertical axis with constant angular velocity ω as shown in Fig. A. What is the magnitude of the reaction induced at B if the angular speed of the turntable is 60 rpm and $\beta = 90°$? *Ans.* $R = 4.95$ lb.

4. Solve Prob. 3 if the angle $\beta = 120°$ and all other data are the same. *Ans.* $R = 7.04$ lb.

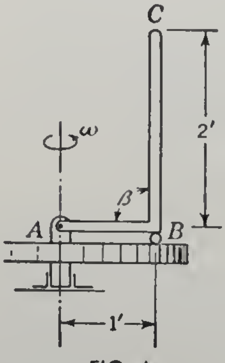


FIG. A

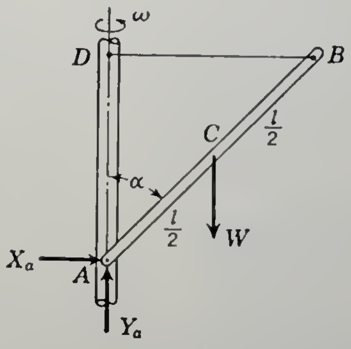


FIG. B

5. A slender bar AB of length l and weight W is attached to a vertical shaft as shown in Fig. B. Calculate the tension S in the string BD when the system rotates with constant angular velocity ω about the vertical axis AD. The following numerical data are given: $W = 25$ lb, $\alpha = 45°$, $l = 4$ ft, and the system rotates at 600 rpm. *Ans.* $S = 2{,}900$ lb.

6. Assuming that the system in Fig. B is at rest, calculate the instantaneous values of the reaction components at the hinge A just after the string DB is cut. Assume $W = 20$ lb, $\alpha = 35°$. *Ans.* $X_a = 7.05$ lb; $Y_a = 15.1$ lb.

7. Referring to Fig. C, calculate the angular speed at which the sliding weight $Q = 2$ lb will begin to lift free of its support if each of the slender prismatic bars weighs 2 lb. *Ans.* 101 rpm.

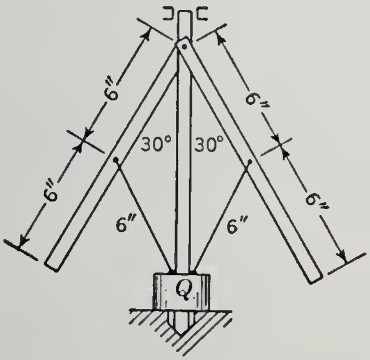


FIG. C

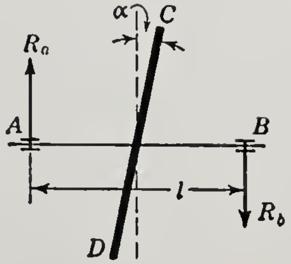


FIG. D

8. A thin circular disk CD of radius r and weight W is attached at its center to a shaft AB, and its plane makes with the plane normal to the axis of the shaft a small angle α (Fig. D). If the disk rotates with a constant angular velocity ω, find the bearing reactions at A and B due to this rotation.

Hint. Use Eq. (d) from Example 4 above for an elemental ring of radius ρ and thickness $d\rho$ and integrate over the full radius r of the disk. *Ans.* $R_a = R_b = W\omega^2r^2\alpha/4gl$.

9. Referring to Fig. E, calculate the bending couple M transmitted to the vertical shaft AB by the bent prismatic rod $DEFG$. The weight per unit length of the rod is $w = 0.10$ lb/in. and the system rotates at a constant angular speed of 600 rpm. *Ans.* $M = 36.8$ in.-lb.

10. An ordinary carpenters square (18 by 24 in.) hangs freely from a pin at the end of the long leg and rotates with constant angular velocity about a vertical axis through this pin. The weight of the short leg is $\frac{1}{4}$ lb; that of the long leg, $\frac{3}{4}$ lb. What is the required rpm to keep the long leg vertical, i.e., on the axis of rotation? *Ans.* 38.2 rpm.

11. A slender prismatic bar of length l and weight W is supported in a horizontal position as shown in Fig. F. Find the instantaneous reaction at A just after the string BC is cut. *Ans.* $R_a = 3W/4$.

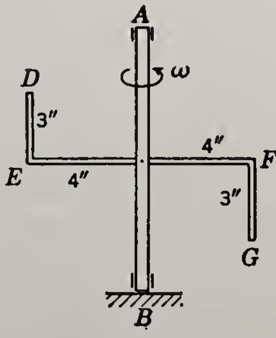


FIG. E

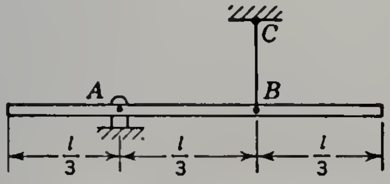


FIG. F

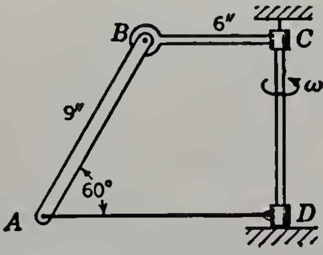


FIG. G

12. A slender prismatic bar AB rotates about a vertical axis CD as shown in Fig. G. Calculate the constant angular velocity ω for which the axial force in the tie bar AD will be zero. Neglect the weight of the tie bar. *Ans.* $\omega = 4.98 \text{ sec}^{-1}$.

13. A slender prismatic bar AB is supported in a vertical plane as shown in Fig. H. Calculate the magnitude of the total reaction R_a at the hinge A: (a) an instant before the string DE is cut and (b) an instant after it is cut *Ans.* (a) $R_a = 0.662W$; (b) $R_a = 0.545W$.

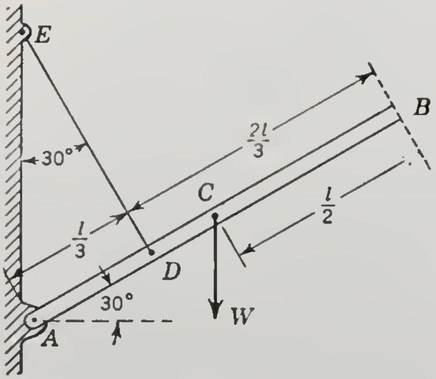


FIG. H

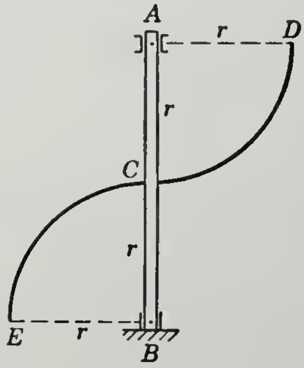


FIG. I

14. Referring to Fig. I, calculate the bending couple M exerted on the vertical shaft AB by the slender prismatic curved bar DCE rigidly attached to the shaft at C. The curves DC and EC are circular quadrants of radius r centered at A and B, respectively, and the weight per unit length of the curved bar is w. The system rotates with constant angular velocity ω about the axis AB. *Ans.* $M = wr^3\omega^2/g$.

8.8. Resultant inertia force in rotation. In Art. 8.7, we have seen that the inertia forces acting on the various particles of a rigid body in rotation represent some form of distributed load. Sometimes the distribution of such forces becomes too complicated to work with directly and it is helpful to reduce the inertia forces to their resultant. In this article, we shall limit our attention to the case of a body having a plane of symmetry and rotating about an axis normal to this plane. Then we can treat the distributed inertia forces as a coplanar system acting in this plane of symmetry and the problem will be much simplified.

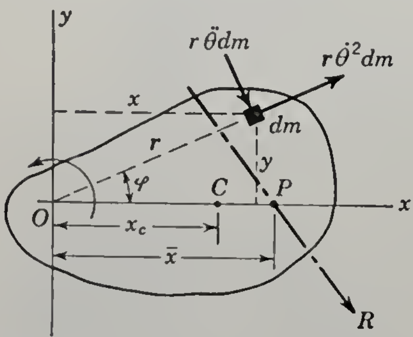


FIG. 293

Referring to Fig. 293, we consider a rigid body rotating about a fixed axis through O normal to the plane of the figure which is a plane of symmetry of the body. In this plane, we take coordinate axes x and y through point O as shown. The x axis is taken through the center of gravity C of the body and both x and y axes are attached to the body so that they rotate with it.

Considering a particle of mass dm at the distance r from the axis of rotation and having coordinates x and y, the normal and tangential inertia forces $r\dot{\theta}^2\,dm$ and $r\ddot{\theta}\,dm$ will be directed as shown in Fig. 293. We wish to determine the magnitude and line of action of their resultant R which, of course, lies in the rotating xy plane. To do this, we denote by X and Y the rectangular components of the force R and by $\bar{x}$ the distance from the axis of rotation to the point P where its line of action cuts the x axis. Then from statics

$$\begin{aligned} X &= \int r\dot{\theta}^2\,dm\cos\varphi + \int r\ddot{\theta}\,dm\sin\varphi \\ Y &= \int r\dot{\theta}^2\,dm\sin\varphi - \int r\ddot{\theta}\,dm\cos\varphi \end{aligned}$$

Noting from the figure that

$$x = r\cos\varphi \qquad \text{and} \qquad y = r\sin\varphi$$

and remembering that $\dot{\theta}$ and $\ddot{\theta}$ are the same for all particles, these expressions become

$$\begin{aligned} X &= \dot{\theta}^2\int x\,dm + \ddot{\theta}\int y\,dm \\ Y &= \dot{\theta}^2\int y\,dm - \ddot{\theta}\int x\,dm \end{aligned} \tag{a}$$

Observing that

$$\int x\,dm = \frac{W}{g}x_c \qquad \text{and} \qquad \int y\,dm = \frac{W}{g}y_c = 0$$

Eqs. (a) reduce to

$$X = \frac{W}{g}\dot{\theta}^2x_c \qquad \text{and} \qquad Y = -\frac{W}{g}\ddot{\theta}\,x_c \tag{75}$$

We see that these components of the resultant inertia force R represent individually the resultant normal inertia force and the resultant tangential inertia force, respectively, and that both of these resultants may be calculated by assuming the entire mass of the body concentrated at its center of gravity C. This is a conclusion that has been noted previously in some of the examples of Art. 8.7.

To find the point P where the resultant inertia force R cuts the x axis, we use the theorem of moments. Noting that the normal inertia forces $r\dot{\theta}^2\,dm$ all intersect at O and using this point as a moment center, we have

$$Y\bar{x} = -\int r^2\ddot{\theta}\,dm = -I_0\ddot{\theta} = -\frac{W}{g}i_0^2\ddot{\theta} \tag{b}$$

where i_0 is the radius of gyration of the body with respect to the axis of rotation. Substituting the value of Y from the second of Eqs. (75), we obtain from Eq. (b)

$$\bar{x} = \frac{i_0^2}{x_c} \tag{76a}$$

Using the parallel axis theorem $i_0^2 = x_c^2 + i_c^2$ (see page A,19), this may also be written in the form

$$\bar{x} = x_c + \frac{i_c^2}{x_c} \tag{76b}$$

Comparing Eqs. (76) with Eqs. (74), page 381, we see that the point of application P of the resultant inertia force is the same as the center of oscillation discussed previously in connection with the theory of the compound pendulum. This point is also called the *center of percussion* of the body.

If the axis of rotation passes through the center of gravity C of the body in Fig. 293, we have $x_c = 0$ and the resultant inertia force as defined by Eqs. (75) vanishes. However, at the same time, the distance $\bar{x}$ as defined by Eq. (76) becomes infinite. This means that the distributed inertia forces reduce to a resultant couple (see page 105). The moment of this resultant couple is represented by Eq. (b), that is,

$$M_R = -I_0\ddot{\theta} \tag{77}$$

which is simply the inertia couple discussed previously in Art. 8.3, page 367.

Summarizing the results of the foregoing discussion, we conclude that in the case of a rigid body rotating about a fixed axis normal to a plane of summetry, the resultant of all inertia forces may be represented as shown in Fig. 294*a*. It should be emphasized that while the components of this resultant are calculated by assuming the entire mass W/g concentrated at the center of gravity C, they do not act at this point, but at the center of percussion P, in order to be statically equivalent to the actual distribution of inertia forces on the individual particles.

We can bring the resultant inertia force in Fig. 294*a* to the center of gravity C by resolving the tangential component $W\ddot{\theta}x_c/g$ into an equal parallel force at C and a couple (see page 67). Then the moment of the couple, opposite in sense to the angular acceleration $\ddot{\theta}$, will be

$$\frac{W}{g}\,\ddot{\theta}x_c\,\frac{i_c^2}{x_c} = I_c\ddot{\theta} \tag{c}$$

and we can represent the resultant of all inertia forces as shown in

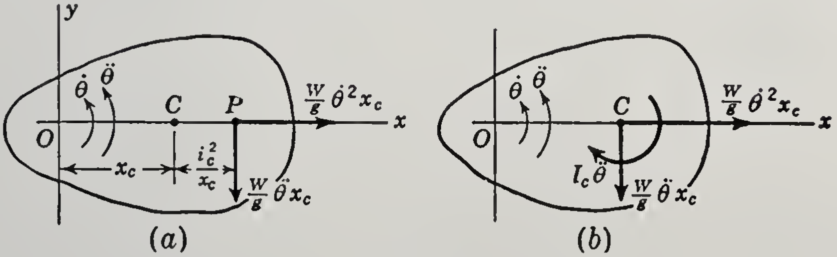

y
$\dot{\theta}$
$\ddot{\theta}$
C
P
O
$\frac{W}{g}\dot{\theta}^2 x_c$
x
x_c
$\frac{i_c^2}{x_c}$
$\frac{W}{g}\ddot{\theta}x_c$
(a)
$\dot{\theta}$
$\ddot{\theta}$
C
O
$\frac{W}{g}\dot{\theta}^2 x_c$
x
$I_c\ddot{\theta}$
$\frac{W}{g}\ddot{\theta}x_c$
(b)

FIG. 294

Fig. 294*b*. Both methods of representation in Fig. 294 will be found useful.

EXAMPLES

1. A homogeneous thin plate in the form of a circular quadrant of radius *r* is supported in a vertical plane by a hinge at *O* and a stop at *B* as shown in Fig. 295*a*. Find the horizontal and vertical components of the reaction at the hinge *O*: (*a*) before the stop at *B* is removed and (*b*) just after it is removed.

Solution. Before the stop at *B* is removed, the plate is in static equilibrium, and we have the free-body diagram as shown in Fig. 295*a*. Writing $\Sigma(M_b)_i = 0$ and $\Sigma Y_i = 0$, we obtain

$$X_0 = \frac{4W}{3\pi} = 0.424W \qquad Y_0 = W \tag{d}$$

directed as shown.

Just after the stop at *B* is removed, the plate has angular acceleration $\ddot{\theta}$ but no angular velocity. Hence the resultant inertia force is $Wc\ddot{\theta}/g$ applied at the center of percussion *P* and acting at right angles to the radius *OC* as shown in Fig. 295*b*. Under the action of this inertia force together with the gravity force *W* and the reaction at *O*, the plate is in dynamic equilibrium.

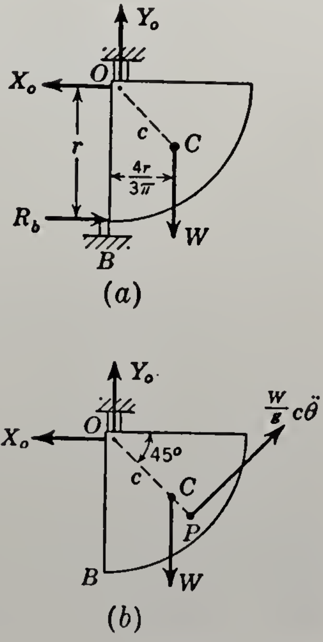

Y_o
X_o
O
r
c
C
$\frac{4r}{3\pi}$
R_b
W
B
(a)
Y_o
$\frac{W}{g}c\ddot{\theta}$
X_o
O
45°
c
C
P
B
W
(b)

FIG. 295

Using the equation $I_0\ddot{\theta} = M$, we find for the initial instant ($t = 0$) that

$$\ddot{\theta} = \frac{M}{I_0} = \frac{W\,4r}{3\pi}\,\frac{8g}{W\,r^2} = \frac{8g}{3\pi r} \tag{e}$$

Also from $\Sigma X_i = 0$ and $\Sigma Y_i = 0$, we have

$$X_0 = \frac{1}{\sqrt{2}} \frac{W}{g} c\ddot{\theta} \qquad Y_0 = W - \frac{1}{\sqrt{2}} \frac{W}{g} c\ddot{\theta} \tag{f}$$

Substituting the value of $\ddot{\theta}$ from expression (e) and noting that the distance $c = \sqrt{2}\,(4r/3\pi)$, expressions (f) reduce to

$$X_0 = 0.36W \qquad Y_0 = 0.64W \tag{g}$$

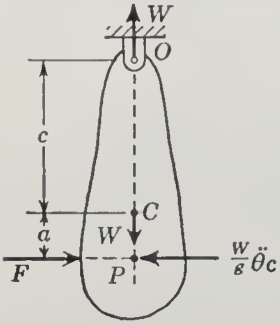


FIG. 296

Comparing these dynamic reactions with the static reactions given by Eqs. (d), we see that there is a considerable change in both components, with the greater change occurring in the case of the vertical component Y_0.

2. A compound pendulum, suspended from O and initially at rest, is to be set in rotation by the action of a suddenly applied horizontal force F as shown in Fig. 296. At what distance a below the center of gravity should the force be applied in order to avoid creating a horizontal reactive force at the hinge O?

Solution. The force F having moment $M = F(c + a)$ about O produces angular acceleration $\ddot{\theta}$ of the pendulum, but at the initial instant there is as yet no angular velocity ($\dot{\theta} = 0$). Hence there will be only tangential inertia forces ($-r\ddot{\theta}\,dm$) on the individual particles and the resultant of these distributed forces is a horizontal force $(W/g)\ddot{\theta}c$ applied at the center of percussion P of the pendulum as shown in the figure. We see now that if the applied force F is collinear with this resultant inertia force, the system is in equilibrium without the need of any horizontal reaction at O. Hence the force F should be applied along a line passing through the center of percussion P, that is, $a = i_c^2/c$.

In practical cases of impact on a body free to rotate about a fixed axis, it is usually very desirable to eliminate the transmission of impact forces to the axis. For this purpose it is necessary to apply the impact at the center of percussion. Thus in the case of an impact testing machine (Fig. 297), the proportions of the hammer OCP used for producing impact are such that the reaction at the point of impact goes through the center of percussion P. The same condition should be fulfilled in the case of a ballistic pendulum or in various kinds of hammers used in forging.

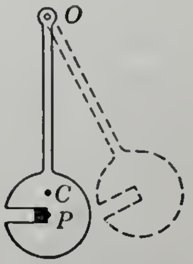


FIG. 297

PROBLEM SET 8.8

1. A slender prismatic bar AB is supported in a horizontal position as shown in Fig. A. At what distance x from the hinge A should the vertical string DE be attached to the bar in order that when it is cut there will be no immediate change in the reaction at A? *Ans.* $x = 2l/3$.

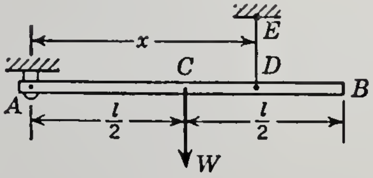


FIG. A

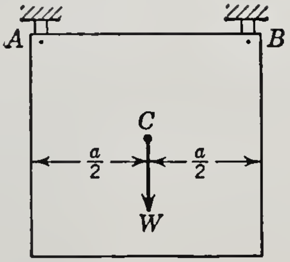


FIG. B

2. A homogeneous square plate of weight W and having dimensions a hangs in a vertical plane by two pins A and B as shown in Fig. B. Calculate the horizontal and vertical components of the reaction at A an instant after the pin at B is removed. *Ans.* $X_a = 3W/8$; $Y_a = 5W/8$.

3. In Fig. C a homogeneous plate having the form of an equilateral triangle with edges of length a hangs in a vertical plane by pins at two of its corners A and B. Calculate the horizontal and vertical components of the reaction at A an instant after the pin at B is removed. The weight of the plate is $W = 40$ lb and the dimension $a = 24$ in. *Ans.* $X_a = 13.88$ lb; $Y_a = 16.0$ lb.

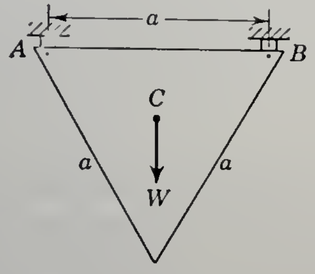


FIG. C

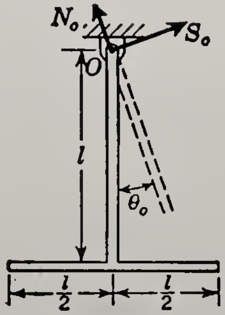


FIG. D

4. The compound pendulum shown in Fig. D consisting of two slender prismatic bars each of length l and weight $W/2$ oscillates in a vertical plane with small amplitude θ_0. Find the rectangular components N_0 and S_0 of the reaction at O when the pendulum is in an extreme position. Assume in calculation that $\sin\theta_0 \approx \theta_0$ and $\cos\theta_0 \approx 1$. *Ans.* $N_0 \approx W$; $S_0 \approx 0.206W\theta_0$

5. Referring again to Fig. D, calculate the reaction components N_0 and S_0 as the pendulum swings through the middle position. Use the same data as given in Prob. 4. *Ans.* $N_0 = W(1 + 0.795\theta_0^2)$; $S_0 = 0$.

6. A locomotive wheel 6 ft in diameter is out of balance to the extent of 200 lb at a radius of 1 ft. The load on the wheel including its own weight is 7 tons. Calculate the maximum and minimum force transmitted to the rail when the speed of the locomotive is 60 mph.

Hint. Uniform translatory motion of the locomotive does not affect accelerations of the particles of the wheel in its rotory motion; thus inertia forces are the same as in the case of rotation of the wheel about a fixed axis. *Ans.* $R_{\max} = 9.67$ tons; $R_{\min} = 4.33$ tons.

8.9. The principle of angular momentum in rotation. In discussing motion of a particle in a plane, it was shown (see page 348) that the rate of change of *moment of momentum* of the particle with respect to any fixed point in its plane of motion is equal to the moment of all forces acting on the particle with respect to the same point. This principle of moment of momentum can be readily extended to the case of a rigid body rotating about a fixed axis.

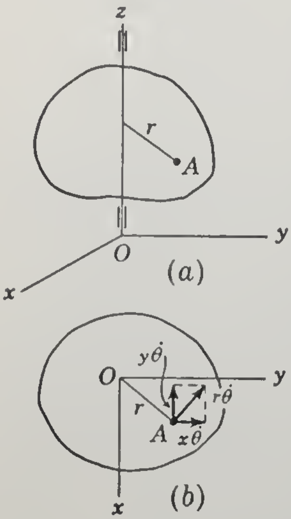


FIG. 298

Conservation of Angular Momentum. We begin with a consideration of moment of momentum about the axis of rotation (Fig. 298). Taking this axis as the z axis and considering an element dm of the body at a point A, distance r therefrom, we see that, during rotation, the element describes a circle in the plane perpendicular to the z axis and having a radius r. If $\dot{\theta}$ is the angular velocity of the body, the velocity of the point A is $r\dot{\theta}$ and the moment of momentum of the element dm with respect to the axis of rotation is

$$r^2\dot{\theta}\, dm \tag{a}$$

The rate of change of this moment of momentum is equal to the moment, about the z axis, of all forces acting on the element. Summing up the moments of momenta (a) together with the corresponding moments of forces for all elements of the rotating body, we find that the rate of change of this sum must be equal to the moment with respect to the axis of rotation of all external forces acting on the body. That is,

$$\frac{d}{dt}\int \dot{\theta} r^2\, dm = M_z \tag{b}$$

where M_z denotes the moment of external forces acting on the body. Noting that the angular velocity $\dot{\theta}$ is the same for all elements of the body and can be taken out from under the integral sign, we write Eq. (b) in the following form:

$$\frac{d}{dt}(I_z\dot{\theta}) = M_z \tag{78}$$

where $I_z = \int r^2\, dm$ is the moment of inertia of the body with respect to the axis of rotation. The expression $I_z\dot{\theta}$ is called the *angular momentum* of the rotating body with respect to its axis of rotation, and Eq. (78) states the *principle of angular momentum.* That is, *the rate of change of angular momentum of a rotating body with respect to its fixed axis of rotation is equal to the moment of all external forces acting on the body with respect to the same axis.* Considering I_z as constant, we see that Eq. (78) coincides with Eq. (67) which was derived previously by using D'Alembert's principle. In using the equation in the form here given, we may consider also cases in which I_z is variable, as might result, for instance, from thermal expansion of a body during rotation.

Equation (78) can be used also in the case of a system of bodies rotating about the same axis and having moments of inertia I_1, I_2, . . . and angular velocities $\dot{\theta}_1$, $\dot{\theta}_2$, . . . about this axis. In this case the equation can be written in the form

$$\frac{d}{dt}(I_1\dot{\theta}_1 + I_2\dot{\theta}_2 + \cdots) = M_z \tag{79}$$

If the moment of external forces with respect to the axis of rotation is zero, we conclude from Eq. (79) that the angular momentum of the system of bodies with respect to that axis remains constant. This is called the *law of conservation of angular momentum.*

Vectorial representation of angular momentum. In the preceding discussion, the principle of angular momentum was established only for the fixed axis of rotation of the body, but it holds also for any other fixed axis. Let us take, for instance, the x axis (Fig. 298). Then resolving the velocity $r\dot{\theta}$ of the element at A into rectangular components $y\dot{\theta}$ and $x\dot{\theta}$ parallel, respectively, to the x and y axes, as shown in Fig. 298*b*, we see that only the component $x\dot{\theta}$ parallel to the y axis should be considered in calculating the moment of momentum of the particle with respect to the x axis. This moment of momentum is[1]

$$-dm\,\dot{\theta}xz \tag{c}$$

and its rate of change is equal to the corresponding moment of the resultant force acting on the particle as before. Then summing moments of momenta (c) and the corresponding moments of forces for all particles of the body as before, we obtain

$$\frac{d}{dt}\left(-\dot{\theta}\int xz\,dm\right) = \frac{d}{dt}(-I_{xz}\dot{\theta}) = M_x \tag{d}$$

---

[1] We use for moment of momentum the same rule of signs as for moment of force.

where M_x denotes the resultant moment with respect to the x axis of all external forces acting on the body and $-I_{xz}\dot{\theta}$ is the angular momentum with respect to the x axis of the body rotating about the z axis. In a similar manner we may write

$$\frac{d}{dt}(-I_{yz}\dot{\theta}) = M_y \tag{e}$$

Using the previously derived equation (78) together with Eqs. (d) and (e), we obtain the complete set of equations defining the system of forces external to a rigid body rotating about a fixed axis as follows:

$$\frac{d}{dt}(I_z\dot{\theta}) = M_z \qquad \frac{d}{dt}(-I_{xz}\dot{\theta}) = M_x \qquad \frac{d}{dt}(-I_{yz}\dot{\theta}) = M_y \tag{80}$$

If the z axis is a principal central axis of inertia of the rotating body, the angular momenta $-I_{xz}\dot{\theta}$ and $-I_{yz}\dot{\theta}$ with respect to the x and y axes will always be zero, since the products of inertia I_{xz} and I_{yz} are always zero. In such cases we conclude that the moments M_x and M_y of all external forces with respect to these axes are also zero. Thus a rigid body rotating about a principal central axis does not produce dynamic reactions at its bearings.

Introducing for the components of angular momentum of a rotating body the notations

$$H_x = -I_{xz}\dot{\theta} \qquad H_y = -I_{yz}\dot{\theta} \qquad H_z = I_z\dot{\theta} \tag{f}$$

and taking their geometric sum, we obtain the resultant angular momentum $\bar{H}_0$ with respect to the fixed point O. In general this resultant angular momentum vector may change both in magnitude and direction and its rate of change is represented by the velocity of its end point. Then from Eqs. (80), we conclude that *the rate of change of the resultant angular momentum $\bar{H}_0$, considered vectorially, is equal to the resultant moment vector $\bar{M}_0$ of external forces.* Thus

$$\frac{d}{dt}\bar{H}_0 = \bar{M}_0 \tag{81}$$

This is a generalized statement of the principle of angular momentum for the case of a body or system of bodies rotating about a fixed axis.

In the particular case where a rigid body rotates about a principal central axis, the components H_x and H_y of angular momentum vanish and the component $H_z = I_z\dot{\theta}$ directed along the axis of rotation becomes the resultant angular momentum.

EXAMPLES

1. Prove that, if the shaft with two disks at the ends (Fig. 299) is twisted and then suddenly released, the disks in their torsional oscillations will rotate always in opposite directions if there is no friction in the bearings.

Solution. The angular momentum of the system with respect to the axis of rotation at the initial instant is zero, since the disks are at rest. The moment of external forces with respect to the same axis is zero, provided there is no friction and the centers of gravity of the disks are on the axis of rotation. In such case the rate of change of angular momentum of the system with respect to the axis of the shaft must be zero. Hence, if this angular momentum was initially zero, it continues to remain zero. This condition can be satisfied only if the disks are at rest or are moving always in opposite directions. If I_1 and I_2 are the moments of inertia of the disks with respect to the axis of the shaft and $\dot{\theta}_1$ and $\dot{\theta}_2$ are their angular velocities at any instant t, the equality to zero of the angular momentum of the system about the axis of rotation requires that

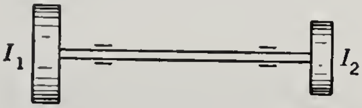


FIG. 299

$$I_1\dot{\theta}_1 + I_2\dot{\theta}_2 = 0$$

Hence

$$\dot{\theta}_1 : \dot{\theta}_2 = -I_2 : I_1$$

i.e., the angular velocities are of opposite sign and are inversely proportional to the moments of inertia of the corresponding disks. The angular displacements of the disks from their equilibrium positions are in the same ratio as the angular velocities. Hence the nodal cross section (see page 376) will have distances from the disks inversely proportional to their moments of inertia.

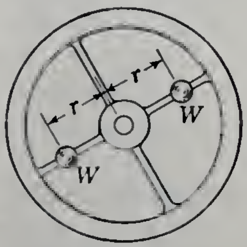


FIG. 300

2. A pulsating torque $M_0 \cos \omega t$ acts on a flywheel that is rotating with constant angular velocity ω. This uniformity of rotation is obtained by properly changing, by means of cams, the radial distances r of two equal weights W sliding along two spokes, as shown in Fig. 300. How should r change with the time?

Solution. From Eq. (78) we have

$$\frac{d}{dt}(I\omega) = M_0 \cos \omega t$$

Hence, observing that ω is constant, we obtain

$$\frac{dI}{dt} = \frac{M_0 \cos \omega t}{\omega} \tag{g}$$

If I_0 is the moment of inertia of the flywheel without the weights W, we can assume

$$I \approx I_0 + \frac{2W}{g} r^2 \tag{h}$$

Substituting (h) for I in Eq. (g), we obtain

$$\frac{2W}{g}\frac{d(r^2)}{dt} = \frac{M_0 \cos \omega t}{\omega}$$

from which, by integration,

$$r^2 = \frac{M_0 g}{2W\omega^2} \sin \omega t + C$$

Denoting by r_0 the initial value of r when $t = 0$, we find $C = r_0^2$ and the above equation gives, for the desired $r = f(t)$,

$$r = \sqrt{r_0^2 + \frac{M_0 g}{2W\omega^2} \sin \omega t}$$

3. As a device for measuring high velocities of bullets, a ballistic pendulum consisting of a heavy weight W, say a block of wood, suspended on a slender rod OA is used (Fig. 301). Establish the relation between the velocity v of the bullet striking the pendulum at its center of percussion P and the angular velocity $\dot{\theta}_0$ that the pendulum acquires owing to impact.

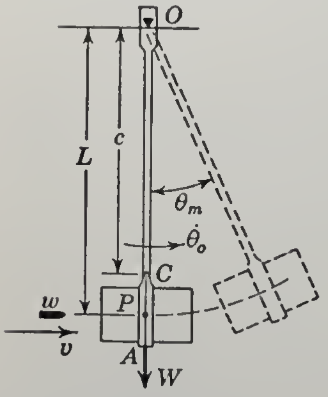


FIG. 301

Solution. If w is the weight of the bullet and I_0 the moment of inertia of the pendulum with respect to its axis of rotation through O, the initial angular momentum of the system consisting of the bullet and pendulum is, with respect to the axis of rotation, wvL/g.

At the end of impact the angular momentum is $I_0\dot{\theta}_0$. It is assumed that the duration of impact is very small in comparison with the period of oscillation of the pendulum, and the angle of rotation of the pendulum during impact is neglected. The mass of the bullet in comparison with that of the pendulum is also neglected, and the moment of inertia I_0 is assumed unchanged. Since the moment of external forces with respect to the axis O remains zero during impact, the angular momentum of the system must remain unchanged and we obtain for determining v the equation

$$\frac{w}{g} vL = I_0\dot{\theta}_0$$

from which

$$v = \frac{gI_0\dot{\theta}_0}{wL}$$

Observing that $I_0 = Wi_0^2/g$ while $L = i_0^2/c$, this reduces to

$$v = \frac{Wc\dot{\theta}_0}{w} \tag{i}$$

Instead of the initial angular velocity $\dot{\theta}_0$ of the pendulum, it is more convenient to measure the maximum angle of rotation θ_m. The relation between this angle and the angular velocity $\dot{\theta}_0$ will be discussed later (see Example 3, page 413).

4. A horizontal turntable (Fig. 302) carries a gun at A and rotates with initial angular velocity ω_0 about its vertical geometric axis. Calculate the increment of angular velocity $\Delta\omega$ that the turntable will obtain if the gun fires a bullet of mass m with tangential muzzle velocity v.

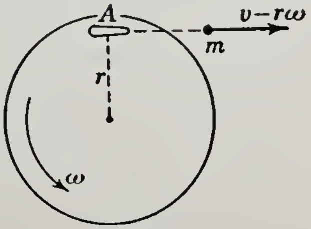


FIG. 302

Solution. Denoting by I_0 the combined moment of inertia of the turntable and gun, the initial angular momentum of the system with respect to the axis of rotation is

$$(I_0 + mr^2)\omega_0 \tag{j}$$

Denoting by ω the angular velocity of the disk after the gun has been fired, the final angular momentum of the system is

$$I_0\omega - m(v - r\omega)r \tag{k}$$

where $v - r\omega$ is the absolute velocity of the bullet to the right.

Equating expressions (j) and (k) in accordance with the law of conservation of angular momentum, we obtain

$$(I_0 + mr^2)(\omega - \omega_0) = mvr$$

from which

$$\Delta\omega = \omega - \omega_0 = \frac{mvr}{I_0 + mr^2} \tag{l}$$

We see that the increment of angular velocity imparted to the disk is independent of its initial angular velocity ω_0.

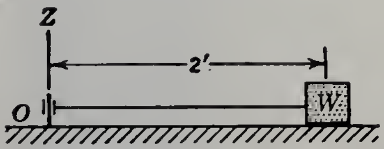


FIG. A

PROBLEM SET 8.9

1. In Fig. A, a wood block of weight $W = 5$ lb rests on a horizontal plane and is attached to a fixed hub O by a horizontal string as shown. A revolver bullet of weight $w = 1$ oz is fired horizontally into the center of the block in the direction normal to the plane of the figure with velocity $v = 2{,}000$ fps. Calculate the number of revolutions that the block will make around the vertical z axis before coming to rest if the coefficient of friction between the block and plane is $\mu = \frac{1}{4}$. Neglect the dimensions of the block. *Ans.* 3 revolutions.

2. A rotor A, initially at rest, is suddenly connected to a rotor B, having the initial angular velocity ω_b, by means of a clutch C (Fig. B). What common angular velocity will the two systems acquire if the moment of inertia of the system A is I_a and that of the system B, I_b? Neglect friction in the bearings. *Ans.* $\omega = \omega_b\,[I_b/(I_a + I_b)]$.

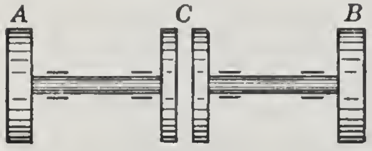


FIG. B

3. A prismatic bar AB of weight W_1 and length $2l_2$ carries two balls M and N of equal weights W and is rotating with uniform angular velocity ω_0 about the vertical z axis perpendicular to the bar and through its center of gravity (Fig. C). The balls are kept at equal distances l_1 from the axis of rotation by strings. What will be the new angular velocity of the system if the strings are suddenly cut and, owing to centrifugal forces, the balls come to the extreme positions P and Q with distances from the axis of rotation equal to l? The following numerical data are given: $\omega_0 = 2\pi\ \text{sec}^{-1}$, $W_1 = 5$ lb, $2l_2 = 70$ in., $W = 12\frac{1}{2}$ lb, $2l_1 = 30$ in., and $2l = 45$ in. Assume in calculation that the mass of the bar is distributed along its axis and the masses of the balls concentrated at their centers. *Ans.* $\omega_1 = 0.52\omega_0 = 3.27\ \text{sec}^{-1}$.

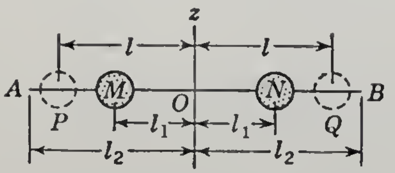


FIG. C

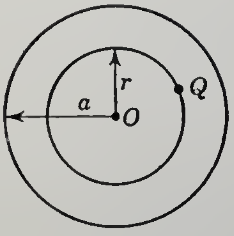


FIG. D

4. A horizontal circular disk of uniform thickness, radius a, and weight W can rotate freely about its vertical geometric axis (Fig. D). Along a circular track of radius r, a particle of weight Q is traveling. Initially the disk and the particle are at rest. What angular velocity ω will the disk acquire if the particle begins to travel along the track with a constant speed v relative to the disk? Assume $W = 10$ lb, $Q = 1$ lb, $a = 10$ in., $r = 6$ in., $v = 4$ fps. *Ans.* $\omega = 0.537\ \text{sec}^{-1}$.

5. Owing to cooling, the linear dimensions of a solid homogeneous sphere, uniformly rotating about a diameter, diminish by 0.1 per cent. What effect will this have on the angular velocity if there is no external moment with respect to the diameter of the sphere? *Ans.* The angular velocity increases by 0.2 per cent.

6. A man weighing 160 lb steps onto the edge of a horizontal circular turntable initially at rest, walks five times around the circumference, and steps off at the same spot on the ground from which he started without having passed it previously. Assuming a perfectly ideal setup and treating the turntable as a solid right circular disk, calculate its weight W. *Ans.* $W = 80$ lb.

7. A horizontal turntable initially at rest carries two men A and B at its circumference. To bring the turntable into rotation, the men run with velocity v relative to the disk and then jump off tangentially. Prove that, if B stands still while A runs and jumps and then subsequently does likewise, the disk will acquire a greater final angular velocity ω than if both men run and jump simultaneously.

8.10. Energy equation for rotating bodies. If a body rotates about a fixed axis (Fig. 303) with angular velocity $\dot{\theta}$, a particle of mass dm at point A describes, during rotation, a circle of radius r and its velocity along this path is $r\dot{\theta}$. Hence the kinetic energy of the particle is

$$dm \frac{(r\dot{\theta})^2}{2} \tag{a}$$

The total kinetic energy T of the rotating body is obtained by summing up expressions (a) for all particles of the body. Thus

$$T = \int \frac{dm}{2} (r\dot{\theta})^2$$

Observing that the angular velocity $\dot{\theta}$ is the same for all particles, we can write

$$T = \frac{\dot{\theta}^2}{2} \int r^2\, dm = \frac{I\dot{\theta}^2}{2} \tag{b}$$

where $I = \int r^2\, dm$ is the moment of inertia of the body with respect to its axis of rotation. Comparing Eq. (b) here with Eq. (a) on page 299, we see that the expression for kinetic energy of rotation has a form similar to that for kinetic energy of translation.

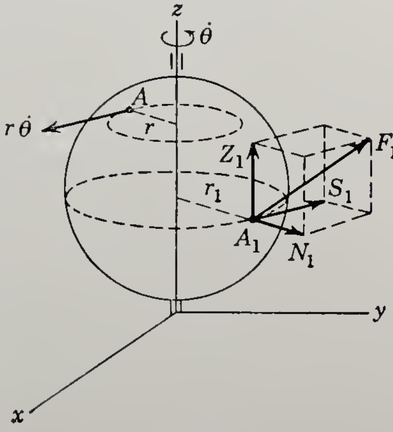


FIG. 303

In discussing curvilinear motion of a particle (see Art. 7.6), it was shown that the change in kinetic energy of a particle during a small displacement ds is equal to the work done by all forces acting on the particle during the same displacement. Summing up such changes in kinetic energy for all particles in a rotating body and also the corresponding work of all forces acting on these particles, we conclude that the change in the kinetic energy of the entire body during a small angle of rotation $d\theta$ is equal to the corresponding work done by all external[1] forces acting on the body.

---

[1] In the case of a rigid body in which the distances between particles do not change, the net work of all internal forces is zero.

In calculating the work done by external forces during rotation, we resolve any force F_1 acting at A_1 into three components S_1, N_1, Z_1, as shown in Fig. 303. During rotation of the body through a small angle $d\theta$, point A_1 moves the distance $ds = r_1\,d\theta$ and we see that only the tangential component S_1 produces work. This work, equal to the product of the force and the corresponding displacement of its point of application, is

$$S_1 \cdot r_1\,d\theta = S_1 r_1 \cdot d\theta$$

We see that this may be interpreted simply as the moment of the force F_1 with respect to the axis of rotation multiplied by the corresponding angular displacement $d\theta$ around this axis. From this, we conclude that the total work of all external forces is

$$M\,d\theta \tag{c}$$

where M is the resultant moment with respect to the axis of rotation.

Using expressions (b) and (c), the above-stated conclusion regarding the relation between change in kinetic energy of a rigid body during a small angle of rotation $d\theta$ about a fixed axis and the corresponding work of all external forces may be expressed as follows:

$$d\left(\frac{I\dot{\theta}^2}{2}\right) = M\,d\theta$$

Then by integration, we obtain

$$\frac{I\dot{\theta}^2}{2} - \frac{I\dot{\theta}_0^2}{2} = \int_{\theta_0}^{\theta} M\,d\theta \tag{82}$$

The right side of this equation represents the work produced by all external forces during a finite angle of rotation from θ_0 to θ in which the angular velocity of the body changes from $\dot{\theta}_0$ to $\dot{\theta}$. Equation (82) then states that the total change in the kinetic energy of the rotating body during any angle of rotation is equal to the total work done by the external forces during the same angle of rotation.

EXAMPLES

1. If the slender prismatic bar in Fig. 304 is released from rest in the horizontal position AB and allowed to fall under the influence of gravity, what angular velocity $\dot{\theta}$ will it acquire by the time it reaches the vertical position AB_1?

Solution. The final kinetic energy of the bar in the vertical position will be

$$\frac{W}{g}\frac{l^2}{3}\frac{\dot{\theta}^2}{2} \tag{d}$$

The initial kinetic energy in the horizontal position is zero; hence expression (d) represents also the change in kinetic energy.

The gravity force W is constant in both magnitude and direction. Hence the work done is simply

$$W\frac{l}{2} \tag{e}$$

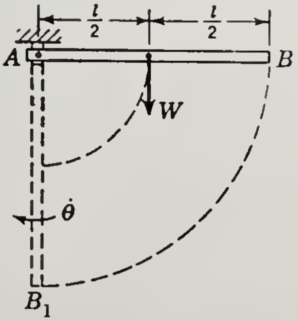


FIG. 304

Equating change in kinetic energy (d) to work (e), we obtain

$$\dot{\theta} = \sqrt{\frac{3g}{l}} \tag{f}$$

2. Determine the velocity v of the falling weight Q of the system shown in Fig. 305 as a function of its displacement from the initial position of rest.

Solution. The initial kinetic energy of the system is zero. After the weight Q has fallen a distance x and acquired a velocity v and the cylinder has acquired an angular velocity $\dot{\theta} = v/r$, the total kinetic energy of the system is

$$\frac{Q}{g}\frac{v^2}{2} + \frac{W}{g}\frac{r^2}{2}\frac{v^2}{2r^2} = \frac{v^2}{2g}\left(Q + \frac{W}{2}\right) \tag{g}$$

which, in this case, also represents the change in kinetic energy. Neglecting friction at the bearings, the corresponding work of all external forces acting on the system is

$$Qx \tag{h}$$

Equating change in kinetic energy (g) to work (h), we obtain

$$\frac{v^2}{2g}\left(Q + \frac{W}{2}\right) = Qx$$

from which

$$v = \sqrt{2gx\frac{Q}{Q + \frac{1}{2}W}}$$

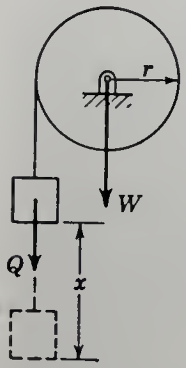


FIG. 305

3. If, owing to the impact of a bullet, the ballistic pendulum shown in Fig. 301 swings up through an angle θ_m before coming to rest, find the muzzle velocity v of the bullet. Assume that the properties of the pendulum and the weight w of the bullet are given.

Solution. Denoting by $\dot{\theta}_0$ the initial angular velocity imparted to the pendulum due to impact of the bullet, the initial kinetic energy of the pendulum is $I_0\dot{\theta}_0^2/2$. When the pendulum reaches its extreme position, indicated in the figure by dotted lines, its kinetic energy is zero. Thus the left side of Eq. (82) becomes $-I_0\dot{\theta}_0^2/2$. In calculating the work done by external forces, which in this case are the gravity force and the reaction at the axis of rotation, we note that the moment of these forces with respect to the axis of rotation, for any angle of rotation θ, is $-Wc \sin \theta$, where the minus sign indicates that the moment acts in the direction opposite to the direction of increasing θ. Thus the total work done during the angle of rotation θ_m is

$$-\int_0^{\theta_m} Wc \sin \theta \, d\theta = -Wc(1 - \cos \theta_m)$$

Substituting this in Eq. (82), we obtain

$$\frac{I_0 \dot{\theta}_0^2}{2} = Wc(1 - \cos \theta_m)$$

from which

$$\dot{\theta}_0 = \sqrt{\frac{2gc(1 - \cos \theta_m)}{i_0^2}}$$

where i_0 is the radius of gyration of the pendulum with respect to its axis of rotation. Substituting this expression for $\dot{\theta}_0$ into the formula for v previously obtained in Example 3, page 408, we find

$$v = \frac{Wc}{wi_0} \sqrt{2gc(1 - \cos \theta_m)}$$

From this formula the muzzle velocity v of the bullet can be calculated if the maximum angle of rotation θ_m of the pendulum is determined experimentally.

PROBLEM SET 8.10

1. The wheel of a small gyroscope is set spinning by pulling on a string wound around the shaft. The moment of inertia of the wheel is $I = 0.045$ lb-sec²-in. and the diameter of the shaft on which the string is wound is $\frac{1}{2}$ in. If 36 in. of string is pulled off with a constant force of 12 lb, what rpm will be imparted to the wheel? *Ans.* $n = 1{,}324$ rpm.

2. A slender prismatic bar of total length $2l$ and weight $2W$ is bent in a right angle at its middle point O and hangs from a hinge at this point as shown in Fig. A. Find the maximum angular velocity $\dot{\theta}$ that the bar will acquire if released from rest in the position shown. *Ans.* $\dot{\theta}_{\max} = \sqrt{0.621 g/l}$.

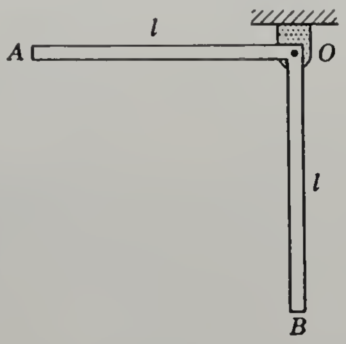


FIG. A

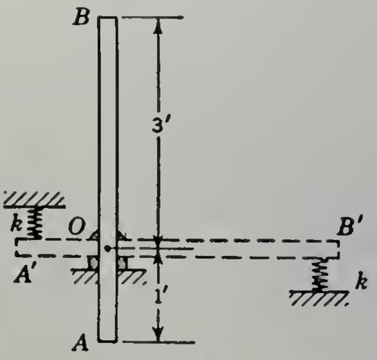


FIG. B

3. In Fig. B, a slender prismatic bar AB is free to rotate in a vertical plane about a fixed axis through O. The bar is released from rest in the unstable position shown by solid lines and falls into the horizontal position $A'B'$, where it is brought to rest by two identical springs having constants k. If the spring at A' is compressed $\frac{1}{2}$ in., before the bar comes to rest, what is the magnitude of the spring constants k? The weight of the bar is $W = 8$ lb. Neglect mass of the springs. *Ans.* $k = 76.8$ lb/in.

4. If the slender prismatic bar AB shown in Fig. C is released from rest in the horizontal position and rotates about A under the influence of gravity, find the reaction at A as it swings through the vertical position. *Ans.* $R_a = 5W/2$.

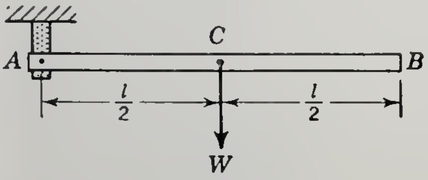


FIG. C

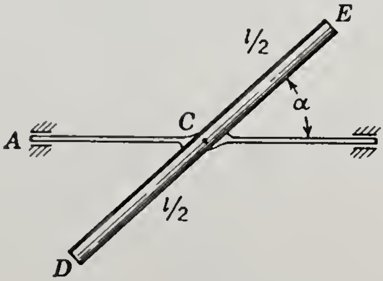


FIG. D

5. A slender prismatic bar DE of length l and weight W is rigidly attached at its mid-point C to a shaft AB and makes therewith an angle α as shown in Fig. D. Determine the kinetic energy T of the bar if it rotates with constant angular velocity ω about AB. The following numerical data are given: $W = 20$ lb, $l = 40$ in., $\alpha = 45°$, and the uniform angular speed is 200 rpm. *Ans.* $T = 755$ in.-lb.

6. In Fig. E, a slender prismatic bar OA of length $l = 3$ ft and weight $W = 3$ lb rotates as a conical pendulum about a vertical axis through the hinge O. If it has such uniform angular velocity ω that the angle $\alpha = 30°$, calculate the kinetic energy T of the bar for this steady state of motion. *Ans.* $T = 0.650$ ft-lb.

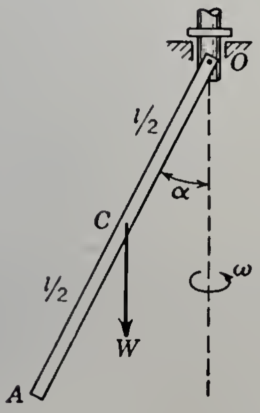


FIG. E

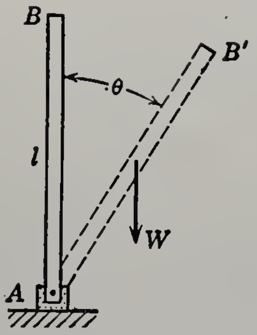


FIG. F

7. A homogeneous slender prismatic bar AB of length l and weight W is hinged at its lower end A and can rotate freely in the vertical plane as shown in Fig. F. If the bar is released from rest in the unstable position and falls

over under the influence of gravity, calculate the horizontal component of the reaction at A when $\theta = 30°$. *Ans.* $X_a = 0.225W$.

8. What initial angular velocity ω should be given to the bar AB in Fig. F in the top position in order that it will swing through the lowest position (B vertically below A) with angular velocity 2ω? *Ans.* $\omega = \sqrt{2g/l}$.

9. In Fig. G, a rotor of weight $W = 386$ lb and radius of gyration $i_0 = 4$ in. is mounted on a horizontal shaft and set in rotation by a falling weight $W = 386$ lb, as shown. If the system is released from rest, find the velocity of the block after it has fallen through a distance of 10 ft. *Ans.* $v = 11.35$ fps.

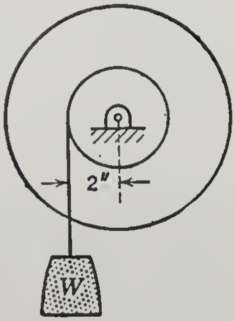


FIG. G

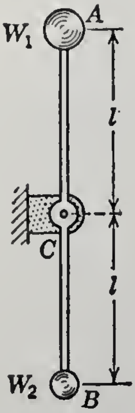


FIG. H

10. A slender weightless rod pivoted at C has balls of weights $W_1 = 5$ lb and $W_2 = 3$ lb attached at its ends, as shown in Fig. H, and is initially at rest in the unstable position shown in the figure. If the system is disturbed slightly, the bar begins to rotate about C. Find the horizontal and vertical components of the reaction at C as the bar passes through the horizontal position. *Ans.* $X_c = -1.00$ lb; $Y_c = +7.50$ lb.

11. A homogeneous plate 1 ft square is supported in a vertical plane as shown in Fig. I. If the pin at B is removed, what angular velocity ω will the plate acquire by the time the diagonal AC becomes vertical? *Ans.* $\omega = 4.47$ radians/sec.

12. Referring again to Fig. I, calculate the vertical reaction R_a for the instant when the diagonal AC of the square is vertical. *Ans.* $R_a = 1.44W$.

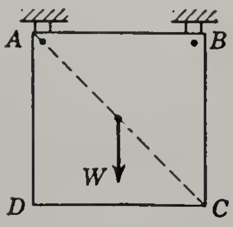


FIG. I

13. The ballistic pendulum shown in Fig. 301, page 408 has a total weight $W = 20$ lb, and the distance from its center of gravity to the center of suspension is found by experiment to be $c = 7.63$ ft. Also its observed period of oscillation for small amplitudes is $\tau = 3.22$ sec. When the pendulum is hanging in its position of stable equilibrium, a rifle bullet of weight $w = 1$ oz is fired horizontally into the block at the center of oscillation, and as a result of the impact the pendulum is observed to swing through an angle $\theta_m = 26°40'$ before coming to rest. Find the muzzle velocity v of the bullet. *Ans.* $v = 2{,}200$ fps.

8.11. Gyroscopes. A *gyroscope* usually has the form of a solid of revolution mounted on an axle coinciding with its geometric axis. If such a body rotates about this axis of symmetry (*axis of spin*) with a high angular velocity, it possesses certain dynamic characteristics that are of practical interest. In discussing these characteristics, it is advantageous to use the principle of angular momentum and the idea of vectorial representation of resultant angular momentum with respect to a fixed point and resultant moment of external forces with respect to the same point as discussed in Art. 8.9. These two vectors, as we have already seen, usually do not coincide, but the vector representing the moment of force gives the magnitude and direction of the rate of change of the vector representing the angular momentum.

If the axis of spin of a gyroscope is fixed in direction and we denote by I_z and ω the moment of inertia and angular velocity with respect to this axis, the relation between the corresponding angular momentum $I_z\omega$ and moment M_z of external forces is (see p. 404)

$$\frac{d}{dt}(I_z\omega) = M_z \tag{83}$$

It follows from the discussion of Art. 8.9 that the angular momentum with respect to any axis perpendicular to and intersecting the axis of spin is zero, since, by virtue of symmetry, the axis of spin is a principal central axis of inertia of the gyroscope. From this it can be concluded that the angular momentum $I_z\omega$ with respect to this axis is the resultant angular momentum, which can be represented by a vector directed along the fixed axis of spin in accordance with the right-hand-screw rule. If there is no friction in the bearings and otherwise no moment of forces about the z axis, we conclude from Eq. (83) that this resultant angular momentum vector maintains a constant magnitude; i.e., the gyroscope maintains a constant angular velocity of spin.

Now let us consider the case where the axis of spin of a gyroscope is not fixed but can change its direction in space by free rotation around a fixed point, for it is under such conditions that the dynamic characteristics referred to above manifest themselves. As an example of a gyroscope the axis of spin of which is free to rotate in any way about a fixed point O, consider a top spinning about its geometric axis OA with angular velocity ω and supported at a point O on this axis (Fig. 306). Then for reference we take through the point O a system of fixed rectangular coordinate axes, x, y, z, the z axis of which coincides with the instantaneous position of the axis of spin of the gyroscope as shown in the figure.

Now when the axis of spin is changing its direction through point O, there will be some rotation of the gyroscope with respect to the x and y axes, and the angular momenta with respect to these axes will no longer be zero. This fact complicates the calculation of the resultant angular momentum of the gyroscope with respect to the fixed point O. To simplify the problem, we shall assume that the angular velocity of spin ω is very large compared with the rate of change of the direction of the moving axis of spin and also that the moment of inertia I of the gyroscope with respect to its axis of spin is not small compared with its moments of inertia with respect to the x and y axes. In such case, the angular momentum of the gyroscope with respect to its axis of spin will be very large in comparison with its angular momenta with respect to the x and y axes and these latter momenta can be neglected. Thus it can be assumed with good accuracy that the angular momentum $I\omega$, as represented by the vector $\overline{OM}$ directed along the moving axis of spin, is the resultant angular momentum. If there is no friction and no other force giving moment about the moving axis of spin, this resultant angular momentum remains constant in magnitude, as in the preceding case, where the axis of spin was fixed.

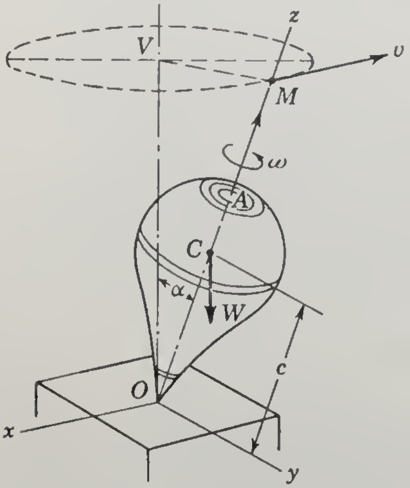


FIG. 306

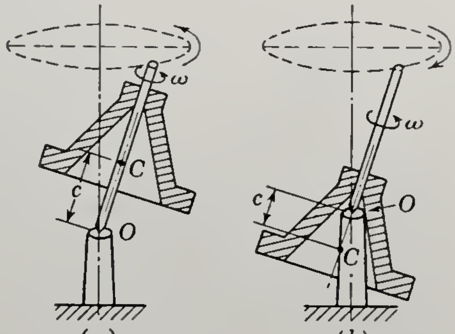
Owing to the weight W of the top acting at its center of gravity C and the reaction at O, we obtain a couple in the plane OVM. The moment of this couple is equal to $Wc \sin \alpha$, and it is represented by a vector perpendicular to the plane OVM and directed away from the reader, i.e., in the negative direction of the x axis. Applying now the principle of angular momentum, we conclude that the end M of the vector $\overline{OM}$ must have a velocity, representing a rate of change of the angular momentum $I\omega$, which is equal to the vector representing the moment of force $Wc \sin \alpha$, that is, perpendicular to the plane OVM so that M is traveling along a horizontal circle of radius

$$\overline{OM} \sin \alpha = I\omega \sin \alpha \tag{a}$$

Hence the axis of spin of the top describes a circular cone of the angle 2α. The angular velocity Ω of the plane OVM rotating around the vertical axis OV is obtained by dividing the velocity $Wc \sin \alpha$ of the point M by the radius $I\omega \sin \alpha$ of the circle. Thus we obtain

$$\Omega = \frac{Wc \sin \alpha}{I\omega \sin \alpha} = \frac{Wc}{I\omega} \tag{b}$$

This angular velocity with which the vertical plane containing the axis of spin of the top revolves is called the angular velocity of *precession* of the top. We see that it is inversely proportional to the angular velocity ω and directly proportional to the distance c of the center of gravity above the support. When c is positive, that is, when the center of gravity of the top is above its support (Fig. 307a), we see that Ω is positive and the top precesses in the same direction that it is spinning. When c is negative, that is, when the center of gravity is below the support (Fig. 307b), Ω is negative and the top precesses in the direction opposite to its direction of spin.



(*a*) (*b*)

FIG. 307

If we support a gyroscope at its center of gravity, the distance c in Eq. (b) becomes zero and the axis of spin of the gyroscope remains immovable in space. This fact was utilized by Foucault to prove the existence of rotation of the earth about its axis. If the axis of spin of a gyroscope supported at its center of gravity is directed toward a fixed star, it retains this direction, and to an observer the axis of the gyroscope appears to rotate describing a cone around the axis directed toward the polar star.

The same characteristic of a spinning body to retain the direction of the axis of rotation is utilized when a high angular velocity of spin about the longitudinal axis is given to a bullet by rifling the barrel of the gun. Owing to air resistance eccentrically applied, some moment will be produced with respect to the axis through the center of gravity of the bullet and perpendicular to the vertical plane containing the initial velocity. This moment produces a gradual change in the direction of the axis of spin of the bullet, which is responsible for some deviation of the trajectory from the above-mentioned vertical plane.

EXAMPLES

1. A homogeneous circular disk rotates with a high angular velocity ω about its horizontal geometric axis, which is supported in bearings A and B (Fig. 308). If the axis AB is fixed and the disk is perfectly balanced, the reactions at the bearings are due to the gravity force alone and can easily be calculated by using equations of statics. What additional reactions will exist at the bearings A and B if the frame supporting these bearings rotates about a vertical axis through the center of gravity of the disk with the angular velocity Ω as indicated in the figure? Assume that the angular velocity Ω is small compared with ω.

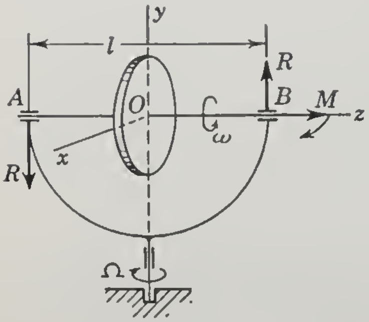


FIG. 308

Solution. In calculating the additional pressures at the bearings due to rotation of the frame, we use the principle of angular momentum. Taking fixed rectangular coordinate axes x, y, z, with origin O at the center of gravity of the disk as shown and the z axis coinciding with an instantaneous position of the axis of spin, we see that the angular momentum of the disk with respect to the y axis is small compared with its angular momentum with respect to the z axis, since ω is large compared with Ω. Thus we are justified in assuming that the angular momentum $I\omega$ with respect to the axis of spin is the resultant angular momentum. With good accuracy the resultant momentum can be represented by the vector $\overline{OM}$ equal to $I\omega$ and directed along the z axis in accordance with the right-hand-screw rule. Then, owing to rotation of the frame around the vertical y axis, the end M of this vector describes a horizontal circle of radius $OM = I\omega$. The velocity of the end M of this vector, perpendicular to the yz plane and equal to $I\omega\Omega$, represents the rate of change of the angular momentum $I\omega$. If the frame has the direction of rotation indicated in the figure, this velocity has the positive direction of the x axis. Then the vector representing the resultant moment of external forces, which in this case are additional reactions from the bearings due to rotation about the y axis, must have the same direction. Hence we conclude that the reactions represent a couple in the yz plane as shown in the figure and of such magnitude that

$$Rl = I\omega\Omega \tag{c}$$

from which

$$R = \frac{I\omega\Omega}{l} \tag{d}$$

By putting the bearings A and B on pistons of hydraulic cylinders, the additional forces R, due to the angular velocity Ω, can be determined by measuring the change of oil pressure in the supporting cylinders. In this way formula (*d*) can be found to be in good agreement with actual conditions.

2. The armature of an electric motor has its axis of spin perpendicular to the longitudinal axis of a rolling ship (Fig. 309). The angular velocity of spin of the rotor is ω, and the rolling motion of the ship is assumed to be represented by the equation $\theta = \theta_0 \cos pt$. Find the maximum magnitude of the additional pressures on the bearings of the rotor due to rolling of the ship.

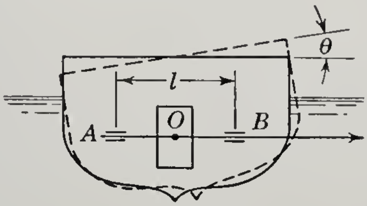


FIG. 309

Solution. Since the rolling motion of a ship has a fairly large period, we are justified again in this case in assuming that the angular momentum $I\omega$ of the rotor with respect to its axis of spin is the resultant angular momentum. This angular momentum is represented by the vector $\overline{OM}$ directed along the axis of spin as shown in the figure. During rolling of the ship the end M of this vector describes the arc of a circle of radius $OM = I\omega$ and its velocity along this path is determined by multiplying this radius by the angular velocity of rolling $\dot{\theta} = -\theta_0 p \sin pt$. Hence the maximum velocity of point M has the magnitude $I\omega\theta_0 p$. This maximum velocity of the point M represents the maximum rate of change of the angular momentum $I\omega$ of the rotor and is equal to the maximum moment of the additional reactions at the bearings due to rolling. Since it occurs when $\sin pt = 1$, $\cos pt = 0$ and the angle of roll $\theta = 0$. At such instants the velocity of point M is vertical, and hence the additional reactions at the bearings represent a couple in the horizontal plane. The magnitude of each reaction is found from the equation

$$R_{\max} l = I\omega\theta_0 p$$

from which

$$R_{\max} = \frac{I\omega\theta_0 p}{l} \tag{e}$$

PROBLEM SET 8.11

1. Calculate the maximum value of the reactions discussed in Example 2 above if the maximum angle of roll of the ship is 15°, the period of roll 30 sec., the speed of the rotor 1,800 rpm, $I = 25$ lb-sec^2-ft, and $l = 36$ in. *Ans.* $R_{\max} = 86.1$ lb.

2. A locomotive travels with constant speed v around a curve of radius r. Due to gyroscopic action of a pair of drivers, there will be an increase of pressure on the outside rail and a corresponding decrease on the inside rail. Evaluate this change in rail pressures if I_z denotes the moment of inertia for a pair of drivers with respect to their axle and a and l are the radius of a driver and the distance between rails, respectively. *Ans.* $R = I_z v^2/arl$.

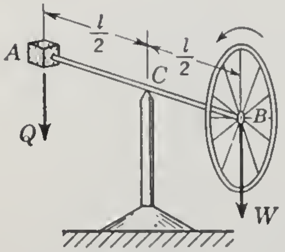


FIG. A

3. A horizontal axle AB of length $l = 36$ in. is pivoted at its mid-point C and carries at A a weight $Q = 5$ lb and at B a spinning wheel of weight $W = 10$ lb, the mass of which is uniformly distributed around the rim of radius $r = 12$ in. (Fig. A). If the velocity of spin of the wheel is 600 rpm in the direction shown in the figure, calculate the angular velocity of precession Ω of the system around the vertical axis through C. *Ans.* $\Omega = 0.384$ radians/sec.

4. The armature of the motor of an electric car weighs 600 lb and rotates in a direction opposite to the rotation of the car wheels. The distance between its bearings is 2 ft, and its radius of gyration is 6 in. The motor makes 4 revolutions to 1 revolution of the car wheels, which have diameters of 33 in. If the car is moving forward around a curve of 100-ft radius with a velocity of 20 fps, what are the total pressures on the bearings of the armature if the center of the curve is to the right? *Ans.* Left bearing, 273 lb; right bearing, 327 lb.

5. A pair of locomotive drivers 6 ft. in diameter weigh 5,000 lb and have a radius of gyration of $2\frac{1}{2}$ ft. Compute the additional gyroscopic pressure on the outer rail as the locomotive travels around a 6° curve at 45 mph. The rails are 60 in. apart. A 6° curve is one for which a 100-ft chord subtends a central angle of 6°. *Ans.* 295 lb.

6. The armature of a big generator located in Philadelphia (latitude $\phi = 40°$N) has moment of inertia $I = 400$ lb-sec^2-in. and rotates at 1,800 rpm so that the positive end of its horizontal axis of spin points due north. Calculate the magnitude of the gyroscopic moment due to rotation of the earth. *Ans.* $M = 3.52$ in.-lb tending to increase north-bearing reaction.

7. Calculate the tension S induced in the vertical drive shaft O_1O_2 of the rolling mill shown in Fig. B as a result of gyroscopic action of the rollers. The angular velocity around the axis O_1O_2 is Ω, and the rollers do not slip on the horizontal plane. Consider each roller as a right circular disk of weight W and radius r. *Ans.* $S = (W/g)\,\Omega^2 r$.

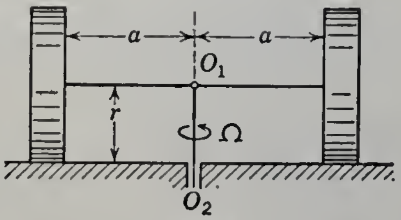


FIG. B

8. An airplane with a three-blade propeller (Fig. C) makes a turn of 1,000-ft radius while traveling at 450 mph. If the propeller, rotating at

1,590 rpm, has weight $W = 100$ lb and radius of gyration $i_c = 3.0$ ft, what bending moment M is created in the propeller shaft during the turn? *Ans.* $M = 3{,}060$ ft-lb.

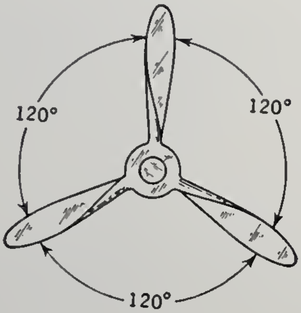


FIG. C

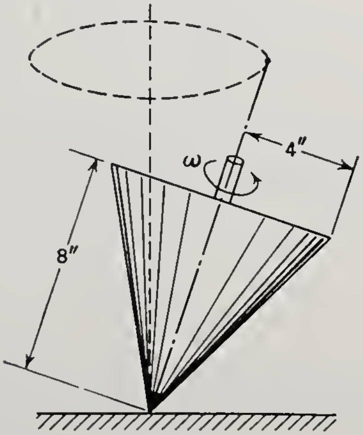


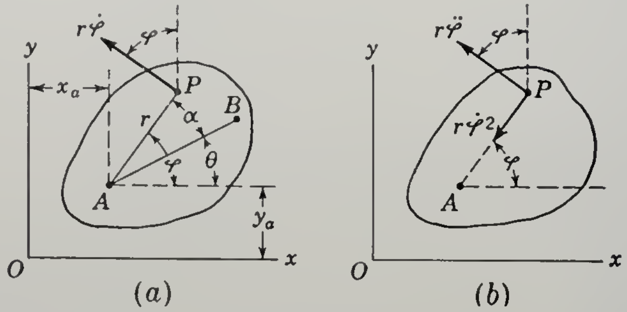
FIG. D

9. A spinning top weighing 1 lb and having the form of a solid cone with dimensions as shown in Fig. D, is observed to precess at the constant rate of 1 rps. What is its rate of spin? *Ans.* 733 rpm.

9

PLANE MOTION OF A RIGID BODY

9.1. Kinematics of plane motion. In engineering problems of dynamics, we often encounter such motion of a rigid body that a certain plane in the body always remains in a fixed plane. Under such conditions, the path of each particle of the body is a plane curve parallel to the fixed plane, and the body is said to have *plane motion.* Plane motion of a rigid body is easily demonstrated by rolling a right circular cylinder or disk along a flat surface in such a way that its geometric axis always remains parallel to itself. Such motion of a rigid body can be completely defined by the motion of a plane figure representing the projection of the body on a fixed plane normal to the axis of rotation. Further, the position of this plane figure is completely defined by the position of one point in it together with the angle of rotation of the figure about an axis through that point and perpendicular to the plane of motion. Thus the position of the plane figure shown in Fig. 310a is completely defined by the coordinates x_a, y_a of a point A and the angle of rotation θ that a line AB in the figure makes with the fixed x axis. The arbitrarily chosen point A in the figure is called a *pole.* As the figure moves in the xy plane, the coordinates x_a, y_a, and the



(a) (b)

FIG. 310

angle θ are changing and the motion is completely defined if we know

$$x_a = f_1(t) \qquad y_a = f_2(t) \qquad \theta = f_3(t) \tag{a}$$

If Eqs. (a) defining any plane motion of a rigid body are given, the velocity and acceleration of any point in the body can easily be found. In the case of the body represented in Fig. 310*a*, for example, let us consider a point P the position of which in the body is defined by its distance r from the chosen pole A and the angle α that the line AP makes with the selected reference line AB. Then using the notation $\varphi = \theta + \alpha$, we see that the coordinates x and y of the particle P are

$$x = x_a + r \cos \varphi \qquad y = y_a + r \sin \varphi \tag{b}$$

Differentiating each of these expressions once with respect to time and remembering that r is constant while x_a, y_a, and φ are changing with time, we find

$$\dot{x} = \dot{x}_a - r\dot{\varphi} \sin \varphi \qquad \dot{y} = \dot{y}_a + r\dot{\varphi} \cos \varphi \tag{c}$$

From these equations we see that the $\dot{x}$ and $\dot{y}$ projections of the velocity of P are obtained by adding algebraically to the corresponding projections of the velocity of the pole A the projections of the velocity $r\dot{\varphi}$ due to rotation of the body around point A. Thus, if Eqs. (a) defining the motion of a rigid body in a plane are given, the velocity of any point P in the body is obtained as the geometric sum of the velocity of the chosen pole A and the relative rotational velocity of P with respect to A. This may be expressed by the vector equation

$$\bar{v}_p = \bar{v}_a \nrightarrow \bar{v}_{p/a} \tag{84}$$

where the sign $\nrightarrow$ means that the vectors $\bar{v}_a$ and $\bar{v}_{p/a}$ are to be added geometrically.

Let us consider now the acceleration of the point P of the body shown in Fig. 310. Differentiating Eqs. (c) with respect to time, we obtain

$$\begin{aligned} \ddot{x} &= \ddot{x}_a - r\dot{\varphi}^2 \cos \varphi - r\ddot{\varphi} \sin \varphi \\ \ddot{y} &= \ddot{y}_a - r\dot{\varphi}^2 \sin \varphi + r\ddot{\varphi} \cos \varphi \end{aligned} \tag{d}$$

The first term on the right side of each of these equations represents a corresponding projection of the acceleration of the arbitrarily chosen pole A. The second term in each case is seen to represent the projection on the corresponding coordinate axis of the normal component $r\dot{\varphi}^2$ of the relative acceleration of P with respect to A, as indicated in Fig. 310*b*. Likewise the last term in each equation represents the corresponding projection of the tangential component $r\ddot{\varphi}$ of the relative acceleration of P with respect to A. Thus, we conclude from Eqs. (d) that, if Eqs. (a), defining the plane motion of a rigid body, are given, the acceleration of any point P of the body is obtained as the geometric sum of the acceleration of the pole A and the relative acceleration of P with respect to A. Expressed in equation form, we have

$$\bar{a}_p = \bar{a}_a \mathrel{+\!\!\!\!\!\rightarrow} \bar{a}_{p/a} \tag{85}$$

Applications of Eqs. (84) and (85) to various problems of plane motion will now be illustrated by the following examples.

EXAMPLES

1. A right circular cylinder rolls without slipping along a horizontal plane AB, and its center has at a certain instant a velocity v_c, as shown in Fig. 311a. Find the velocities at the same instant of the points D and E on the rim of the cylinder.

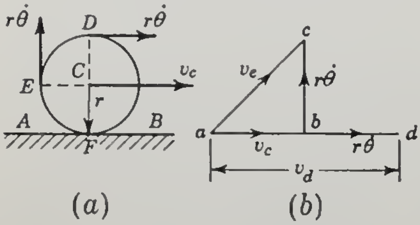


FIG. 311

Solution. Since the cylinder rolls without slipping, the velocity of point F in contact with the plane must be zero. Thus using Eq. (84) with point C as a pole, we have $v_f = v_c - r\dot{\theta} = 0$ from which $\dot{\theta} = v_c/r$. Then owing to rotation the points D and E have, with respect to the moving pole C, relative velocities $r\dot{\theta} = rv_c/r = v_c$ directed as shown in Fig. 311a. Adding geometrically each of these relative velocities to the velocity v_c of the pole C, we obtain the vectors $\overline{ad}$ and $\overline{ac}$ as shown in Fig. 311b and representing, respectively, the velocities of points D and E. From Fig. 311b, we see that the velocity of point D has the magnitude $2v_c$ and the velocity of point E, the magnitude $\sqrt{2}v_c$.

2. A prismatic bar AB has its ends A and B constrained to move horizontally and vertically as shown in Fig. 312a. If the end A of the bar moves with constant velocity v_a, find the angular velocity $\dot{\theta}$ of the bar and the velocity v_b of the end B for the instant when the axis of the bar makes the angle θ with the horizontal x axis.

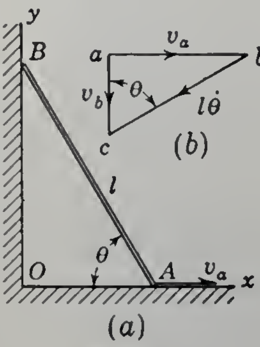


FIG. 312

Solution. Considering the end A of the bar as a pole, we know that the velocity of the end B is equal to the geometric sum of the velocity v_a of point A and the relative velocity $l\dot{\theta}$ of point B with respect to A as a center of rotation. Then to construct the diagram of vectors (Fig. 312b), we begin by laying out the horizontal vector $\overline{ab}$ representing the given velocity v_a of the point A. Since the relative velocity of B with respect to A is due to rotation of the bar around A as a center, it must have the direction perpendicular to the axis of the bar, and since B is constrained to move only vertically, its total velocity must be vertical. Thus the directions of the vectors $\overline{bc}$ and $\overline{ac}$ are both known and the diagram of vectors is completed. From this diagram we find

$$v_b = v_a \cot\theta \qquad \dot{\theta} = \frac{v_a}{l}\csc\theta \qquad \text{(e)}$$

3. If the crank OA of the engine shown in Fig. 313a has a length $r = 10$ in. and a constant angular velocity $\omega = 60\pi$ sec^{-1} and the connecting rod AB has a length $l = 24$ in., find graphically the velocity $\dot{y}_b$ and acceleration $\ddot{y}_b$ of the piston B at the instant when the angle BOA is equal to 45°.

Solution. The diagram of the engine is first constructed carefully to scale as shown in Fig. 313a. Taking point A as a pole, the vector diagrams for determining the velocity $\dot{y}_b$ and acceleration $\ddot{y}_b$ of the piston are shown in Figs. 313b and 313c, respectively. From these diagrams, we find $\dot{y}_b = 1{,}740$ ips and $\ddot{y}_b = 258{,}000$ in./sec^2.

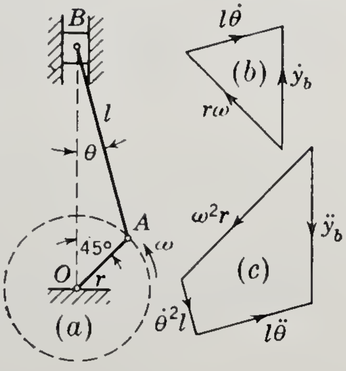


FIG. 313

PROBLEM SET 9.1

1. A locomotive runs along a straight level track with constant acceleration $a = 0.2g$. Find the total acceleration of a point at the top of the rim of a driver of radius $r = 3$ ft when the speed of the locomotive is 15 mph. *Ans.* 161.6 ft/sec^2.

2. A circular roller of radius $a = 8$ in. is contacted at the top and bottom points of its circumference by two conveyor belts AA and BB as shown in Fig. A. If the belts run with uniform speeds $v_1 = 6$ fps and $v_2 = 4$ fps, find the linear velocity v_c of the roller and also its angular velocity ω. *Ans.* $v_c = 5$ fps; $\omega = 1.5$ sec^{-1}.

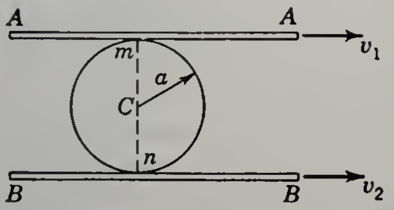


FIG. A

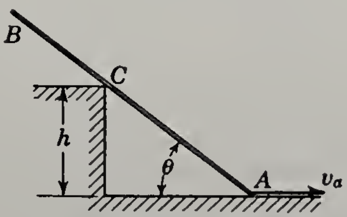


FIG. B

3. If the end A of the prismatic bar shown in Fig. B moves with constant horizontal velocity v_a, find the angular velocity $\dot{\theta}$ of the bar as a function of the angle θ that its axis makes with the horizontal. *Ans.* $\dot{\theta} = (v_a/h)\sin^2\theta$.

4. In Fig. C the end A of a slender bar AB is constrained to follow the horizontal straight line OA while point C is attached to a crank OC. For the configuration shown, point A has a known velocity v_a. Find the velocity of point B. *Ans.* $v_b = 1.29v_a$.

5. Referring to Fig. C and assuming that the crank OC has a constant angular velocity $\omega = 10\ \text{sec}^{-1}$, find the total acceleration of point B for the configuration shown. *Ans.* $a_b = 1{,}230\ \text{in./sec}^2$.

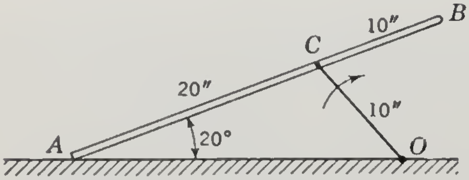


FIG. C

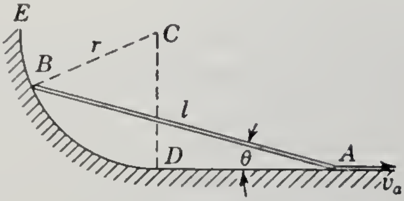


FIG. D

6. In Fig. D, the end A of the bar AB moves with constant speed $v_a = 20$ fps along the horizontal plane DA, while the end B follows the circular arc DE. Find the velocity of point B for the instant when $\theta = 15°$. The following numerical data are given: $l = 4$ ft, $r = 2$ ft. *Ans.* $v_b = 28$ fps.

7. Referring again to Fig. D, and using the same data as in Prob. 6, find the total acceleration of point B when $\theta = 15°$. *Ans.* $a_b = 231\ \text{ft/sec}^2$.

9.2. Instantaneous center. Let us consider any displacement of a plane figure from the position AB to the position A_1B_1 (Fig. 314). As pointed out in Art. 9.1, this displacement can be accomplished by a translatory motion of the figure together with a pole and a rotation of the figure around this pole. We can take, for instance, as a pole the point A and move the body parallel to itself from the position AB to the position A_1B' and then rotate it around A_1 through the angle θ so as to bring it to the position A_1B_1. Again, if we take as a pole the point B, we bring the body first to the position B_1A' and then rotate it about B_1 through the same angle θ, as before, to come to the position A_1B_1. It is seen that the displacement of the pole depends upon what point in the body is chosen as a pole but that the angle of rotation θ of the figure is independent of the choice of a pole. We can, as a matter of fact, always find in the plane of the figure such a point C that, for displacement of the figure from the position AB to the position A_1B_1, only rotation about C will be required. It is evident from Fig. 314 that

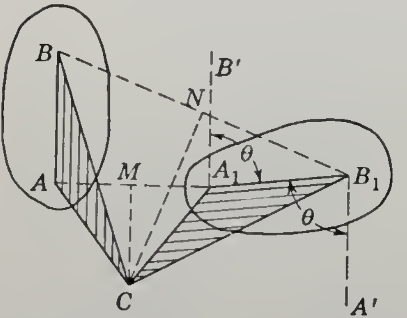


FIG. 314

this center is obtained as the intersection of the perpendicular bisectors CM and CN of the lines AA_1 and BB_1, respectively. For, from the construction, it follows that the two shaded triangles ABC and A_1B_1C are equal. Thus they can be brought into coincidence by rotation θ about point C. If the line AB coincides with the line A_1B_1, the entire figure AB coincides with the figure A_1B_1.

In studying the motion of a plane figure, representing a cross section of a body performing plane motion, we may consider the continuous motion of the figure as made up of successive infinitesimal displacements from one instantaneous position to the next. Each such infinitesimal displacement may be considered as a rotation of the figure about a certain point in its plane. This point is called the *instantaneous center of rotation* for that particular position of the figure. Considering, for instance, the cylinder shown in Fig. 311a, which we assume rolls without slipping, it is seen that at any instant the point of contact with the plane has zero velocity. Hence this point is the instantaneous center of the rolling cylinder, and we conclude that the instantaneous velocity of any point on the rim is directed perpendicular to the chord joining it with the point of contact and is equal to the product of the length of this chord and the angular velocity of the cylinder. It should be noted that, as the cylinder moves, the point of contact, i.e., the instantaneous center of rotation, is continually changing its position;[1] in this case it is evidently moving along the line AB.

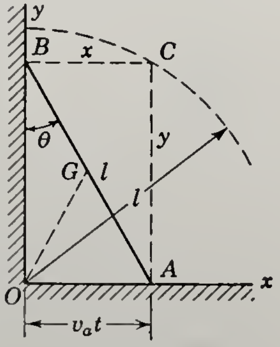


FIG. 315

If we know the directions of the velocities of two points of a moving plane figure for any instantaneous position of the figure, the location of the instantaneous center can always be determined. Consider, for example, a bar AB of length l, the ends of which are constrained to move along the x and y axes (Fig. 315). Then the velocity of point A at any instant must be horizontal, and we conclude that, if this velocity is a result of rotation about a certain instantaneous center, then this center must lie on a vertical line through point A. In the same way we conclude that, if the end B of the bar is moving vertically, the instantaneous center must lie on a horizontal line through point B.

---

[1] The notion of *instantaneous center of rotation* should not be confused with the notion of a *pole* as discussed in Art. 9.1. While a pole is always a point fixed in the body, the instantaneous center is not fixed in the body.

Thus its position is determined by the intersection C of these two lines. Denoting by x and y the coordinates of the instantaneous center C, we see from the figure that

$$x^2 + y^2 = l^2$$

Thus the instantaneous center C in this case is moving along a circle of radius l and center at O.

As another example, we take the case of the connecting rod AB of an engine (Fig. 316). Since the point A moves perpendicular to the radius OA and since the point B moves vertically, we obtain the instantaneous center as the intersection C of the prolongation of the radius OA and the horizontal line through point B.

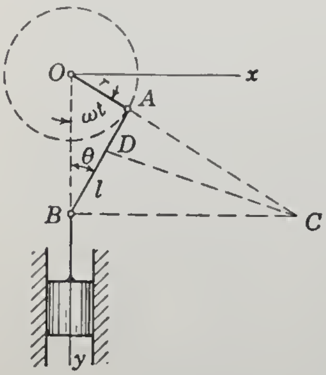


FIG. 316

The instantaneous center of rotation of a body performing plane motion is very useful in finding the angular velocity of the body and the lineal velocities of its various points when the magnitude and direction of the velocity of one point and the direction of the velocity of another point are known. In the case of the connecting rod AB (Fig. 316), for example, if the angular velocity ω of rotation of the crank OA is given, we know the magnitude and direction of the velocity of point A and also the direction of the velocity of point B. Hence, using the instantaneous center C, located as explained above, we conclude, since motion of the connecting rod for the instantaneous position shown is equivalent to a rotation about point C, that

$$AC \cdot \dot{\theta} = r\omega$$

from which

$$\dot{\theta} = \frac{r\omega}{AC}$$

If the diagram in Fig. 316 has been constructed to scale, the length AC can be measured from it and $\dot{\theta}$ is determined. As soon as the angular velocity $\dot{\theta}$ of the rod is known, the magnitude of the velocity of point B can be obtained from the equation

$$v_b = BC \cdot \dot{\theta}$$

where again the distance BC must be scaled from the figure or computed trigonometrically. Likewise the velocity of any point D on the axis of the rod is

$$v_d = DC \cdot \dot{\theta}$$

and has the direction perpendicular to DC.

EXAMPLES

1. Find the ratio of the angular velocities $\dot{\theta}_1$ and $\dot{\theta}_2$ of the cranks O_1A and O_2B of the system shown in Fig. 317 for the instantaneous positions shown.

Solution. The instantaneous center C for the bar AB is obtained as shown. Then we have

$$\frac{r_1\dot{\theta}_1}{AC} = \frac{r_2\dot{\theta}_2}{BC}$$

from which

$$\frac{\dot{\theta}_1}{\dot{\theta}_2} = \frac{r_2}{r_1}\frac{AC}{BC}$$

For given numerical values of r_1, r_2, θ_1, θ_2, and a, the construction shown in Fig. 317 can be made to scale and the lengths AC and BC measured from the figure.

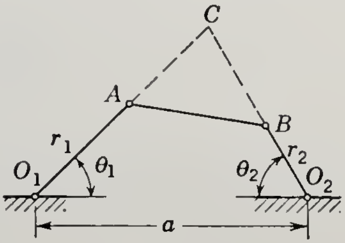


FIG. 317

2. The ends A and B of a slender bar of length l are constrained to follow the straight lines OA and OB with exterior angle α as shown in Fig. 318. Prove that, for motion of the bar in the plane of the figure, the instantaneous center of rotation describes a circle of radius $l/\sin\alpha$ with center at O.

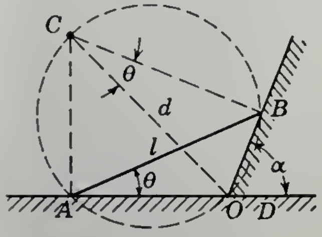


FIG. 318

Proof. First, we draw normals to the given lines OA and OB through A and B, respectively, and obtain the instantaneous center C as shown. Then since the angles OAC and OBC are right angles, it follows that O, A, C, B all lie on a circle with OC as a diameter. Thus, the angles OAB and OCB which subtend the same chord OB are equal. Denoting this angle by θ and the diameter OC by d, we may write

$$BD = l\sin\theta = d\sin\theta\sin\alpha$$

from which $d = l/\sin\alpha$. We see that regardless of the value of θ, the instantaneous center C remains a constant distance d from O.

PROBLEM SET 9.2

1. A locomotive wheel of radius $r = 3$ ft rolls without slip along a horizontal rail AB with constant speed $v_c = 60$ fps as shown in Fig. A. Using the notion of instantaneous center, find the velocity of point E on the rim of the wheel. *Ans.* $v_e = 84.8$ fps.

2. If the end A of the bar AB in Fig. 318 above moves to the left with constant speed $v_a = 10$ fps along OA, find the velocity of point B for the instant when $\theta = 30°$. Assume $l = 4$ ft and $\alpha = 60°$. *Ans.* $v_b = 10$ fps.

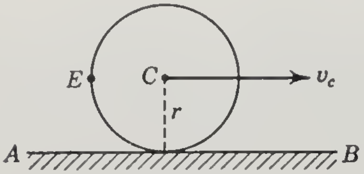


FIG. A

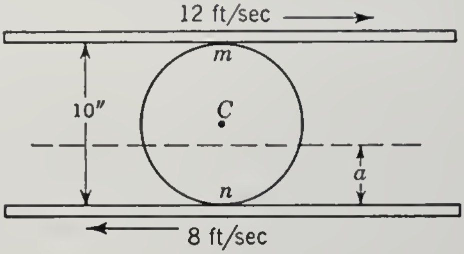


FIG. B

3. A roller of radius $r = 5$ in. rides between two horizontal bars moving in opposite directions as shown in Fig. B. Calculate the distance a defining the position of the horizontal path of the instantaneous center of rotation of the roller. Assume that there is no slip at the points of contact m and n. *Ans.* $a = 4$ in.

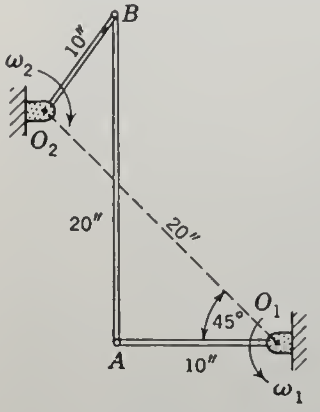


FIG. C

4. For the mechanism shown in Fig. C, the crank centers O_1 and O_2 are 20 in. apart and the line O_1O_2 makes an angle of 45° with the horizontal. Find the angular velocity $\dot{\theta}$ of the bar AB for the instant when the crank O_1A, having constant angular velocity ω_1, is horizontal as shown. *Ans.* $\dot{\theta} = 0.364\omega_1$.

5. Find the angular velocity ω_2 of the crank O_2B of the mechanism shown in Fig. C for the configuration shown. *Ans.* $\omega_2 = 1.36\ \omega_1$.

6. The disk shown in Fig. D rolls without slip on two level rails with constant speed $v_c = 40$ ips. Find the velocities v_d and v_e of points D and E at the top and bottom of the rim. *Ans.* $v_d = 120$ ips; $v_e = -40$ ips.

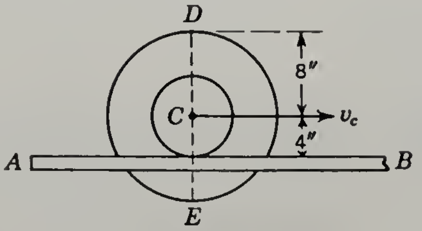


FIG. D

7. A ladder AB of length $l = 16$ ft rests on a horizontal floor at A and leans against a vertical wall at B. If the lower end A is pulled away from the wall with constant velocity $v_a = 10$ fps, what is the angular velocity $\dot{\theta}$ of the ladder at the instant when A is 8 ft from the wall? *Ans.* $\dot{\theta} = 0.722\ \text{sec}^{-1}$.

8. In Fig. E, the wheel is moving to the left with velocity $v_c = 2$ fps and the bar OA is vertical when CB is horizontal as shown. Find the angular velocity $\dot{\theta}$ of the bar OA for the configuration shown if the wheel rolls on the horizontal plane without slip. *Ans.* $\dot{\theta} = 0.66 \text{ sec}^{-1}$.

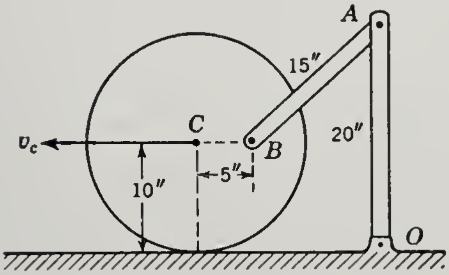


FIG. E

9. The wheel in Fig. F rolls to the left along a horizontal plane with constant speed $v_c = 3.0$ fps. For the configuration shown, find the velocity v_a of the block A which slides in a groove cut in the surface of the plane. Assume that the wheel rolls without slip. *Ans.* $v_a = 5.52$ fps.

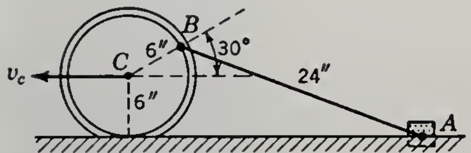


FIG. F

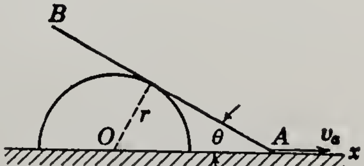


FIG. G

10. If the bar AB (Fig. G) always remains tangent to a circle of radius r and the end A moves with constant velocity v_a along the horizontal x axis, find the angular velocity $\dot{\theta}$ of the bar as a function of the angle of rotation θ. *Ans.* $\dot{\theta} = v_a \sin^2 \theta/(r \cos \theta)$.

9.3. Equations of plane motion. In the two preceding articles, we have considered only the kinematics of plane motion, without regard as to how this motion was produced. We come now to the problem of finding the relation between applied external forces and the motion that they produce, i.e., the derivation of the equations of plane motion. For this purpose we shall use D'Alembert's principle as we did in the derivation of Eq. (67) for rotation of a rigid body about a fixed axis.

Referring to Fig. 319, let C be the center of gravity of a body that moves parallel to the xy plane under the action of applied forces and let A be any particle of mass dm at the distance r from an axis through C normal to the plane of motion. Let x_c and y_c be the coordinates of point C, considered as a pole, and θ, the angle that AC makes with the x axis. These three coordinates completely define the position of the body at any instant t as already discussed in Art. 9.1, and point A may now be regarded as having two motions. (1) translation together with the pole C and (2) rotation about the axis through C normal to the plane of motion.

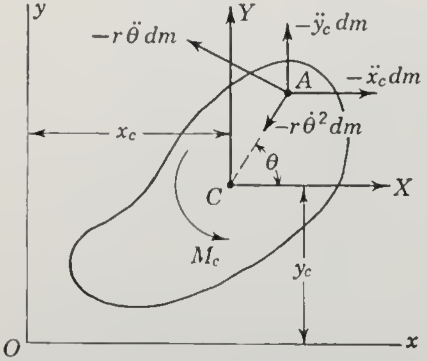


FIG. 319

As a result of the translation, the particle at A has two components of inertia force $-\ddot{x}_c\,dm$ and $-\ddot{y}_c\,dm$. As a result of rotation about the normal axis through C, it has tangential and normal components of inertia force $-r\ddot{\theta}\,dm$ and $-r\dot{\theta}^2\,dm$. These four inertia forces are now applied to the particle as shown in Fig. 319. Then in accordance with D'Alembert's principle, this system of inertia forces for all particles in the body is in equilibrium with the external applied forces and we may write the following equations of dynamic equilibrium:

$$\begin{aligned} X - \ddot{x}_c\int dm + \ddot{\theta}\int r\sin\theta\,dm + \dot{\theta}^2\int r\cos\theta\,dm &= 0 \\ Y - \ddot{y}_c\int dm - \ddot{\theta}\int r\cos\theta\,dm + \dot{\theta}^2\int r\sin\theta\,dm &= 0 \\ M_c - \ddot{y}_c\int r\cos\theta\,dm + \ddot{x}_c\int r\sin\theta\,dm - \ddot{\theta}\int r^2\,dm &= 0 \end{aligned} \tag{a}$$

where X, Y, and M_c represent the applied external forces. Noting that the statical moments

$$\int r\sin\theta\,dm = 0 \qquad \int r\cos\theta\,dm = 0 \tag{b}$$

since C is the center of gravity of the body, Eqs. (a) become greatly simplified and we have

$$\frac{W}{g}\ddot{x}_c = X \qquad \frac{W}{g}\ddot{y}_c = Y \qquad I_c\ddot{\theta} = M_c \tag{86}$$

where $W/g = \int dm$ is the total mass of the body and $I_c = \int r^2\,dm$ is its moment of inertia about the centroidal axis normal to the plane of motion.

From the above system of equations, we make a very important observation: In the case of a rigid body performing plane motion under the action of applied external forces defined by X, Y, M_c, the center of gravity of the body moves exactly as if the entire mass were concentrated there and acted upon by forces X, Y. The moment M_c of external forces with respect to the center of gravity has no effect on the motion of that point. At the same time, the body rotates about the axis through the moving center of gravity exactly as if this were a fixed axis. This is sometimes referred to as the principle of independence of translation and rotation in the case of plane motion. It must be observed, however, that this is true only so long as the center of gravity C is taken as a pole. If any other point in the body is chosen as a pole, conditions (b) will not be fulfilled and Eqs. (a) cannot be reduced to the simple form (86).

Equations (86) can be used to solve two kinds of problems involving the motion of a rigid body parallel to a fixed plane: (1) The motion of the body is given, and the forces required to produce this motion are to be determined; (2) the acting forces are given, and it is required to determine the kind of motion of the body that they will produce. In most practical problems of plane motion of a rigid body we have a combination of the above two cases. Various combinations that may be encountered can best be illustrated by several examples.

EXAMPLES

1. Under the action of gravity, a solid of revolution of weight W rolls without sliding down a plane inclined to the horizontal by the angle α (Fig. 320). Determine the acceleration $\ddot{x}_c$ of its center of gravity down the plane and the maximum angle of inclination α of the inclined plane for which the body will roll without sliding if the coefficient of friction at the point of contact is μ. Assume that the body has a plane of symmetry normal to its axis of revolution.

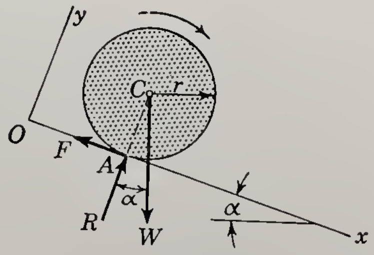


FIG. 320

Solution. Choosing the vertical plane of symmetry of the body as the xy plane and taking as the x axis the line of intersection of this plane with the inclined plane on which the body rolls, we may limit our attention to the motion in the xy plane of a circular figure of radius r and Eqs. (86) become

$$\begin{aligned} \frac{W}{g}\ddot{x}_c &= W\sin\alpha - F \\ \frac{W}{g}\ddot{y}_c &= W\cos\alpha - R = 0 \\ I_c\ddot{\theta} &= Fr \end{aligned} \tag{c}$$

where F represents the total friction at the point of contact A.

From the condition that there is no sliding, we conclude that the velocity of the point of contact at any instant is zero. That is, point A is the instantaneous center of rotation. From this it follows that the angular velocity of rotation of the body is $\dot{\theta} = \dot{x}_c/r$, from which, by differentiation, $\ddot{\theta} = \ddot{x}_c/r$. Substituting this in the last of Eqs. (c), we obtain

$$I_c \frac{\ddot{x}_c}{r} = Fr \tag{d}$$

Eliminating the friction force F between Eq. (d) and the first of Eqs. (c), we obtain

$$\left(\frac{W}{g} + \frac{I_c}{r^2}\right) \ddot{x}_c = W \sin \alpha \tag{e}$$

from which it is seen that the center of gravity C is moving down the plane with a constant acceleration

$$\ddot{x}_c = \frac{g \sin \alpha}{1 + i_c^2/r^2} \tag{f}$$

where i_c is the radius of gyration of the body with respect to its geometric axis.

If the body were to slide down the plane without friction and hence without rolling, we know that its acceleration would be simply $g \sin \alpha$. Hence we conclude that, owing to friction which causes the body to roll without sliding, the acceleration is reduced in the ratio

$$\frac{r^2}{r^2 + i_c^2} \tag{g}$$

In the case of a solid cylinder, $i_c^2 = r^2/2$ and the ratio (g) is equal to $\frac{2}{3}$. For a thin cylindrical shell, we can take $i_c^2 \approx r^2$ and the ratio (g) becomes $\frac{1}{2}$. For a sphere, $i_c^2 = 2r^2/5$ and the ratio (g) is equal to $\frac{5}{7}$. Thus, if a sphere, a solid cylinder, and a hollow cylinder all roll down the same incline without sliding, the sphere will reach the bottom first, the solid cylinder second, and the hollow cylinder last.

Having the acceleration $\ddot{x}_c$ of the center of gravity of the body, the friction force F is determined from the first of Eqs. (c) from which

$$F = \frac{W i_c^2 \sin \alpha}{r^2 + i_c^2} \tag{h}$$

The condition that the body rolls without sliding can be satisfied only if this force F is not greater than the total friction $\mu W \cos \alpha$ that can be developed at the point of contact A. Thus the condition of rolling without sliding requires

$$\frac{W i_c^2 \sin \alpha}{r^2 + i_c^2} < \mu W \cos \alpha$$

from which

$$\tan \alpha < \frac{\mu(r^2 + i_c^2)}{i_c^2} \tag{i}$$

If the angle of inclination α of the plane does not satisfy expression (i), there will be rolling of the body together with sliding. In such a case there is no geometric relationship between the acceleration $\ddot{x}_c$ of the center of gravity C and the angular acceleration $\ddot{\theta}$, and these two accelerations are obtained by substituting $F = \mu W \cos \alpha$ into the first and last of Eqs. (c).

2. A sphere of radius r and weight W is projected along a horizontal plane surface with initial linear velocity v_0 and initial angular velocity $\dot{\theta}_0$ (Fig. 321). Discuss the subsequent motion of the sphere assuming that the coefficient of friction at the point of contact is μ.

Solution. If $v_0 = r\dot{\theta}_0$, the point A has zero velocity and becomes the instantaneous center of rotation. In such case the sphere rolls without sliding and its velocity remains constant.

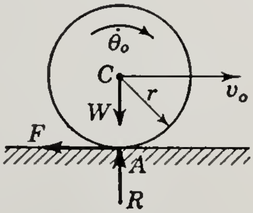


FIG. 321

If $v_0 > r\dot{\theta}_0$, there must be sliding at the point of contact and a friction force $F = \mu W$ will act as shown in the figure. In this case the equations of motion are

$$\frac{W}{g}\ddot{x}_c = -\mu W \qquad \frac{2}{5}\frac{W}{g}r^2\ddot{\theta} = \mu W r$$

from which, by integration,

$$\dot{x}_c = v_0 - \mu g t \qquad \dot{\theta} = \dot{\theta}_0 + \frac{5\mu g}{2r}t \tag{j}$$

These expressions hold until $\dot{x}_c = r\dot{\theta}$, that is, until

$$v_0 - \mu g t = r\dot{\theta}_0 + \frac{5\mu g}{2}t$$

from which

$$t = \frac{v_0 - r\dot{\theta}_0}{\mu g(1 + \frac{5}{2})} \tag{k}$$

After this time the sphere continues to move with constant velocity

$$\dot{x}_c = \tfrac{5}{7}v_0 + \tfrac{2}{7}r\dot{\theta}_0 \tag{l}$$

which is obtained by substituting the value of t from Eq. (k) into the first of Eqs. (j) above.

If $v_0 < r\dot{\theta}_0$, there is again sliding at the point of contact and a friction force $F = \mu W$ acts in the direction opposite to that shown in the figure. It is left as an exercise for the student to determine the time that must elapse in this case before the velocity of the sphere along the plane becomes constant and equal to the same value (l) as for the previous case.

3. A circular cylinder of radius r and weight W can roll without sliding in a fixed hollow cylinder of inner radius a (Fig. 322). The axes of the two cylinders are horizontal and parallel. Find the period of oscillation of the rolling cylinder, assuming that its initial displacement from the position of

stable equilibrium is small so that the angle φ defining an instantaneous position of its center of gravity C is always small.

Solution. We begin by noting that the center of gravity C of the rolling cylinder follows a circular path of radius $a - r$. The forces acting on the cylinder are its gravity force W and the reaction at the point of contact A which we resolve into a normal component R and a tangential or friction component F. In discussing the motion of the center of gravity C, which can be done without consideration of rotation of the cylinder, we shall find it convenient to resolve the resultant acceleration of point C into radial and tangential components at any point on its path. These components evidently are

$$a_t = (a - r)\ddot{\varphi} \qquad \text{and} \qquad a_n = \dot{\varphi}^2(a - r)$$

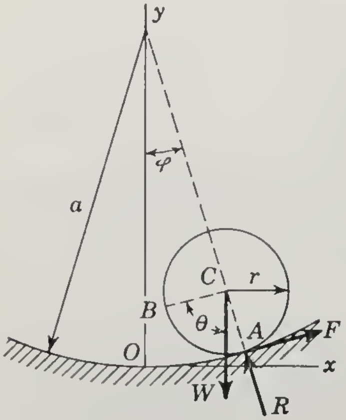


FIG. 322

Then the equations of motion for the body considered as a particle concentrated at its center of gravity C, to which all forces are applied parallel to their actual directions, become

$$\frac{W}{g}(a - r)\ddot{\varphi} = -W \sin \varphi + F \qquad \frac{W}{g}(a - r)\dot{\varphi}^2 = R - W \cos \varphi \tag{m}$$

Considering now rotation of the cylinder about its geometric axis and using the third of Eqs. (86), we may write

$$\frac{W}{g} i_c^2 \ddot{\theta} = -Fr \tag{n}$$

where θ is the angle of rotation of the cylinder corresponding to the displacement of its center of gravity as defined by φ, and i_c is its radius of gyration with respect to the geometric axis. Since we assume rolling without sliding, we conclude that the arc OA is equal to the arc AB where B is the point that was originally in contact with point O. Hence between θ and φ we obtain the relationship, $r(\theta + \varphi) = a\varphi$, from which

$$\theta = \frac{a - r}{r} \varphi \tag{o}$$

and Eq. (n) above may be written

$$\frac{W}{g} i_c^2 \frac{a - r}{r} \ddot{\varphi} = -Fr \tag{n$'$}$$

Eliminating F between the first of Eqs. (m) and Eq. (n′), we obtain

$$(a - r)\left(1 + \frac{i_c^2}{r^2}\right)\ddot{\varphi} = -g \sin \varphi \tag{p}$$

Equation (p) is seen to have the same form as the differential equation of motion of a mathematical pendulum having the length

$$L = (a - r)\left(1 + \frac{i_c^2}{r^2}\right) \tag{q}$$

Hence we conclude that for small displacements the oscillations of the rolling cylinder are harmonic and have the period

$$\tau = 2\pi\sqrt{\frac{L}{g}}$$

PROBLEM SET 9.3

1. A solid circular cylinder and a sphere are started from rest at the top of an inclined plane at the same time, and both roll without sliding down the plane. If, when the sphere reaches the bottom of the incline, the cylinder is 12 ft behind it, what is the total length s of the incline? *Ans.* $s = 180$ ft.

2. A solid cylinder and a thin hoop of equal weights W and radii r are connected by a bar AB and roll down an inclined plane as shown in Fig. A. Find the acceleration of the system down the plane and also the force S in the bar AB. Assume that no slipping occurs and neglect the mass of the bar and friction at the axles. *Ans.* $\ddot{x} = \frac{4}{7}\, g \sin \alpha$; $S = (W/7) \sin \alpha$, compression.

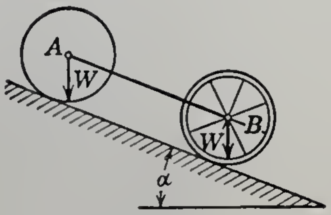


FIG. A

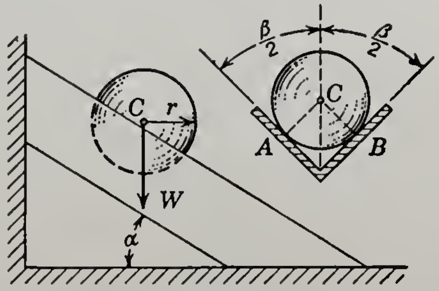


FIG. B

3. A sphere of weight W and radius r rolls without sliding down a V-shaped trough, the axis of which is inclined to the horizontal by the angle α (Fig. B). The cross section of the trough perpendicular to its longitudinal axis is also shown in Fig. B. Assuming that there is sufficient friction to prevent slipping at the points of contact A and B, find the acceleration $\ddot{x}_c$ of the center C of the sphere down the length of the trough. Numerical data are given as follows: $W = 10$ lb, $r = 4$ in., $\alpha = 30°$, $\beta = 90°$. *Ans.* $\ddot{x}_c = 0.278g$.

4. What is the least coefficient of friction at the points of contact A and B in Fig. B for which the sphere will roll without slipping? Use all data as given in Prob. 3. *Ans.* $\mu \geqq 0.181$.

5. A right circular cylinder of radius r and weight W is suspended by a cord that is wound around its surface (Fig. C). If the cylinder is allowed to fall, prove that its center of gravity C will follow a vertical rectilinear path and find the acceleration $\ddot{x}_c$ along this path. Determine also the tensile force S in the cord. *Ans.* $\ddot{x}_c = \frac{2}{3}g$; $S = W/3$.

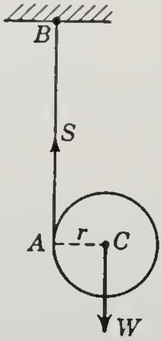


FIG. C

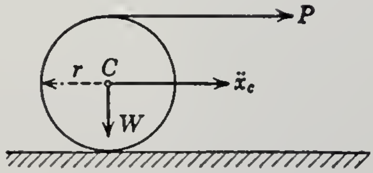


FIG. D

6. A solid right circular cylinder of weight W and radius r is pulled along a horizontal plane by a horizontal force P applied to the end of a string wound around the cylinder as shown in Fig. D. (*a*) Assuming no slip, find the acceleration $\ddot{x}_c$ of the center C. (*b*) What coefficient of friction μ is required between the cylinder and plane to prevent slipping under these conditions? *Ans.* (*a*) $\ddot{x}_c = 4Pg/3W$; (*b*) $\mu \geqq P/3W$.

7. A solid homogeneous right circular cylinder of weight W and diameter d is pulled up a 30° incline by a constant force $P = \frac{1}{2}W$ applied to the end of a string wound around its circumference as shown in Fig. E. Assuming no slip at the point of contact A, find the acceleration $\ddot{x}_c$ of the center of gravity C up the plane. *Ans.* $\ddot{x}_c = g/3$.

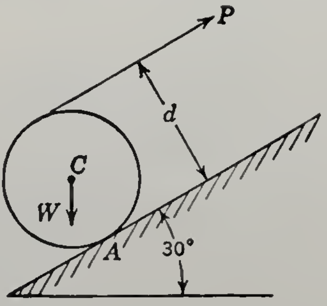


FIG. E

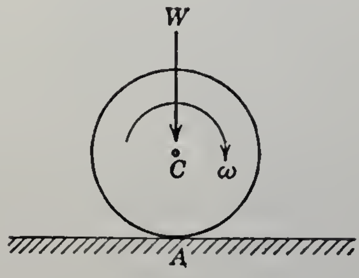


FIG. F

8. A homogeneous solid right circular cylinder of weight W and radius r, initially spinning about its geometric axis with angular velocity ω, is suddenly set down on a horizontal plane without any foreward velocity (Fig. F). What velocity v_c along the plane will the cylinder finally acquire if the coefficient of friction at the point of contact A is μ? *Ans.* $v_c = r\omega/3$.

9. A solid homogeneous right circular cylinder of weight W and radius r is supported on an inclined plane as shown in Fig. G. The string wound around its circumference is attached to the foundation at A and is parallel to the plane. Find the acceleration $\ddot{x}_c$ of the cylinder down the plane if the coefficient of friction at the point of contact B is $\mu = \frac{1}{3}$. *Ans.* $\ddot{x}_c = 0.355g$.

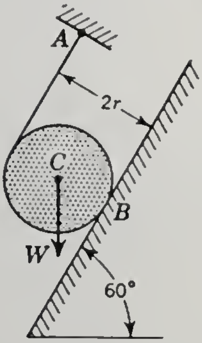


FIG. G

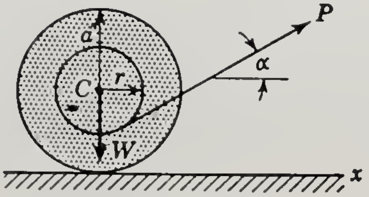


FIG. H

10. A spool of weight W and radius of gyration i_c with respect to its geometric axis is pulled along a horizontal plane by a force P applied to the end of a string wound around the shaft of the spool as shown in Fig. H. Find the acceleration $\ddot{x}_c$ of the spool if it rolls along the plane without slip. Numerical data are given as follows: $P = 10$ lb, $W = 20$ lb, $a = 6$ in., $r = 3$ in., $i_c = a/3$, $\alpha = 30°$. *Ans.* $\ddot{x}_c = 0.165g$.

11. Referring again to Fig. H, find the limiting value of the angle α below which the spool will roll to the right and above which it will roll to the left. Use all other data as given in Prob. 10 above. *Ans.* $\alpha = 60°$.

12. A solid right circular roller of weight $W = 500$ lb and radius $r = 1$ ft rolls on two inclined rails ($\alpha = 10°$) and is made to roll up the incline by a falling weight $Q = 200$ lb arranged as shown in Fig. I. Find the acceleration of the roller up the incline if it rolls without slip. Assume that the length of string between roller and weight remains approximately vertical. *Ans.* $\ddot{x}_c = 2.84$ ft/sec².

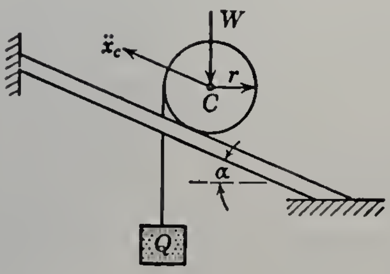


FIG. I

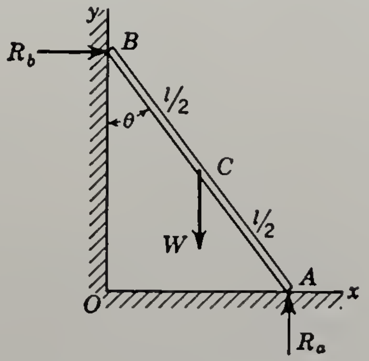


FIG. J

***13.** If the bar AB in Fig. J starts from rest in the vertical position ($\theta = 0$) and falls in the vertical xy plane under the action of its own weight W, find the reactions R_a and R_b as functions of the angle θ. The ends A and B of the bar are constrained to follow the x and y axes, respectively, without friction. *Ans.* $R_a = (W/4)(3\cos\theta - 1)^2$; $R_b = (W/4)(9\cos\theta - 6)\sin\theta$.

9.4. D'Alembert's principle in plane motion. Writing Eqs. (86) in the form

$$X - \frac{W}{g}\ddot{x}_c = 0 \qquad Y - \frac{W}{g}\ddot{y}_c = 0 \qquad M_c - I_c\ddot{\theta} = 0 \tag{87}$$

we conclude that, in the case of a body performing plane motion, we need only to add to the applied external forces, defined by X, Y, M_c, the inertia forces $-(W/g)\ddot{x}_c$ and $-(W/g)\ddot{y}_c$ and the inertia couple $-I_c\ddot{\theta}$ to obtain dynamic equilibrium of the body. The components of inertia force $-(W/g)\ddot{x}_c$ and $-(W/g)\ddot{y}_c$ must be applied to the body at its center of gravity; the inertia couple $-I_c\ddot{\theta}$, anywhere in the plane of motion. Then writing equations of statics in the usual way, we obtain the solution to our problem. This is D'Alembert's principle.

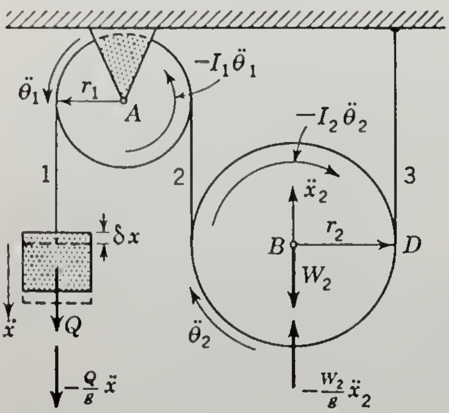


FIG. 323

So long as we are dealing with the plane motion of a single body, Eqs. (87) hold no particular advantage over Eqs. (86), but in the case of a system of connected bodies moving in one plane, the use of D'Alembert's principle together with the principle of virtual work may result in great simplification of the problem. Consider, for example, the system of pulleys shown in Fig. 323. The weight Q falls vertically with acceleration $\ddot{x}$, thus rotating the fixed pulley A with angular acceleration $\ddot{\theta}_1$ and lifting the free pulley B with acceleration $\ddot{x}_2$. The pulley B also rotates clockwise with angular acceleration $\ddot{\theta}_2$.

If we try to write separate equations of motion for each of the three bodies involved, we have to introduce the tensions S_1, S_2, and S_3 in the three vertical branches of the string and the algebraic work becomes somewhat involved. To avoid this, we introduce the inertia forces $-(Q/g)\ddot{x}$, $-(W_2/g)\ddot{x}_2$ and the inertia couples $-I_1\ddot{\theta}_1$, $-I_2\ddot{\theta}_2$, as shown, thus putting the system as a whole in dynamic equilibrium. Then making a small virtual displacement δx of the weight Q and denoting by $\delta\theta_1$, $\delta\theta_2$, and δx_2 the corresponding virtual displacements of the pulleys, we may write one equation of virtual work as follows:

$$\left(Q - \frac{Q}{g}\ddot{x}\right)\delta x - I_1\ddot{\theta}_1\,\delta\theta_1 - \left(W_2 + \frac{W_2}{g}\ddot{x}_2\right)\delta x_2 - I_2\ddot{\theta}_2\,\delta\theta_2 = 0 \tag{a}$$

Assuming that such quantities as the weights and dimensions of the pulleys and block are given, Eq. (a) contains as unknowns the following four quantities: x, θ_1, θ_2, x_2, as well as their time derivatives. Thus to obtain a solution of the problem, we need three more equations relating these four quantities. These will be obtained from the geometrical conditions of constraint of the system which has only one degree of freedom. This means that when a displacement x of the weight Q is specified, then the angular displacements θ_1, θ_2 of the two pulleys and the displacement x_2 of point B are also known. In this case, we see that if there is no slip between the string and the pulleys, point D is the center of rotation for the pulley B and we must have

$$\theta_1 = \frac{x}{r_1} \qquad x_2 = \frac{x}{2} \qquad \theta_2 = \frac{x}{2r_2} \tag{b}$$

Equations (b) relating the coordinates of the several parts of the system are sometimes called *equations of constraint.* Differentiating each of Eqs. (b) twice with respect to time, we obtain also

$$\ddot{\theta}_1 = \frac{\ddot{x}}{r_1} \qquad \ddot{x}_2 = \frac{\ddot{x}}{2} \qquad \ddot{\theta}_2 = \frac{\ddot{x}}{2r_2} \tag{c}$$

Substituting the values of θ_1, θ_2, x_2 and their time derivatives from Eqs. (b) and (c) into Eq. (a) and solving for $\ddot{x}$, we obtain

$$\ddot{x} = g\frac{Q - \frac{1}{2}W_2}{Q + W_1\frac{i_1^2}{r_1^2} + \frac{1}{4}W_2\left(1 + \frac{i_2^2}{r_2^2}\right)} \tag{d}$$

where i_1 and i_2 are the radii of gyration of the two pulleys. For any given numerical data, this equation defines the acceleration of the weight Q. We note that, when $Q = \frac{1}{2}W_2$, the system is in equilibrium.

The separate equations of motion for the three parts of the system will be as follows:

1. For the weight Q,

$$\frac{Q}{g}\ddot{x} = Q - S_1$$

2. For the fixed pulley,

$$\frac{W_1}{g}i_1^2\ddot{\theta}_1 = (S_1 - S_2)r_1$$

3. For the movable pulley,

$$\frac{W_2}{g}\ddot{x}_2 = S_2 + S_3 - W_2$$

$$\frac{W_2}{g}i_2^2\ddot{\theta}_2 = (S_2 - S_3)r_2$$

Eliminating the three unknowns S_1, S_2, and S_3 from these equations, Eq. (d) will be obtained. The student is advised to carry this out and judge the advantage of the use of the principle of virtual work.

EXAMPLES

1. A prismatic timber of weight W rests on two rollers, each of weight $W/2$ and radius r, and is pulled along a horizontal plane by a force P as shown in Fig. 324*a*. Assuming that there is no slipping and treating the rollers as solid right circular cylinders, find the acceleration $\ddot{x}$ of the timber.

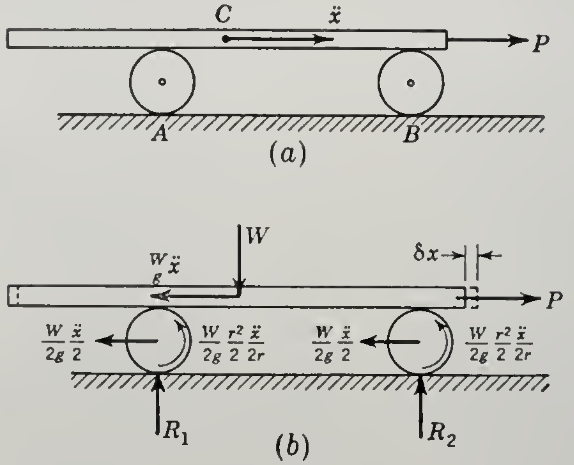


FIG. 324

Solution. Since there is no slipping, points A and B are the instantaneous centers of rotation of the rollers. Thus, the rollers have linear accelerations $\ddot{x}/2$ and angular accelerations $\ddot{x}/2r$. The corresponding inertia forces and inertia couples together with the applied forces are shown in Fig. 324*b*, and the system is in dynamic equilibrium. Defining a virtual displacement of the system by a small forward movement δx of the timber, we see that the roller centers move forward through the distance $\delta x/2$ and rotate through angles $\delta x/2r$. Hence the equation of virtual work is

$$\left(P - \frac{W}{g}\ddot{x}\right)\delta x - 2\left(\frac{W}{2g}\frac{\ddot{x}}{2}\right)\frac{\delta x}{2} - 2\left(\frac{W}{2g}\frac{r^2}{2}\frac{\ddot{x}}{2r}\right)\frac{\delta x}{2r} = 0$$

from which

$$\ddot{x} = \frac{8}{11}\frac{P}{W}g \qquad \text{(e)}$$

2. Find the acceleration $\ddot{x}$ of the timber in Fig. 324 by writing equations of motion directly without using the principle of virtual work.

Solution. For this purpose, we make separate free-body diagrams of the timber and one roller as shown in Fig. 325. In so doing, it becomes necessary to introduce unknown friction forces F_1 and F_2 as shown. It makes no difference whether we assume the correct directions for these forces or not, but the fact that the force $F_1/2$ acting at one end of the timber is equal and opposite to the force acting on the top of the corresponding roller is important.

We are now ready to write the equations of motion as follows:

1. For the timber,

$$\frac{W}{g}\ddot{x} = P - F_1$$

2. For either roller,

$$\frac{W}{2g}\frac{\ddot{x}}{2} = \frac{F_1}{2} - \frac{F_2}{2}$$

$$\frac{W}{2g}\frac{r^2}{2}\frac{\ddot{x}}{2r} = \left(\frac{F_1}{2} + \frac{F_2}{2}\right)r$$

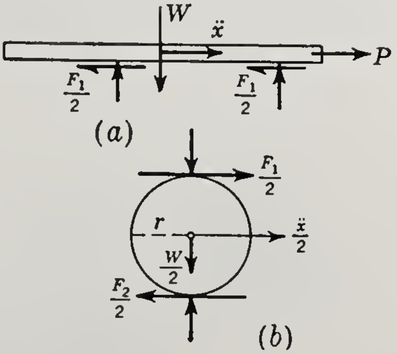


FIG. 325

Upon elimination of the friction forces F_1 and F_2, these three equations will yield the same result as obtained in Example 1 above.

PROBLEM SET 9.4

1. Calculate the acceleration $\ddot{x}$ of the falling weight Q in Fig. 323 for the following numerical data: $Q = 5$ lb, $W_1 = 10$ lb, $r_1 = 5$ in., $W_2 = 4$ lb, $r_2 = 8$ in. Treat the pulley A as a solid right circular disk and the pulley B as a thin hoop. *Ans.* $\ddot{x} = g/4$.

2. Referring to Fig. A, find the acceleration $\ddot{x}_c$ of the solid right circular roller of weight W which is pulled along a horizontal plane by means of the weight Q on the end of a string wound around the circumference of the roller. The following numerical data are given: $W = 20$ lb; $Q = 10$ lb. Assume sufficient friction at A to prevent slipping. *Ans.* $\ddot{x}_c = 9.2$ ft/sec².

3. In Fig. B, the weight of the two-step pulley is $W = 40$ lb, and its radius of gyration is $i_c = 5$ in. The small pulley has weight $Q = 10$ lb and radius $r = 3$ in. and may be considered as a solid right circular disk. Find the

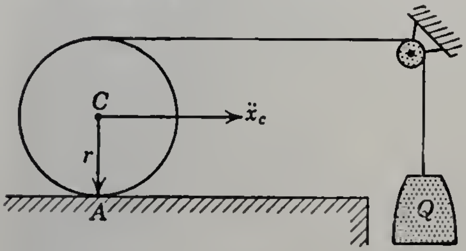


FIG. A

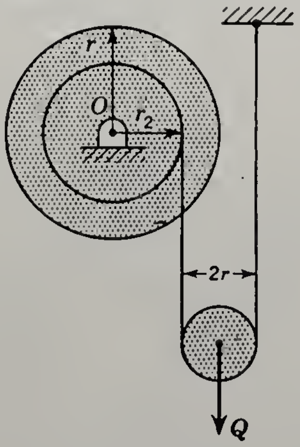


FIG. B

angular acceleration $\ddot{\theta}$ of the large pulley if $r_1 = 8$ in. and $r_2 = 6$ in. *Ans.* $\ddot{\theta} = 10.2 \text{ sec}^{-1}$.

4. Two identical right circular disks are arranged in a vertical plane as shown in Fig. C. Neglecting friction, find the acceleration $\ddot{x}_c$ of the center C of the falling disk. *Ans.* $\ddot{x}_c = 4g/5$.

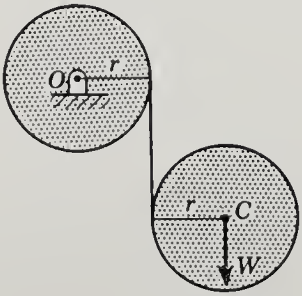


FIG. C

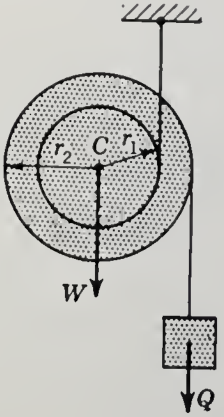


FIG. D

5. Referring to Fig. D and assuming that the two-step pulley of weight $W = 40$ lb has radii $r_1 = 4$ in., $r_2 = 6$ in., and radius of gyration $i_c = 3$ in., find the acceleration a of the rising weight $Q = 60$ lb. There is another hub and vertical string on the far side of the pulley to make the vertical plane of the figure a plane of symmetry. *Ans.* $a = 0.0645g$.

6. A prismatic timber of weight W supported on rollers each of weight $W/2$ and radius r rolls down an inclined plane as shown in Fig. E. Assuming no slip and treating the rollers as solid right circular cylinders, find the acceleration $\ddot{x}$ of the timber down the plane. *Ans.* $\ddot{x} = \frac{12}{11}g \sin \alpha$.

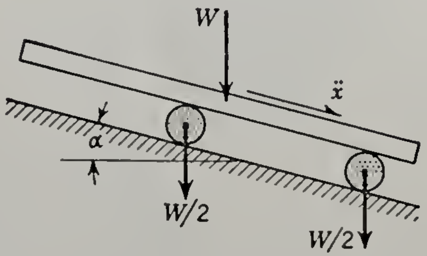


FIG. E

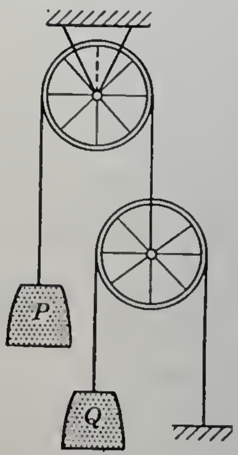


FIG. F

7. For the system of pulleys in Fig. F, we have $P = Q = 10$ lb and each pulley is a thin hoop of weight $W = 4$ lb and radius $r = 8$ in. Find the acceleration of the falling weight Q, neglecting the inertia of the spokes in the pulleys. *Ans.* $a = 0.452g$.

8. Referring to Fig. G, find the acceleration $\ddot{x}_1$ of the block W_1 down the inclined plane for the following numerical data: $W_1 = 200$ lb, $W_2 = 100$ lb, $\mu = 0.2$. Each pulley is a solid right circular disk of weight $W = 50$ lb and radius $r = 1$ ft. *Ans.* $\ddot{x}_1 = 4.57$ ft/sec^2.

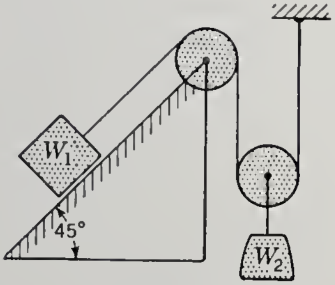


FIG. G

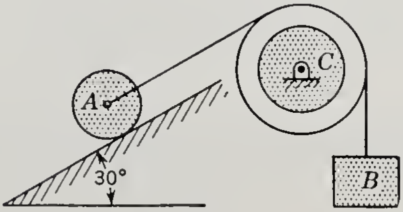


FIG. H

9. A small car body of weight W has four solid disk wheels, each of weight $W/4$ and radius r. What acceleration will the car have in coasting down a straight track inclined to the horizontal by an angle α, if rolling resistance is neglected? Assume that the wheels roll without slip. *Ans.* $\ddot{x} = \frac{4}{5}g \sin \alpha$.

10. In Fig. H, the block B has weight $W_b = 322$ lb, the fixed pulley C has weight $W_c = 32.2$ lb, radius $r_c = 2.5$ ft, and radius of gyration $i_c = 2$ ft. The right circular cylinder A has weight $W_a = 64.4$ lb and radius $r_a = 1$ ft. Find the acceleration a of the falling block B if the cylinder rolls on the inclined plane without slip. What is the least coefficient of friction between the rolling cylinder and the inclined plane for which this will be possible? Neglect friction in the bearing of the pulley C. *Ans.* $a = 21.2$ ft/sec^2; $\mu = 0.38$.

9.5. The principle of angular momentum in plane motion. Let us consider a body moving in the xy plane as shown in Fig. 326. Choosing the center of gravity C as a pole, we define the position of this point at any instant by the coordinates x_c, y_c. Through the moving center of gravity, we take also coordinate axes ξ, η, parallel, respectively, to x and y.[1] Then the coordinates of any point A of the body will be

$$x = x_c + \xi \qquad y = y_c + \eta \qquad \text{(a)}$$

In these expressions, both sets of coordinates x_c, y_c and ξ, η are changing with time; x, y are called absolute coordinates, of point A; ξ, η, relative coordinates.

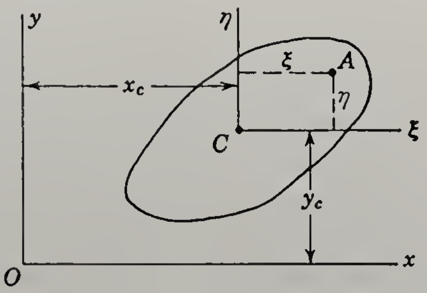


FIG. 326

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[1] These axes move with point C, but they do not rotate with the body.

Differentiating Eqs. (a) with respect to time, we obtain

$$\dot{x} = \dot{x}_c + \dot{\xi} \qquad \dot{y} = \dot{y}_c + \dot{\eta} \tag{b}$$

With these expressions for the velocity components of any point A in the body, we can easily calculate the total angular momentum of the body with respect to the fixed point O in its plane of motion. By definition, the moment of momentum, with respect to point O, of a particle of mass dm at A is

$$dm(\dot{y}x - \dot{x}y) \tag{c}$$

which, by using Eqs. (a) and (b) above, becomes

$$dm[(\dot{y}_c + \dot{\eta})(x_c + \xi) - (\dot{x}_c + \dot{\xi})(y_c + \eta)] \tag{d}$$

Summing up such expressions for all particles of the body, we obtain for the resultant angular momentum with respect to the z axis the following expression.

$$\begin{aligned} H_z &= \Sigma\, dm\,[(\dot{y}_c + \dot{\eta})(x_c + \xi) - (\dot{x}_c + \dot{\xi})(y_c + \eta)] \\ &= \Sigma\, dm(x_c\dot{y}_c - y_c\dot{x}_c) + \Sigma\, dm(\dot{\eta}\xi - \dot{\xi}\eta) \\ &\qquad + x_c\Sigma\dot{\eta}\, dm + \dot{y}_c\Sigma\xi\, dm - y_c\Sigma\dot{\xi}\, dm - \dot{x}_c\Sigma\eta\, dm \end{aligned}$$

Since C is the center of gravity of the body,

$$\begin{aligned} \Sigma\xi\, dm &= 0 \qquad & \Sigma\eta\, dm &= 0 \\ \Sigma\dot{\xi}\, dm &= 0 \qquad & \Sigma\dot{\eta}\, dm &= 0 \end{aligned} \tag{e}$$

and the expression for angular momentum reduces to

$$H_z = \frac{W}{g}\,(x_c\dot{y}_c - y_c\dot{x}_c) + \Sigma(\dot{\eta}\xi - \dot{\xi}\eta)\, dm \tag{f}$$

where $W/g = \Sigma\, dm$ is the total mass of the body. The first term in expression (f) is seen to represent the moment of momentum of the mass W/g moving as a particle concentrated at the center of gravity C of the body. The second term is the angular momentum of the body with respect to an axis parallel to the z axis and through the center of gravity C. This angular momentum is due to rotation of the body around point C. Thus we see that, in the case of a body performing plane motion, the total angular momentum with respect to a fixed axis perpendicular to the plane of motion is made up of two parts: (1) the moment of momentum, with respect to that axis, of the body considered as a particle of mass W/g concentrated at the center of gravity C and (2) the angular momentum of the body with respect to a parallel axis through the moving center of gravity C.

Denoting by $\dot{\theta}$ the angular velocity of rotation of the body with respect to the axis through C, the second term in expression (f) can be written in the same form as in the case of rotation of a rigid body with respect to a fixed axis. Then expression (f) becomes

$$H_z = \frac{W}{g}(x_c\dot{y}_c - y_c\dot{x}_c) + I_c\dot{\theta} \tag{88}$$

in which I_c is the moment of inertia of the body with respect to an axis through C and parallel to the fixed z axis, that is, perpendicular to the plane of motion.

Using now the principle of angular momentum and equating the rate of change of H_z to the moment of all external forces with respect to the z axis, we may write

$$\frac{d}{dt}(H_z) = M_z \tag{89}$$

This states the principle of angular momentum for a rigid body in plane motion. The rate of change of angular momentum (88) with respect to any fixed point in the plane of motion is equal to the moment of applied external forces with respect to the same point.

Equation (89) is particularly useful in studying the sudden changes produced in the motion of a rigid body under the action of impulsive forces. In our further discussion, we assume that the body has a plane of symmetry and that it moves only in this plane. If such a body is given a quick blow or impact also in the plane of motion, very large accelerations will be produced momentarily and the motion just after the blow will be quite different from what it was before, even though there has been no appreciable change in the position of the body. In dealing with such problems, it is customary to neglect entirely all ordinary forces such as gravity in comparison with the very large impact forces and also to neglect change in position of the body during the impact. In short, knowing the motion of a body just before impact and the nature of the impact, what will be the new motion of the body just after impact?

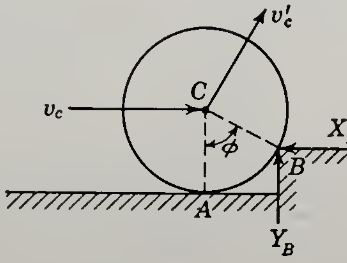

FIG. 327

As an example, let us consider a solid right circular cylinder that rolls with constant velocity along a horizontal plane and suddenly strikes an obstruction at B as shown in Fig. 327. Just before impact at B, the cylinder has a horizontal velocity v_c and angular velocity

$\dot{\theta} = v_c/r$, since point A is its instantaneous center of rotation. Just after the impact, point B becomes the new instantaneous center, and we wish to know the magnitude of the new velocity v_c' of the cylinder. To find this, we observe that during the interval of impact very large forces X_b and Y_b act on the cylinder at point B. In comparison with such forces, we can neglect the gravity force W entirely and assume that X_b and Y_b are the only forces acting during the impact. Then if we take B as a moment center, the moments of these forces vanish and we conclude from Eq. (89) that the angular momentum of the cylinder with respect to point B does not change. Before impact it was

$$H_b = \frac{W}{g}\frac{r^2}{2}\frac{v_c}{r} + \frac{W}{g} v_c r \cos\varphi \tag{g}$$

while just after impact it is

$$H_b' = \frac{W}{g}\frac{r^2}{2}\frac{v_c'}{r} + \frac{W}{g} v_c' r \tag{h}$$

Equating expressions (g) and (h), we obtain

$$v_c' = \frac{v_c}{3}(1 + 2\cos\varphi) \tag{i}$$

The above discussion assumes that there is sufficient friction at B to prevent slipping during the change in motion.

EXAMPLES

1. A thin homogeneous plate of any shape has a prescribed motion in its own plane defined by the velocity components $\dot{x}_c$, $\dot{y}_c$ of its center of gravity C and its angular velocity ω (Fig. 328). If a certain point O in the plate at the distance r from C is suddenly fixed by means of a pin, find the new angular velocity of the plate around this point.

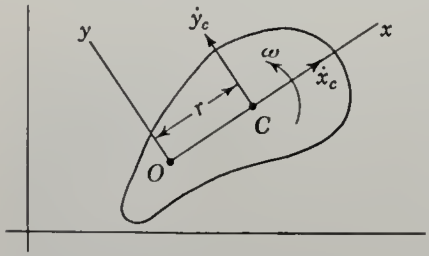

FIG. 328

Solution. Through point O, we choose fixed coordinate axes x, y, in the plane of motion such that the x axis coincides with the instantaneous position of the line OC as shown. Then by Eq. (88), the angular momentum H_z of the plate just before point O is fixed is

$$H_z = \frac{W}{g}(r\dot{y}_c + i_c^2\omega) \tag{j}$$

where i_c is the radius of gyration of the plate with respect to the centroidal axis normal to the plane of the plate. The velocity component $\dot{x}_c$ does not appear in Eq. (j) because $y_c = 0$.

When point O is suddenly fixed by the pin, there will be a large impact force created at O. However, the moment of this force with respect to the z axis through O will be zero because the arm of the force is zero. Hence, by Eq (89), the angular momentum of the plate after impact must remain unchanged.

To calculate the angular momentum of the plate about O after impact, we note that it is now a rigid body rotating about a fixed axis. Hence the new angular momentum is simply

$$H'_z = I_z\omega' = \frac{W}{g}\,(i_c^2 + r^2)\omega' \tag{k}$$

where ω' is the new angular velocity.

Equating expressions (j) and (k) in accordance with the principle of conservation of angular momentum, we obtain

$$\omega' = \frac{\dot{y}_c r + i_c^2\omega}{i_c^2 + r^2} \tag{l}$$

2. A slender prismatic bar AB of length l and weight W is allowed to fall vertically in an inclined position but without rotation as shown in Fig. 329. If the center of gravity C of the bar has velocity v_c at the instant when the end A strikes a smooth horizontal plane, find the new motion of the bar just after the impact assuming no rebound at A.

Solution. Just before impact, the bar has only translatory motion and its angular momentum with respect to point A, by Eq. (88), will be

$$H_a = \frac{W}{g}\frac{v_c l}{2}\cos\alpha \tag{m}$$

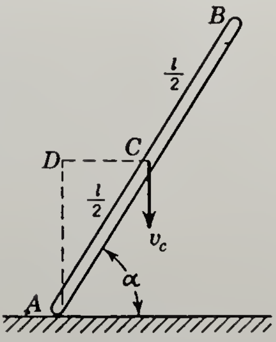

FIG. 329

Now since the impact force at A is vertical in the absence of friction, point C will continue to move vertically after impact but with some new velocity v'_c. Also point A will move horizontally so that the bar acquires some angular velocity ω'. Then, again by Eq. (88), the angular momentum of the bar about A just after impact will be

$$H'_a = \frac{W}{g}\frac{v'_c l}{2}\cos\alpha + \frac{W}{g}\frac{l^2}{12}\,\omega' \tag{n}$$

Equating expressions (m) and (n), we obtain

$$\omega' = \frac{6}{l}\,(v_c - v'_c)\cos\alpha \tag{o}$$

We see that in this case the condition of conservation of angular momentum alone is not sufficient to determine the two unknown velocities ω' and v'_c defining the new motion of the bar. However, since C continues to move vertically and A begins to move horizontally, we conclude that point D as shown in the figure is the instantaneous center of rotation for the bar just after impact. Hence, between ω' and v_c' there exists the relationship

$$v_c' = \frac{\omega' l}{2} \cos \alpha \tag{p}$$

Substituting this into Eq. (o), we find

$$\omega' = \frac{6v_c}{l} \frac{\cos \alpha}{1 + 3 \cos^2 \alpha} \tag{q}$$

after which Eq. (p) becomes

$$v_c' = 3v_c \frac{\cos^2 \alpha}{1 + 3 \cos^2 \alpha} \tag{r}$$

Equations (q) and (*r*) define the new motion of the bar just after impact.

PROBLEM SET 9.5

1. At what height h above the table surface should a billiard ball of radius c be struck by a horizontal impact X in order to have no sliding at the point of contact O (Fig. A)? *Ans.* $h = \frac{7}{5}c$.

2. A prismatic bar AB of length l attached to a string OA of length $l/2$ is supported by a smooth horizontal plane and rotates with uniform angular velocity ω around a vertical axis through O, as indicated by dotted lines in Fig. B. At what radius r should a peg P be inserted in the plane in order that, on striking it, the bar will come exactly to rest? *Ans.* $r = \frac{13}{12}l$.

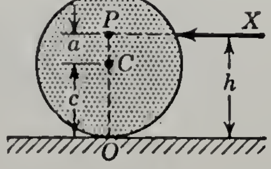

FIG. A

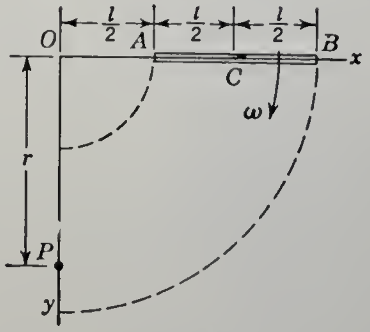

FIG. B

3. A solid right circular disk of radius r supported by a perfectly smooth horizontal table spins with angular velocity ω about its vertical geometric axis as shown in Fig. C. What new angular velocity ω' will the disk have if a point A on its circumference is suddenly pinned to the table? *Ans.* $\omega' = \omega/3$.

4. Solve Prob. 3 for the case of a square plate initially spinning about a vertical axis through its center of gravity if one corner is suddenly pinned down. *Ans.* $\omega' = \omega/4$.

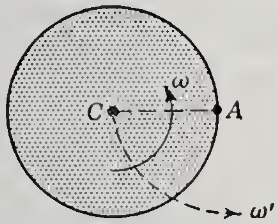

FIG. C

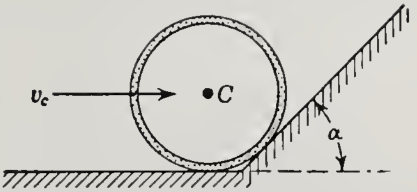

FIG. D

5. A homogeneous thin steel hoop of radius r rolls without slipping along a horizontal plane with velocity v_c and strikes an inclined plane as shown in Fig. D. With what velocity v_c' will the hoop start up the inclined plane if $\alpha = 45°$? Neglect any tendency to rebound or slip. *Ans.* $v_c' = 0.853v_c$.

6. A horizontal prismatic bar AB of length l and weight W is falling in a vertical plane with velocity v when suddenly the end A makes connections with a fixed pivot as shown in Fig. E. With what angular velocity ω' will the bar begin its rotation around the pivot A? *Ans.* $\omega' = 3v/2l$.

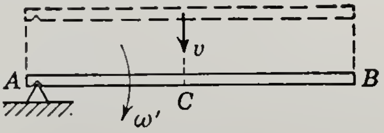

FIG. E

7. An arrow (slender prismatic bar) of length $l = 30$ in. traveling with velocity $v = 50$ fps strikes a smooth hard wall obliquely as shown in Fig. F. Assuming that the end A of the arrow does not penetrate but slides downward along the wall without friction or rebound, find the angular velocity ω' that it will acquire as a result of the impact if $\alpha = 30°$. *Ans.* $\omega' = 16.0\ \text{sec}^{-1}$.

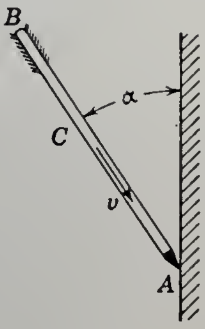

FIG. F

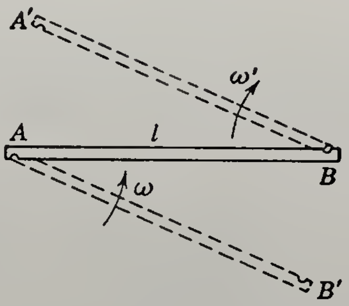

FIG. G

8. A slender prismatic bar AB of weight W and length l rotates on a smooth horizontal plane with constant angular velocity ω about a fixed pin at one end A (Fig. G). At an opportune moment another pin B is inserted in the plane so that, on coming round, the end B of the bar engages this pin and at the same time disengages the pin at A. What new angular velocity ω' will the bar have in its rotation around B? *Ans.* $\omega' = \frac{1}{2}\omega$.

9. Initially, a slender prismatic bar AB of weight W and length l is falling in a vertical plane with velocity v_c and without rotation (Fig. H). The end

B of the bar slides freely along a smooth vertical wall and the bar makes an angle α with the wall. Suddenly the end *A* of the bar comes into contact with a smooth horizontal floor and there is an impact. Assuming no tendency to rebound and that the ends *A* and *B* of the bar are frictionless, find the angular velocity ω' with which the bar begins its new motion after impact, if $\alpha = 30°$. *Ans.* $\omega' = 3v_c/4l$.

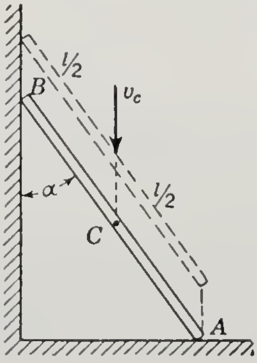

FIG. H

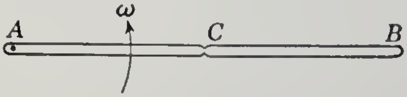

FIG. I

***10.** A slender prismatic bar *AB* of weight *W* and length *l* rotates in a horizontal plane with constant angular velocity ω about a fixed vertical axis through one end *A* (Fig. I). Suddenly, due to centrifugal tension, the bar snaps in two at the middle. What angular velocity ω' will the remaining portion *AC* have? How will the free portion *CB* move after fracture?

9.6. Energy equation for plane motion. Considering motion of a body parallel to the *xy* plane (Fig. 330), the kinetic energy of a particle of mass *dm* at point *A* is

$$\frac{dm}{2}(\dot{x}^2 + \dot{y}^2) \tag{a}$$

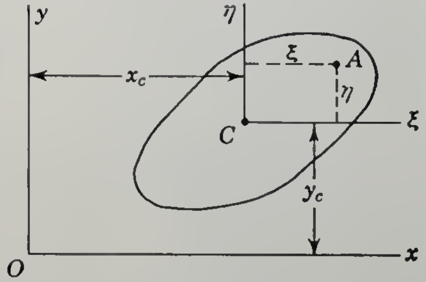

FIG. 330

Summing up such expressions for all particles in the body and using, for the components of the velocity, their expressions

$$\dot{x} = \dot{x}_c + \dot{\xi} \qquad \dot{y} = \dot{y}_c + \dot{\eta}$$

from Eqs. (b), page 448, we obtain, for the total kinetic energy of the body, the expression

$$\begin{aligned} T &= \tfrac{1}{2}\Sigma\, dm[(\dot{x}_c + \dot{\xi})^2 + (\dot{y}_c + \dot{\eta})^2] \\ &= \tfrac{1}{2}(\dot{x}_c^2 + \dot{y}_c^2)\Sigma\, dm + \tfrac{1}{2}\Sigma(\dot{\xi}^2 + \dot{\eta}^2)\, dm + \dot{x}_c\Sigma\dot{\xi}\, dm + \dot{y}_c\Sigma\dot{\eta}\, dm \end{aligned} \tag{b}$$

The last two terms in this expression vanish by virtue of Eqs. (c), page 448 and we have simply

$$T = \tfrac{1}{2}(\dot{x}_c^2 + \dot{y}_c^2)\Sigma\, dm + \tfrac{1}{2}\Sigma(\dot{\xi}^2 + \dot{\eta}^2)\, dm \tag{c}$$

Observing that $\dot{x}_c$ and $\dot{y}_c$ are the components of the resultant velocity v_c of the center of gravity *C* and that the second term in expression (c) represents the kinetic energy of rotation of the body with respect to C, we can write the expression for total kinetic energy in the simplified form

$$T = \frac{W}{g}\frac{v_c^2}{2} + I_c\frac{\dot{\theta}^2}{2} \tag{90}$$

It is seen from Eq. (90) that the total kinetic energy of a body performing plane motion consists of two parts. (1) the kinetic energy of the mass of the body supposedly concentrated at its center of gravity C and moving with this point and (2) the kinetic energy of rotation of the body about the axis through its center of gravity. For example, in the case of a solid right circular cylinder of weight W and radius a that rolls along a horizontal plane with velocity v and without slipping, we have for the total kinetic energy

$$T = \frac{W}{g}\frac{v^2}{2} + \frac{W}{g}\frac{r^2}{2}\frac{v^2}{2r^2} = \frac{3}{2}\frac{W}{g}\frac{v^2}{2}$$

Having the kinetic energy T in the case of motion of a rigid body or system of bodies parallel to a plane, we obtain the energy equation of motion by equating the change in total kinetic energy between any two positions or configurations of the system to the corresponding work of all external forces. Formulation of the energy equation for various cases of plane motion of a rigid body will now be illustrated by several examples.

EXAMPLES

1. Formulate the energy equation of motion for the solid of revolution rolling without sliding down an inclined plane (Fig. 331) and use this equation to find the velocity $\dot{x}_c$ of the body down the plane. Assume the initial conditions $(x_c)_0 = (\dot{x}_c)_0 = \dot{\theta}_0 = 0$.

Solution. The total kinetic energy of the body at any instant is

$$T = \frac{W}{g}\frac{\dot{x}_c^2}{2} + I_c\frac{\dot{\theta}^2}{2} = \frac{W\dot{x}_c^2}{2g}\left(1 + \frac{i_c^2}{r^2}\right) \tag{d}$$

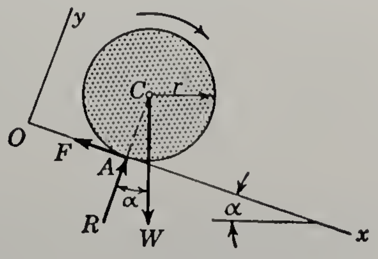

FIG. 331

Since in the initial position the body was at rest, we conclude that expression (d) is also the total change in kinetic energy.

In calculating the work of all external forces, we note that the normal reaction R acts always perpendicular to the direction of motion and therefore does not produce work. Likewise, if there is no slipping at the point of contact, the friction force F does not produce work. Thus the work

$$Wx_c \sin \alpha \tag{e}$$

of the gravity force W represents the total work of all forces.

Equating change in kinetic energy (d) to work (e), we obtain the energy equation of motion

$$\frac{W\dot{x}_c^2}{2g}\left(1 + \frac{i_c^2}{r^2}\right) = Wx_c \sin \alpha \tag{f}$$

from which

$$\dot{x}_c = \sqrt{\frac{2gx_c \sin \alpha}{1 + i_c^2/r^2}}$$

Differentiating Eq. (f) once with respect to time, we obtain

$$\frac{W}{g}\ddot{x}_c\left(1 + \frac{i_c^2}{r^2}\right) = W \sin \alpha$$

which coincides with Eq. (f) of Art. 9.3, page 436.

It is seen that by using the energy equation here we obtain a simplification due to the fact that it is not necessary to consider the reactive forces F and R, since they do no work. If the determination of these forces is required, the equations of motion discussed in Art. 9.3 must be used.

2. Formulate the energy equation of motion for the prismatic bar AB (Fig. 332), falling under the action of gravity from its vertical unstable position of equilibrium and use the equation to find the angular velocity $\dot{\theta}$ of the bar as a function of its angle of rotation θ. Neglect friction at A and B.

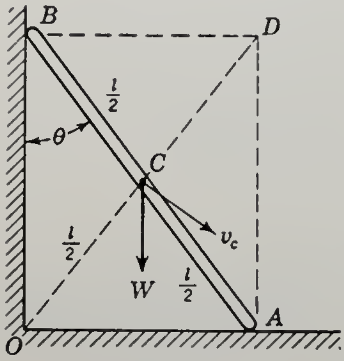

FIG. 332

Solution. We see from the figure that the velocity of the center of gravity C along its circular path of radius $l/2$ and center at O is $v_c = l\dot{\theta}/2$. Hence the total kinetic energy of the bar is

$$T = \frac{W}{g}\frac{(l\dot{\theta})^2}{8} + \frac{W}{g}\frac{l^2}{12}\frac{\dot{\theta}^2}{2} = \frac{1}{6}\frac{W}{g}l^2\dot{\theta}^2 \tag{g}$$

Since the bar started from rest, the initial kinetic energy is zero and Eq. (g) represents also the total change in kinetic energy.

In calculating the work of all external forces, we note that in the absence of friction the reactions R_a and R_b do not produce work and we have simply the work

$$W\frac{l}{2}(1 - \cos \theta) \tag{h}$$

of the vertical gravity force W. Equating change in kinetic energy (g) to work (h), we obtain

$$\frac{1}{6}\frac{W}{g}l^2\dot{\theta}^2 = W\frac{l}{2}(1 - \cos\theta) \tag{i}$$

from which

$$\dot{\theta}^2 = \frac{3g}{l}(1 - \cos\theta) \tag{j}$$

3. A body bounded at the bottom by a circular cylindrical surface of radius r rests on a horizontal plane as shown in Fig. 333. If rolled slightly from its position of stable equilibrium and then released, the body begins to oscillate. Formulate the energy equation of motion and find the period of oscillation, assuming that the body rolls without sliding and that the initial angle of rotation θ_0 is small.

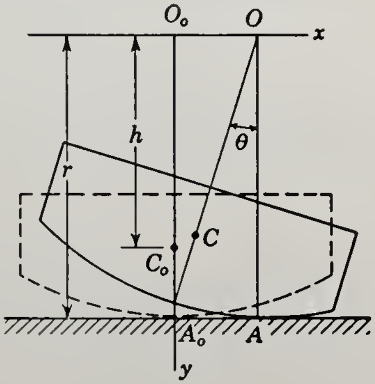

FIG. 333

Solution. We consider two positions of the body during its motion: (1) an extreme position in which the body is at rest and has the maximum angle of rotation θ_0 from the middle position of stable equilibrium (indicated by dotted lines in the figure) and angular velocity $\dot{\theta}_0 = 0$ and (2) any position defined by the angle of rotation θ from the same middle position. Choosing the coordinate axes x and y through the point O_0 as shown in the figure, we see that for any angle of rotation θ the coordinates x_c and y_c of the center of gravity C are

$$x_c = r\theta - h\sin\theta \qquad y_c = h\cos\theta \tag{k}$$

Thus, during motion of the body from an extreme position defined by the angle of rotation θ_0 to any other position defined by the angle of rotation θ, we conclude from the second of Eqs. (k) that the work of all acting forces is equal to

$$Wh(\cos\theta - \cos\theta_0) \tag{l}$$

since only the gravity force W produces work if there is rolling without sliding.

In the extreme position the body is at rest and has kinetic energy equal to zero. We conclude then that the kinetic energy in the position defined by the angle of rotation θ represents the change in kinetic energy. Denoting by W the weight of the body and by i_c its radius of gyration with respect to the axis through the center of gravity C, this change in kinetic energy is

$$\frac{W}{g}\frac{\dot{x}_c^2 + \dot{y}_c^2}{2} + \frac{W}{g}i_c^2\frac{\dot{\theta}^2}{2} \tag{m}$$

Equating change in kinetic energy, Eq. (m), to work, Eq. (l), and substituting for x_c and y_c their values from Eqs. (k), we obtain for the energy equation of motion

$$\dot{\theta}^2[(r^2 - 2rh\cos\theta + h^2) + i_c^2] = 2gh(\cos\theta - \cos\theta_0) \tag{n}$$

For small values of θ we have with good accuracy

$$\cos\theta = 1 - \frac{\theta^2}{2}$$

Substituting this into Eq. (n) and neglecting small quantities of second order, except where their differences are involved, we obtain

$$\dot{\theta}^2[(r - h)^2 + i_c^2] = gh(\theta_0^2 - \theta^2)$$

which, by differentiation, becomes

$$\ddot{\theta} + \frac{gh}{(r - h)^2 + i_c^2}\theta = 0 \tag{o}$$

From this equation we conclude that for small amplitudes, the oscillations of the body are harmonic and that the period of oscillation is

$$\tau = \frac{2\pi}{p} = 2\pi\sqrt{\frac{(r - h)^2 + i_c^2}{gh}} \tag{p}$$

i.e., the same as for a simple pendulum of length

$$L = \frac{(r - h)^2 + i_c^2}{h} \tag{q}$$

PROBLEM SET 9.6

1. A car body of weight $W = 100$ lb has four solid disk wheels each of weight $Q = 25$ lb and rolls along a horizontal plane with constant speed $v = 20$ fps. Calculate the total kinetic energy of the system if the wheels roll without slipping. *Ans.* $T = 1{,}554$ ft-lb.

2. Referring to Fig. A, find the velocity v_c that the right circular cylinder of weight W and radius r will acquire after falling from rest through a vertical distance h. *Ans.* $v_c = \sqrt{4gh/3}$.

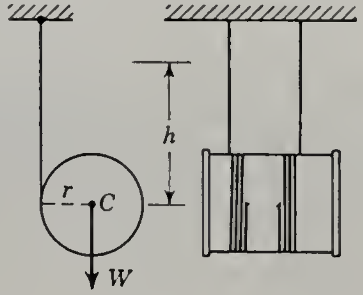

FIG. A

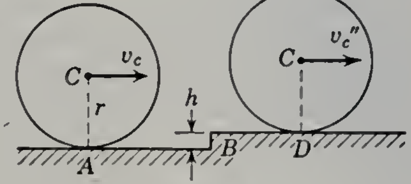

FIG. B

3. In Fig. B, a homogeneous solid right circular cylinder of weight $W = 100$ lb and radius $r = 4$ in. rolls to the right on a horizontal plane with constant speed $v_c = 5$ fps. At B, it hits a curb of height $h = \frac{1}{2}$ in. Assuming no rebound or slip at B, find the velocity v_c'' with which the cylinder will roll out along the horizontal plane at D. How much kinetic energy did the cylinder lose between A and D? *Ans.* $v_c'' = 4.38$ fps.

4. Two prismatic bars AC and BC of equal weights W and lengths l are hinged together at C and are supported on a perfectly smooth horizontal plane as shown in Fig. C. If, owing to sliding of the ends A and B, the bars fall in their own vertical plane, find the velocity v with which the hinge C will strike the plane. The initial height of C above AB is h. *Ans.* $v = \sqrt{3gh}$.

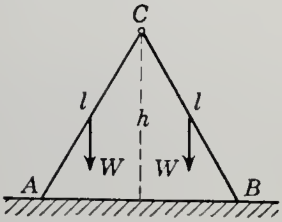

FIG. C

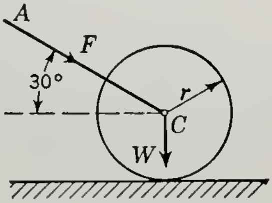

FIG. D

5. A workman moves a cylindrical lawn roller of weight $W = 1{,}000$ lb and radius $r = 1$ ft along a horizontal plane by pushing with a constant force F in the direction AC (Fig. D). What is the magnitude of this force if, after a horizontal displacement $x = 12$ ft, the roller has a velocity $v = 4$ fps? Assume that the cylinder rolls without sliding. *Ans.* $F = 35.9$ lb.

6. A prismatic bar AB of weight W and length $l = \sqrt{2}\,r$ starts from rest in the position shown in Fig. E and under the action of gravity slides without friction along the constraining vertical plane curve ABD, the portion AB of which is a quadrant of a circle of radius r and the portion BD of which is a horizontal tangent to this circle. With what uniform velocity v will the bar move along the horizontal portion BD? *Ans.* $v = \sqrt{gr}$.

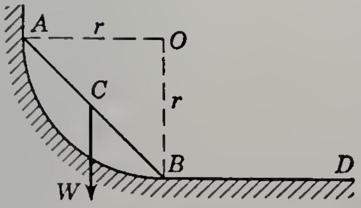

FIG. E

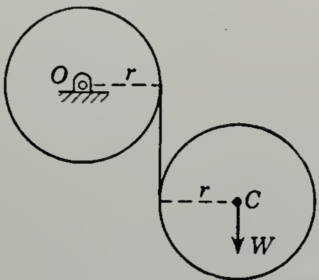

FIG. F

7. Using the method of work and energy, find the velocity v_c of the falling disk in Fig. F as a function of the height h through which it has fallen from rest. *Ans.* $v_c = \sqrt{8gh/5}$.

8. (*a*) Find the period of oscillation of a homogeneous right semicircular cylinder of radius $r = 1$ ft for small amplitudes of rolling on a horizontal plane. (*b*) Repeat for a homogeneous hemisphere of radius $r = 1$ ft. *Ans.* $\tau_a = 1.37$ sec; $\tau_b = 1.46$ sec.

9. For the one-cylinder gas engine shown in Fig. 316, page 430, the following numerical data are given: weight of piston and piston rod, 32.2 lb; weight of connecting rod, 21.5 lb; length of connecting rod, 3 ft. The center of gravity of the connecting rod is 1 ft from A, and its radius of gyration with respect to the centroidal axis normal to the plane of motion is $i_c = 0.85$ ft. The crank radius $r = 1$ ft, and the engine runs uniformly at 600 rpm. Calculate the kinetic energy T_1 of the piston and piston rod and the kinetic energy T_2 of the connecting rod for the configuration shown in the figure where the angle $\omega t = 60°$. *Ans.* $T_1 = 2{,}035$ ft-lb; $T_2 = 1{,}280$ ft-lb.

***10.** A thin steel hoop of weight W and radius r starts from rest at A and rolls down along a circular cylindrical surface of radius a as shown in Fig. G. Determine the angle φ defining the position of point B where the hoop will begin to slip if the coefficient of friction at the point of contact is $\mu = \frac{1}{3}$. *Ans.* $\varphi = 29°32'$.

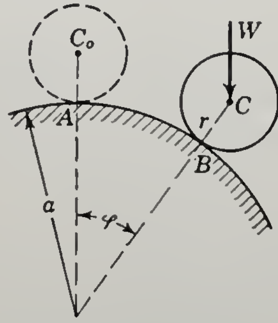

FIG. G

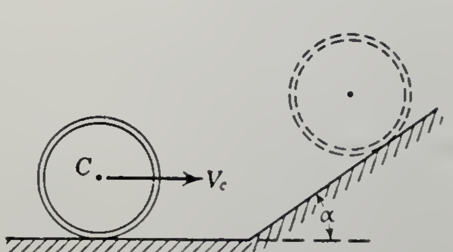

FIG. H

***11.** Referring to Fig. G, assume that the roller is a gear wheel with pitch radius r and radius of gyration i_c rolling on a cylindrical rack with pitch radius a so that there is no possibility of slipping even without friction. Under these conditions, find the value of the angle φ at which the roller will jump clear of the rack if it starts from rest at A. Data are given as follows: $a = 12$ in., $r = 4$ in., $i_c = 3$ in. *Ans.* $\varphi = 55°51'$.

12. The thin cylindrical shell in Fig. H approaches the inclined plane at the right with velocity v_c as shown, rolls up this plane until brought to rest by gravity, and then rolls back down again. What will be its velocity v_c'' as it finally rolls to the left along the horizontal plane? Assume that the shell at all times rolls without slipping and that there is no rebound after impacts. *Ans.* $v_c'' = \frac{1}{4}v_c(1 + \cos\alpha)^2$.

10

RELATIVE MOTION

10.1. Kinematics of relative motion. In all our previous discussions we have considered the motion of a particle or of a body with respect to a system of fixed coordinate axes. There are cases, however, where it is necessary to investigate the motion of a particle or body relative to another body that is in motion. Strictly speaking, we never observe any motion with respect to fixed axes. In all of our laboratory experiments we are observing motion with respect to coordinate axes that are moving with the earth. Although the motion of axes connected to the earth is usually of no practical significance in the solution of engineering problems and can be disregarded, there are many other cases where motion with respect to moving axes must be considered.

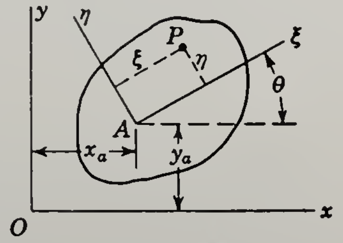

FIG. 334

We begin our discussion of *relative motion* with the case of motion parallel to the fixed xy plane (Fig. 334). If a rigid body moves in this plane, its motion is completely defined by the coordinates x_a and y_a of a pole A and the angle of rotation θ around this pole. Through the pole A and in the plane of motion, we take rectangular coordinate axes ξ and η that are fixed in the body and move with it as shown in the figure. Then if a particle P is moving in the $\xi\eta$ plane, its position in the body is defined by the *relative coordinates* ξ and η which are changing with time. Taking the derivatives $\dot{\xi}$ and $\dot{\eta}$, we obtain the components of the *relative velocity* of the particle P as they would be measured by an observer moving with the body. In the same way, by taking the second derivatives $\ddot{\xi}$ and $\ddot{\eta}$, we obtain the components of the *relative acceleration* of the particle P.

The *absolute coordinates* of the particle P are obtained by taking the coordinates of its projections on the fixed coordinate axes x and y.

Thus we obtain

$$\begin{aligned} x &= x_a + \xi \cos\theta - \eta \sin\theta \\ y &= y_a + \xi \sin\theta + \eta \cos\theta \end{aligned} \tag{a}$$

Taking the derivatives with respect to time of these expressions, we obtain for the components, in the x and y directions, of the *absolute velocity* of the particle P

$$\begin{aligned} \dot{x} &= \dot{x}_a - (\xi \sin\theta + \eta \cos\theta)\dot{\theta} + \dot{\xi}\cos\theta - \dot{\eta}\sin\theta \\ \dot{y} &= \dot{y}_a + (\xi \cos\theta - \eta \sin\theta)\dot{\theta} + \dot{\xi}\sin\theta + \dot{\eta}\cos\theta \end{aligned} \tag{b}$$

The first two terms in each of these expressions are obtained by considering ξ and η as constants in differentiating expressions (a). Thus they represent the components of the velocity of that point of the body with which the moving particle P coincides at the instant under consideration. The last two terms in each of expressions (b) represent the projections on the fixed axes x and y of the relative velocity of the particle P. Thus we conclude from Eqs. (b) that the absolute velocity of the particle P may be considered as the geometric sum of two velocities: (1) the velocity of that point of the body with which the particle coincides and which we call the *base velocity* and denote by v_b and (2) the relative velocity of the particle, with respect to the moving body, which we denote by v_r. Thus vectorially the absolute velocity of the particle P can be put in the following form:

$$\bar{v} = \bar{v}_b \mathbin{+\!\!\!\!\rightarrow} \bar{v}_r \tag{91}$$

Taking the second derivatives with respect to time of expressions (a), we obtain for the components, in the x and y directions, of the *absolute acceleration* of the particle P

$$\begin{aligned} \ddot{x} &= \ddot{x}_a - (\xi \sin\theta + \eta \cos\theta)\ddot{\theta} - (\xi \cos\theta - \eta \sin\theta)\dot{\theta}^2 \\ &\quad - 2(\dot{\xi}\sin\theta + \dot{\eta}\cos\theta)\dot{\theta} + \ddot{\xi}\cos\theta - \ddot{\eta}\sin\theta \\ \ddot{y} &= \ddot{y}_a + (\xi \cos\theta - \eta \sin\theta)\ddot{\theta} - (\xi \sin\theta + \eta \cos\theta)\dot{\theta}^2 \\ &\quad + 2(\dot{\xi}\cos\theta - \dot{\eta}\sin\theta)\dot{\theta} + \ddot{\xi}\sin\theta + \ddot{\eta}\cos\theta \end{aligned} \tag{c}$$

The first three terms in each of these expressions are obtained by considering ξ and η as constants in differentiation. Thus we conclude that they represent the components of the acceleration of that point of the body with which the moving particle P coincides at any given instant. This is called the *base acceleration*, denoted by a_b. The remaining terms in each expression are due to the relative motion of the particle with respect to the body. We can subdivide these terms into two groups. (1) the terms containing $\ddot{\xi}$ and $\ddot{\eta}$ and representing the projections on the fixed axes of the *relative acceleration* of the particle and (2) the terms having the factor 2 and representing the so-called *supplementary acceleration.*[1]

The supplementary acceleration can be visualized by the following geometric representation. Let $\overline{OB}$ (Fig. 335) be a vector representing the relative velocity $\bar{v}_r$ of the particle, the projections of which as shown in the figure are represented by the last two terms in each of Eqs. (b). Now as the system shown in Fig. 334 moves in the xy plane, imagine that the vector $\overline{OB}$ rotates around the origin O with angular velocity $\dot{\theta}$. Thus the end B of this vector has the velocity $v_r\dot{\theta}$ perpendicular to OB as shown. This velocity of the end B of the relative velocity vector $\bar{v}_r$ has the dimension of acceleration. To calculate its projection on the fixed x axis, we note from Fig. 335 that the cosine of the angle that it makes with the x axis is $-(\dot{\xi} \sin \theta + \dot{\eta} \cos \theta)/v_r$. Thus the required projection is

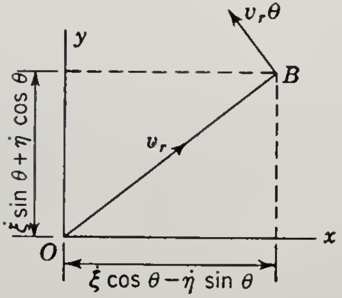
$v_r\dot{\theta}$
y
B
$\dot{\xi} \sin \theta + \dot{\eta} \cos \theta$
v_r
O
x
$\dot{\xi} \cos \theta - \dot{\eta} \sin \theta$

FIG. 335

$$-v_r\dot{\theta}\,\frac{\dot{\xi} \sin \theta + \dot{\eta} \cos \theta}{v_r} = -\dot{\theta}(\dot{\xi} \sin \theta + \dot{\eta} \cos \theta) \tag{d}$$

In the same manner we find the projection on the y axis to be

$$v_r\dot{\theta}\,\frac{\dot{\xi} \cos \theta - \dot{\eta} \sin \theta}{v_r} = \dot{\theta}(\dot{\xi} \cos \theta - \dot{\eta} \sin \theta) \tag{e}$$

Comparing expressions (d) and (e) with the components of the supplementary acceleration in expressions (c), we conclude that this acceleration is represented by the doubled velocity of the end of the vector representing the relative velocity of the particle and rotating with angular velocity $\dot{\theta}$.

Denoting by a_b the base acceleration, by a_r the relative acceleration, and by a_s the supplementary acceleration, we conclude from Eqs. (c) that the absolute acceleration can be represented vectorially as follows:

$$\bar{a} = \bar{a}_b \mathbin{+\!\!\!\rightarrow} \bar{a}_r \mathbin{+\!\!\!\rightarrow} \bar{a}_s \tag{92}$$

In the foregoing discussion it has been assumed that the particle P moves in the $\xi\eta$ plane. If the particle has also some motion in the direction perpendicular to the $\xi\eta$ plane, this motion will not be affected by the motion of the body parallel to the xy plane and that component of the absolute acceleration of P perpendicular to the xy plane is equal to its relative acceleration in the same direction. Thus Eq. (92) holds also in this case, provided a_r is taken as the complete acceleration in relative motion of the particle P.

---

[1] The existence of this supplementary acceleration in the case of relative motion was first pointed out by Coriolis, and it is sometimes called the Coriolis acceleration.

EXAMPLES

1. A particle P moves with constant relative velocity v_r along the circumference of a circular disk of radius r (Fig. 336), while the disk rotates with uniform angular velocity ω in the opposite direction. Find the absolute acceleration a of the particle.

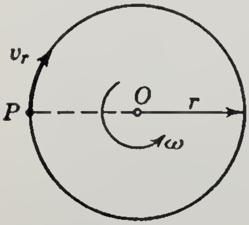

FIG. 336

Solution. The relative acceleration of the particle is directed toward the center of the disk and is

$$a_r = \frac{v_r^2}{r} \tag{f}$$

The acceleration of a point on the disk coinciding with an instantaneous position of the particle is also directed toward the center of the disk, and this base acceleration is

$$a_b = \omega^2 r \tag{g}$$

Using the graphical conception of supplementary acceleration as discussed in connection with Fig. 335, we conclude that in this case the supplementary acceleration is

$$a_s = 2\omega v_r \tag{h}$$

and that it has the radial direction away from the center. Thus the absolute acceleration, from (f), (g), and (h), is

$$a = \frac{v_r^2}{r} + \omega^2 r - 2\omega v_r = r\left(\frac{v_r}{r} - \omega\right)^2 \tag{i}$$

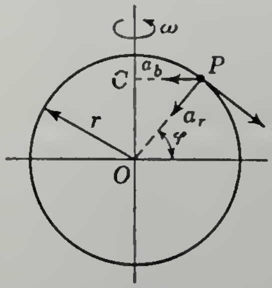

FIG. 337

directed toward the center of the disk. When $v_r = r\omega$, the particle is immovable with respect to fixed axes, and in such case, as is seen from Eq. (i), its absolute acceleration is zero.

2. A particle P moves with uniform relative velocity v_r along a meridian of a sphere of radius r which rotates about a fixed vertical diameter with uniform angular velocity ω (Fig. 337). Find its absolute acceleration for the position defined by the angle φ.

Solution. The point of the sphere with which the instantaneous position of the particle P coincides is evidently rotating in a circular path of radius

$r \cos \varphi$ and with uniform angular velocity ω. Hence the base acceleration is

$$a_b = \omega^2 r \cos \varphi \tag{j}$$

directed as shown in the figure.

Since the relative motion of the particle is uniform along the meridian, the acceleration due to relative motion is directed toward the center of the sphere as shown in the figure and has the magnitude

$$a_r = \frac{v_r^2}{r} \tag{k}$$

The supplementary acceleration of the particle, equal to the doubled velocity of the end of the vector v_r rotating with uniform angular velocity ω around a vertical through P, is perpendicular to the plane of the figure and directed away from the reader. Its magnitude is

$$a_s = 2\omega v_r \sin \varphi \tag{l}$$

The absolute acceleration of the particle P is obtained as the geometric sum of the accelerations (j), (k), and (l).

PROBLEM SET 10.1

1. Assuming that the disk shown in Fig. A has angular velocity $\dot{\theta} = 6 \text{ sec}^{-1}$ and angular acceleration $\ddot{\theta} = 1 \text{ sec}^{-2}$ and that the particle P moves in the opposite direction around the circumference with uniform relative velocity $v_r = 3$ fps, find its absolute acceleration if $r = 1$ ft. *Ans.* $a = 9.06 \text{ ft/sec}^2$.

2. A particle P_1 moves with uniform relative velocity $v_r = 12$ fps along the chord AB of the disk shown in Fig. A while the disk rotates with uniform angular velocity $\omega = 20\pi \text{ sec}^{-1}$. Determine the absolute velocity and acceleration of the particle for the position shown if $r = 12$ in. and $h = 6$ in. Assume that at this instant the particle is moving in the direction of rotation of the disk. *Ans.* $v = 43.4$ fps, in tangential direction; $a = 3{,}483 \text{ ft/sec}^2$, toward the center.

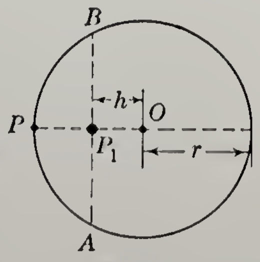

FIG. A

3. If the particle P_1 shown in Fig. A performs simple harmonic motion along the full length of the chord AB and with the period $\tau = 2\pi/\omega$ while the disk rotates with uniform angular velocity ω, find its maximum absolute acceleration for the position shown. Use same data as in Prob. 2. *Ans.* $a = 8{,}812 \text{ ft/sec}^2$, toward the center.

4. If the particle P (Fig. A) starts from O and moves with uniform relative velocity v_r along the radius OP, find its absolute acceleration when it reaches the edge of the disk, which rotates with uniform angular velocity ω. *Ans.* $a = \omega \sqrt{(\omega r)^2 + (2v_r)^2}$

5. If the particle P (Fig. A) performs simple harmonic motion along the full length of a diameter of the disk and with the period $\tau = 2\pi/\omega$, find its absolute acceleration when it is in one of its extreme positions at the edge of the disk. The disk rotates with uniform angular velocity ω. *Ans.* $a = 2r\omega^2$, toward the center.

6. If P (Fig. 337) represents a particle of water in a river flowing south with velocity of flow equal to 3 fps and the latitude $\varphi = 50°$, find the supplementary acceleration of the particle. Assume the radius of the earth to be $r = 209 \times 10^5$ ft. *Ans.* $a_s = 0.000334$ ft/sec², directed east.

7. A racing car in Indianapolis (latitude $\varphi = 40°$N) travels due north along a straight level track with constant speed $v_r = 240$ mph. Calculate the supplementary acceleration a_s, assuming the radius of the earth to be $r = 209 \times 10^5$ ft. *Ans.* $a_s = 0.033$ ft/sec², directed west.

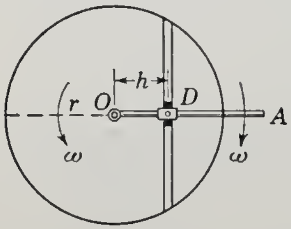

FIG. B

8. In Fig. B, a disk of radius r rotates counterclockwise about a fixed axis through O with constant angular velocity ω. At the same time the arm OA rotates clockwise about the same fixed axis with the same angular velocity ω. For the configuration shown, find the absolute acceleration a of the block D which slides freely in a slot cut in the face of the disk and also along the axis of the bar OA. Assume that $h = \frac{1}{2}r$. *Ans.* $a = 1.5\omega^2 r$, outward.

***9.** Referring again to the system in Fig. B, find the magnitude of the absolute acceleration of the block D when it reaches the end of the slot in the disk. *Ans.* $a = 27.9\omega^2 r$.

10.2. Equations of relative motion. In the formulation of the equations of relative motion of a particle, we observe that in accordance with Newton's second law of motion (see page 256) the resultant force F acting on the particle is equal to the product of its mass and its absolute acceleration a and acts in the direction of this acceleration. Multiplying both sides of Eq. (92) by the mass W/g of the particle, we obtain

$$\bar{F} = \frac{W}{g}\,(\bar{a}_b \mathrel{+\!\!\!\!\rightarrow} \bar{a}_r \mathrel{+\!\!\!\!\rightarrow} \bar{a}_s)$$

from which

$$\frac{W}{g}\,\bar{a}_r = \bar{F} \longrightarrow \frac{W}{g}\,\bar{a}_b \longrightarrow \frac{W}{g}\,\bar{a}_s \tag{93}$$

The second and third terms on the right side of Eq. (93) can be regarded as inertia forces corresponding to the accelerations a_b and a_s. Thus the equation of relative motion of a particle has the same form as the equation of absolute motion of a particle, provided we consider, in addition to the resultant force F, the inertia forces due to the accelerations a_b and a_s.

Equation (93) is given in vectorial form. Projecting the force F and the accelerations a_r, a_b, and a_s on rectangular coordinate axes ξ, η, fixed in the body with respect to which the particle is moving, we can obtain two equations of relative motion of a particle in the same form as Eqs. (57) of Art. 7.2 for absolute motion of a particle. In writing these equations for any particular case of relative motion, we must consider not only the projections of all real forces on the coordinate axes ξ, η, but also the projections on these axes of the two inertia forces due to the base and supplementary accelerations a_b and a_s.

If the body with respect to which we are investigating the relative motion of a particle has a uniform translatory motion, the accelerations a_b and a_s vanish, as may be seen from the discussion of the previous article, and Eq. (93) reduces to

$$\frac{W}{g}\,\bar{a} = \bar{F}$$

which coincides with Eq. (33) for absolute motion of a particle. Thus the relative motion in such case will be the same as if the body on which the particle is moving were at rest. Dynamical experiments made in a car moving uniformly along a straight level track will give the same results as in a stationary laboratory. However, if the car enters a curve, the effect of the accelerations a_b and a_s will at once be perceived.

From the equation of relative motion [Eq. (93)] an equation of *relative equilibrium* of a particle can be obtained. For such equilibrium we need only to assume that the relative velocity v_r and relative acceleration a_r vanish. Then in Eq. (93) only the base acceleration a_b remains and the equation reduces to

$$\bar{F} \longrightarrow \frac{W}{g}\,\bar{a}_b = 0 \tag{94}$$

Thus we conclude that for *relative equilibrium,* not the resultant force F, but the resultant force F together with the inertia force due to the base acceleration must be zero. Equation (94) like Eq. (93) is expressed in vectorial form. We conclude from it that in the case of relative equilibrium of a particle with respect to a moving body, the algebraic sum of the projections, on any axis, of all real forces acting on the particle together with the inertia force due to acceleration a_b must be zero.

As an example of the formulation of the equations of relative motion for a particular case, let us consider motion of the system shown in

Fig. 338. This system consists of two equal weights W that can slide without friction along two spokes of a flywheel uniformly rotating in a horizontal plane with angular velocity ω as shown. Each weight is attached to the rim of the wheel by a helical spring having a spring constant k.

In investigating possible relative motion of the two weights W, we denote by b the distance of the center of gravity of each weight from the axis of rotation of the flywheel when the springs are unstrained. We assume also that the weights are always equidistant from the axis of rotation. Then their positions along the spokes at any instant are completely defined by the relative displacement $b + \xi$ from the center of rotation, ξ being considered positive in the direction away from the axis. The relative acceleration of each weight then is $\ddot{\xi}$. The base acceleration a_b is directed toward the center of the wheel and has the magnitude $\omega^2(b + \xi)$. The supplementary acceleration a_s is directed perpendicular to the diameter AB and is equal to $2\omega\dot{\xi}$. The direction of this acceleration for each weight is shown in the figure, assuming that $\dot{\xi}$ is positive, that is, that the weights are moving toward the rim of the wheel.

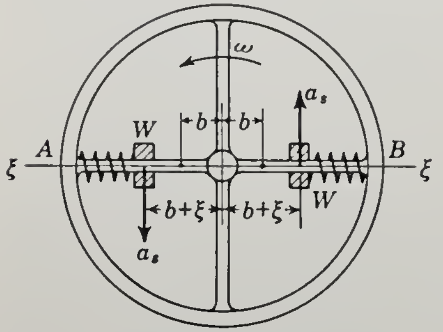

FIG. 338

Using Eq. (93) and projecting all forces and accelerations onto the diameter AB, we find for the equation of relative motion of either weight

$$\frac{W}{g}\ddot{\xi} = -k\xi + \frac{W}{g}(b + \xi)\omega^2 \tag{a}$$

Projecting the same forces and accelerations onto an axis perpendicular to AB, we find

$$0 = R - 2\frac{W}{g}\omega\dot{\xi} \tag{b}$$

where R is the reaction exerted by the guiding spoke on the moving weight W. We see from Eq. (b) that for either weight this reaction R has the same direction as the supplementary acceleration a_s. The actions of the weights on the spokes have opposite directions, and we see that, when the weights are moving apart, a reactive couple is acting on the flywheel in the direction opposing rotation.

From Eq. (a) we may write

$$\frac{W}{g}\ddot{\xi} = -\left(k - \frac{W}{g}\omega^2\right)\xi + \frac{W}{g}\omega^2 b \tag{c}$$

Then setting $\ddot{\xi} = 0$, we obtain for the condition of relative equilibrium

$$\left(k - \frac{W}{g}\omega^2\right)\xi_s = \frac{W}{g}\omega^2 b \tag{d}$$

in which ξ_s denotes the corresponding radial displacements of the weights W from the positions corresponding to zero stress in the springs. From Eq. (d), the magnitude of ξ_s is

$$\xi_s = \frac{(W/g)b\omega^2}{k - (W/g)\omega^2} \tag{e}$$

When this quantity is positive, i.e., for

$$k > \frac{W}{g}\omega^2$$

Eq. (c) represents simple harmonic vibrations having the period

$$\tau = 2\pi\sqrt{\frac{W}{g[k - (W/g)\omega^2]}} \tag{f}$$

From this expression we conclude that the effect of rotation of the wheel on the period of vibration of the weights W is represented by an effective reduction of the spring constants by the amount $W\omega^2/g$.

For such angular velocity ω of the flywheel that

$$k \leqq \frac{W}{g}\omega^2$$

the weights will always press against the rim of the wheel and there will be no vibrations.

As a second example of the formulation of equations of relative motion of a particle, let us consider the effect of rotation of the earth on the motion of a body falling under the action of gravity (Fig. 339). At point A, with latitude φ, we take the system of rectangular coordinate axes ξ, η, ζ which are fixed in the earth and moving with it. The ξ axis is tangent to the meridian at A and the η axis is tangent at the same point to a parallel circle while the ζ axis has the radial direction OA. In writing Eq. (93) for a falling body at A, we shall consider only accelerations a_r and a_s.* In calculating the supplementary acceleration a_s, we resolve the relative velocity of the falling body into components $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$ and consider the supplementary acceleration due to each component. Assuming these components to be represented by vectors as shown at the bottom of the figure and considering the velocities of the ends of these vectors due to rotation of the earth with angular velocity ω, we see that the components of supplementary acceleration corresponding to $\dot{\xi}$ and $\dot{\zeta}$ are perpendicular to the $\xi\zeta$ plane and are equal to $2\omega\dot{\xi}\sin\varphi$ and $2\omega\dot{\zeta}\cos\varphi$, respectively. The component of supplementary acceleration corresponding to $\dot{\eta}$ is in the $\xi\zeta$ plane and perpendicular to the earth's axis. Its magnitude is $2\omega\dot{\eta}$.

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* Calculations show that the effect of base acceleration a_b on the relative motion of a falling body is negligible and that only the supplementary acceleration a_s and relative acceleration a_r need be considered.

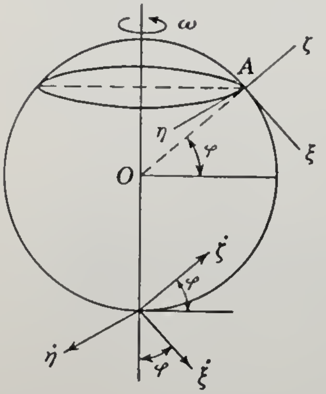


FIG. 339

Keeping in mind the directions of the three components of supplementary acceleration, the differential equations of relative motion of a falling body at A become

$$
\begin{aligned}
\frac{W}{g}\ddot{\xi} &= -2\frac{W}{g}\omega\dot{\eta}\sin\varphi \\
\frac{W}{g}\ddot{\eta} &= +2\frac{W}{g}\omega(\dot{\xi}\sin\varphi + \dot{\zeta}\cos\varphi) \qquad \text{(g)} \\
\frac{W}{g}\ddot{\zeta} &= -W - 2\frac{W}{g}\omega\dot{\eta}\cos\varphi
\end{aligned}
$$

Assuming that the initial velocity of the falling body is zero and integrating the first and third of Eqs. (g), we obtain

$$\dot{\xi} = -2\omega\eta\sin\varphi \qquad \dot{\zeta} = -gt - 2\omega\eta\cos\varphi \tag{h}$$

Substituting these values into the second of Eqs. (g), we obtain

$$\ddot{\eta} = -4\omega^2\eta - 2\omega gt\cos\varphi \tag{i}$$

Observing that ω is a small quantity and neglecting the term containing ω^2 in this equation, we obtain by integration

$$\eta = -\tfrac{1}{3}\omega gt^3\cos\varphi \tag{j}$$

Substituting this into Eqs. (h) and neglecting again terms containing the factor ω^2, we find

$$\xi = 0 \qquad \zeta = h - \tfrac{1}{2}gt^2 \tag{k}$$

where h is the initial height from which the body starts to fall. Equations (j) and (k) define the relative motion of the falling body, and we see that the last two of these equations coincide with what would be obtained on the assumption that the earth is at rest. Equation (j) indicates that the falling body deviates from the vertical ζ axis toward the east. To obtain this deviation for the instant when the falling body reaches the earth, we substitute $\zeta = 0$ in the second of Eqs. (k) and obtain

$$t = \sqrt{\frac{2h}{g}}$$

which when substituted for t in Eq. (j) gives

$$\eta = -\frac{2}{3}\omega\sqrt{\frac{2h^3}{g}}\cos\varphi \tag{l}$$

Actual measurements of the deviation from the vertical of falling bodies are in good agreement with Eq. (l).

EXAMPLES

1. Determine the condition of relative equilibrium of a particle P that is constrained to move without friction along a meridian AOB of a vessel having the form of a generated body and rotating about its geometric axis Oz with uniform angular velocity ω (Fig. 340).

Solution. From Eq. (94) we conclude that to the real forces W and R acting on the particle P as shown, the inertia force $-(W/g)\,a_b$ must be added. In this case $a_b = \omega^2 r$ and the corresponding inertia force is directed as shown in the figure. Projecting all forces onto the tangent to the meridian through P, we find

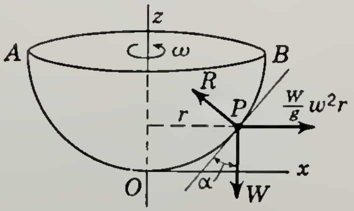


FIG. 340

$$W\cos\alpha - \frac{W}{g}\omega^2 r\sin\alpha = 0 \tag{m}$$

Thus the condition of relative equilibrium is

$$\tan\alpha = \frac{g}{\omega^2 r} \tag{n}$$

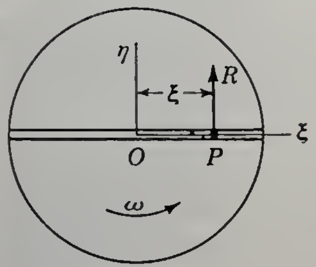


FIG. 341

2. Investigate the relative motion of a particle P along a smooth diametral groove in a horizontal disk rotating with constant angular velocity ω about its vertical geometric axis (Fig. 341) and determine the reaction of the side of the groove on the moving particle. Assume that the particle has a small initial relative displacement ξ_0 from the center of the disk but no initial velocity.

Solution. Taking the center O of the disk as the origin and the axis of the groove as the ξ axis, the equations of relative motion of the particle P become

$$\frac{W}{g}\ddot{\xi} = \frac{W}{g}\omega^2\xi \qquad R = 2\frac{W}{g}\omega\dot{\xi} \tag{o}$$

where R is the reaction of the groove on the particle. Multiplying both sides of the first of Eqs. (o) by $\dot{\xi}\, dt$, it may be represented in the following form:

$$d(\dot{\xi})^2 = \omega^2\, d(\xi)^2 \tag{p}$$

Integrating and remembering that $\dot{\xi}_0 = 0$, we find

$$\dot{\xi} = \omega \sqrt{\xi^2 - \xi_0^2} \tag{q}$$

from which

$$\frac{d\xi}{\sqrt{\xi^2 - \xi_0^2}} = \omega\, dt \tag{r}$$

Integration of Eq. (r) gives

$$\operatorname{arccosh} \frac{\xi}{\xi_0} = \omega t + C$$

or

$$\xi = \xi_0 \cosh (\omega t + C)$$

and remembering that for $t = 0$, $\xi = \xi_0$, we conclude that $C = 0$. Thus the equation of relative motion of the particle is

$$\xi = \xi_0 \cosh \omega t \tag{s}$$

From this equation we conclude that the particle has a position of unstable relative equilibrium at the center of the disk and that, if displaced slightly from this position, it continues to move radially outward.

The second of Eqs. (o) expresses the reaction R as a function of the relative velocity $\dot{\xi}$ of the particle. Replacing $\dot{\xi}$ in this equation by its value (q), we obtain the reaction R as a function of the relative displacement ξ of the particle. Thus

$$R = 2 \frac{W}{g} \omega^2 \sqrt{\xi^2 - \xi_0^2}$$

Differentiating expression (s) once with respect to time, we obtain

$$\dot{\xi} = \xi_0 \omega \sinh \omega t$$

and substituting this for $\dot{\xi}$ in the second of Eqs. (o), we obtain

$$R = 2 \frac{W}{g} \omega^2 \xi_0 \sinh \omega t$$

which expresses the reaction R as a function of time.

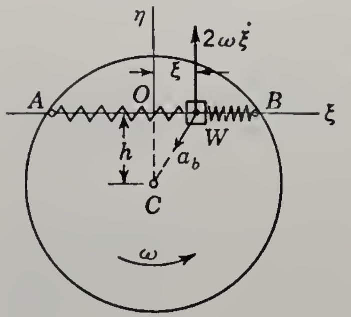


FIG. 342

3. A particle W can slide without friction along the chord AB of a circular disk which rotates in a horizontal plane about its vertical geometric axis (Fig. 342). Determine the period of vibration of the particle if it is attached to points A and B by two springs each having the spring constant $k/2$. Assume that the mid-point O on the chord AB is the position of relative equilibrium of the particle.

Solution. Taking OB as the ξ axis and considering any position of the particle as defined by the relative coordinate ξ, we see that its relative velocity and acceleration are $\dot{\xi}$ and $\ddot{\xi}$, respectively. The point on the disk with which the particle coincides has a uniform circumferential velocity $v_b = \omega\sqrt{h^2 + \xi^2}$, where h is the distance of the chord AB from the center C of the disk. The corresponding base acceleration is

$$a_b = \omega^2\sqrt{h^2 + \xi^2}$$

The supplementary acceleration has the magnitude $2\omega\dot{\xi}$ and is perpendicular to AB. Its direction for counterclockwise rotation of the disk and for positive $\dot{\xi}$ is shown in the figure. Projecting all forces and accelerations onto the ξ axis and using Eq. (93), we obtain

$$\frac{W}{g}\ddot{\xi} = -k\xi + \frac{W}{g}\omega^2\xi = -\left(k - \frac{W}{g}\omega^2\right)\xi \tag{t}$$

This is the differential equation of simple harmonic motion having the period

$$\tau = 2\pi\sqrt{\frac{W}{g[k - (W/g)\omega^2]}} \tag{u}$$

Since the distance h does not appear, we conclude that the motion is the same for any chord AB of the disk.

4. Derive the differential equation of motion of the particle in Fig. 342 assuming that there is friction opposing sliding along the chord AB and that this friction is proportional to the normal pressure on the chord.

Solution. Normal pressure on the chord is due to inertia forces corresponding to the accelerations a_s and a_b. The first force is opposite to the supplementary acceleration and equal to $2(W/g)\omega\dot{\xi}$. The normal component of the second force is directed outward and is equal to $(W/g)\omega^2 h$. Thus, if μ is the coefficient of friction, the friction force is

$$F = -\mu\frac{W}{g}(2\omega\dot{\xi} \pm \omega^2 h) \tag{v}$$

where the sign of the second term in the parentheses is plus when $\dot{\xi}$ is minus, and vice versa. The equation of motion now becomes

$$\frac{W}{g}\ddot{\xi} = -k\xi + \frac{W}{g}\omega^2\xi - \mu\frac{W}{g}(2\omega\dot{\xi} \pm \omega^2 h) \tag{w}$$

It is interesting to note that in the particular case where the chord AB is a diameter, that is, where $h = 0$, the second term in the parentheses of Eq. (v) vanishes and we obtain, from ordinary coulomb friction, a viscous damping.

PROBLEM SET 10.2

1. Find the equation of the generator of the vessel shown in Fig. 340, and show that it must be a paraboloid of revolution in order that the particle P

will be in a condition of relative equilibrium for any position along the meridian *AOB*. *Ans.* $r^2 = 2gz/\omega^2$.

*2. The axes of two ordinary chemist's test tubes make equal angles α with the vertical axis Oz about which they rotate with angular velocity ω (Fig. A). Each tube is completely filled with water and sealed at the end. In one is a steel ball P_1, and in the other a hollow celluloid ball P_2. Prove that the condition of relative equilibrium of the celluloid ball is stable while that of the steel ball is unstable.

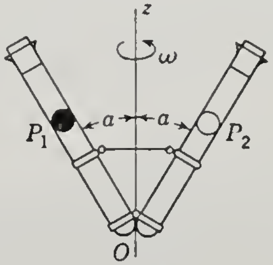


FIG. A

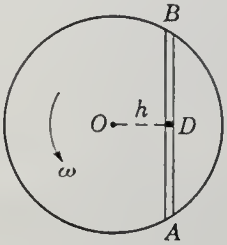


FIG. B

3. In Fig. B, a small ball of weight W rides in a groove cut in the top face of a disk rotating with constant angular velocity ω about its vertical geometric axis. The position shown in the figure is one of unstable relative equilibrium for the ball so that, owing to a slight disturbance, it begins to move along the axis of the groove. Assuming that it starts toward A, find the absolute velocity v with which it will be cast off if $h = \frac{1}{2}r$. Neglect friction between the ball and the groove. *Ans.* $v = 0.940r\omega$.

4. Referring again to Fig. B, find the reactive force R exerted by the ball of weight $W = 1$ lb on the wall of the groove when it has just reached the rim of the disk. Assume $h = \frac{1}{2}r = 1$ ft and that the disk rotates at 300 rpm. *Ans.* $R = 75.5$ lb.

5. An antiaircraft gun fires a shell of weight $W = 75$ lb with muzzle velocity $v_r = 3{,}000$ fps while rotating about a vertical axis at 0.1 rps (Fig. C). If the gun barrel of length $l = 15$ ft is inclined to the horizontal by an angle $\alpha = 45°$, what additional torque M about the z axis must the drive supply just as the shell reaches the muzzle in order to maintain the uniform rotation of the gun? *Ans.* $M = 65{,}700$ ft-lb.

6. The straight pipe in Fig. D discharges 70 gal of water per minute from each end while rotating with constant angular speed of 600 rpm. Calculate the magnitude of the driving torque M required to maintain this uniform rotation. Water weighs 8.34 lb/gal. *Ans.* $M = 152$ ft-lb.

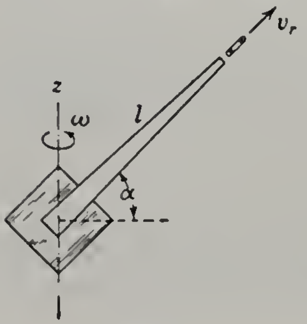


FIG. C

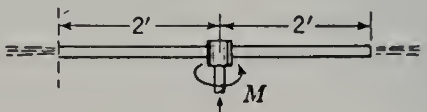


FIG. D

7. A horizontal circular disk of radius r rotates with constant angular velocity ω about a fixed vertical axis through its center O (Fig. E). In the top face of the disk, a shallow circular well of radius $\frac{1}{2}r$ is cut as shown. Inside this well is a small steel ball of weight W which can roll freely around the circumference of the well. Prove that the position D of the ball is one of stable relative equilibrium, and find the period of oscillation of the ball for small values of the relative angular displacement θ away from this stable position. *Ans.* $\tau = 2\pi/\omega$.

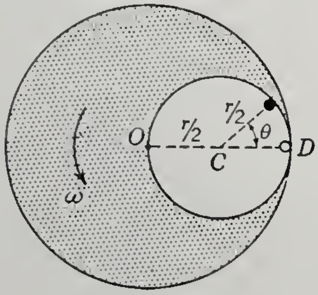


FIG. E

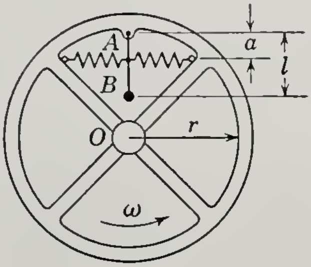


FIG. F

***8.** For small amplitudes find the period of oscillation of the pendulum AB of weight W and length l attached to a rotating wheel by a hinge A and two springs as shown in Fig. F. Each spring has a spring constant $k/2$, and they are attached to the pendulum at the distance a from the hinge A, which is at the radius r from the center of the wheel. The wheel rotates in a horizontal plane with constant angular velocity ω. Numerical data are given as follows: $W = 2$ lb, $k = 60$ lb/in., $a = 4$ in., $l = 8$ in., $r = 12$ in., $\omega = 10\sqrt{2}\pi$ sec^{-1}.

Hint. Assume the entire mass of the pendulum concentrated in the ball B and in calculation neglect the effect of supplementary acceleration, which, for small amplitudes, will have practically no effect on the relative motion of the pendulum. *Ans.* $\tau = 0.206$ sec.

10.3. D'Alembert's principle in relative motion. Applying D'Alembert's principle in the discussion of motion of a particle or system of particles, we simply consider, in addition to all forces acting on the system, the inertia forces of the various particles and then write equations of motion as equations of statics. In the case of relative motion of a particle, we shall have three components of the inertia force, $-ma_r$, $-ma_b$, and $-ma_s$ corresponding, respectively, to the relative, base, and supplementary accelerations of the particle. In the case of relative motion of a rigid body, we apply to each of its particles the above-mentioned three components of inertia force. Then the system of forces consisting of all such inertia forces together with the external forces acting on the body constitute a system of forces in equilibrium, and we write the equations of relative motion of the body as equations of statics.

To illustrate the application of D'Alembert's principle to the case of relative motion of a rigid body, we take the gyroscope shown in Fig. 343 and assume that, while the gyroscope rotates with uniform angular velocity ω around its geometric axis AB, the frame ADB supporting this axis rotates with uniform angular velocity Ω around a vertical axis. We shall assume also that the entire mass of the wheel is concentrated in its rim of radius r.

Considering a particle P of the rim (Fig. 343b) and noting that the relative motion of the wheel is rotation with uniform angular velocity ω about the axis AB, we conclude that the relative velocity of P is $v_r = r\omega$ directed as shown in the figure. Hence the relative acceleration of P is $a_r = \omega^2 r$, directed toward the center C of the wheel. The corresponding inertia forces for all such particles as P are symmetrically distributed around the rim and form a system of forces in equilibrium. Thus they may be disregarded.

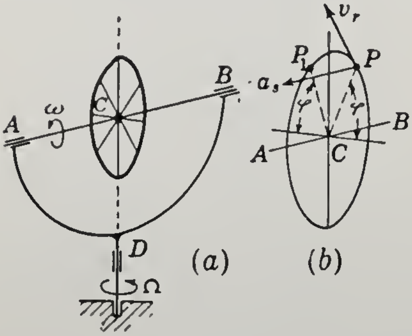


FIG. 343

In calculating the base acceleration a_b for the particle P, we consider only its motion due to rotation of the frame around the vertical axis. Denoting by φ the angle between the radius CP and the horizontal diameter of the wheel (Fig. 343b), we see that, owing to rotation of the frame around the vertical axis, the point describes a horizontal circle of radius $r \cos \varphi$. Hence the base acceleration of P is $a_b = \Omega^2 r \cos \varphi$. The corresponding inertia force of the particle P will be in equilibrium with that of a symmetrically situated particle P_1 (Fig. 343b). Thus for all particles of the rim the inertia forces due to base acceleration represent again a system of forces in equilibrium and can be disregarded.

There remain to be considered inertia forces due to supplementary accelerations of the various particles of the rim. From the direction and magnitude of the relative velocity v_r of the particle P, we conclude that for this particle the acceleration a_s is parallel to the axis AB, equal to $2r\omega\Omega \sin \varphi$, and directed as shown in Fig. 343b. The corresponding inertia force is acting in the opposite direction and gives a moment with respect to the horizontal diameter of the wheel equal to

$$2\, dm\, r\omega\Omega \sin \varphi\, r \sin \varphi = 2\, dm\, \omega\Omega r^2 \sin^2 \varphi \qquad \text{(a)}$$

where dm is the mass of the particle. Denoting by W the total weight of the rim of the wheel and assuming that dm represents the mass of an element of the rim corresponding to the central angle $d\varphi$, we have

$$dm = \frac{W\,d\varphi}{2\pi g}$$

Substituting this in expression (a) and summing up such expressions for all elements of the rim, we find that the inertia forces due to supplementary acceleration have a resultant moment about the horizontal diameter of the wheel of magnitude

$$M = 4\frac{W}{2\pi g}2\omega\Omega r^2\int_0^{\pi/2}\sin^2\varphi\,d\varphi = \frac{W}{g}r^2\omega\Omega = I\omega\Omega \qquad \text{(b)}$$

where I is the moment of inertia of the wheel with respect to its geometric axis AB.

From Eq. (b) we conclude that all external forces acting on the wheel must constitute an equal and opposite moment with respect to the horizontal diameter. Denoting by R the magnitudes of the reactions on the axle at A and B, we have then

$$Rl = I\omega\Omega$$

which is the same result obtained previously in Art. 8.11, by using the principle of angular momentum. For the directions of rotation indicated in Fig. 343, the reaction at A will be down and that at B, up.

PROBLEM SET 10.3

1. A prismatic bar DE of weight W and length $2r$ is attached at right angles to a horizontal axis AB of length l supported in bearings at A and B (Fig. A). The bar DE rotates about AB with uniform angular velocity ω while the frame to which the bearings A and B are attached rotates about a vertical axis with uniform angular velocity Ω. Calculate the bearing reactions when the bar DE coincides with the vertical plane of the frame. Numeri-

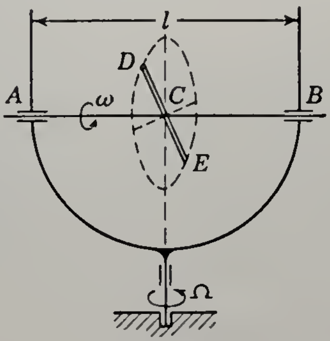


FIG. A

cal data are given as follows: $W = 6$ lb, $r = 6$ in., $l = 24$ in. The bar rotates at 4,000 rpm; the frame at 100 rpm. *Ans.* $R_a = 68.2$ lb, up; $R_b = 68.2$ lb, down.

2. Two equal weights W of a governor (Fig. B) come into oscillation when the angular speed of the governor is $n = 120$ rpm, and for the positions shown the weights have relative velocities $v_r = 6$ ips, as shown. What torque M is transmitted to the vertical shaft at A if $l = 12$ in., $\alpha = 45°$, and $W = 20$ lb? The weights of the bars are to be neglected. *Ans.* $M = -93.8$ in.-lb.

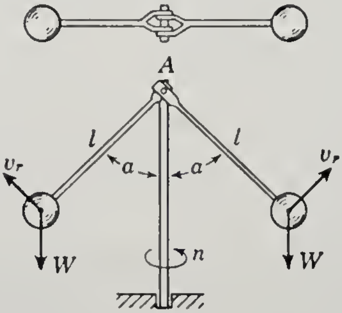


FIG. B

3. Owing to rotation of the earth, the surface of a river flowing due south with uniform relative velocity $v_r = 3$ fps will not be level. Find the angle α that the surface makes with the horizontal if the latitude of the place is $\varphi = 50°$. *Ans.* $\alpha = 0.0000104$ radian.

4. In Fig. C, a bar AB of length $2a$ rotates in a horizontal plane with constant angular velocity ω about a fixed vertical axis through its mid-point C. Attached to the ends of this bar by pins A and B are two identical slender bars AD and BE, each of length l as shown. These bars are in stable relative equilibrium when their axes are in a straight line with AB, that is, when $\theta = 0$. If they come into oscillation during rotation of the system, find their period τ for small values of the relative angular displacements θ. *Ans.* $\tau = (2\pi/\omega)\sqrt{2l/3a}$.

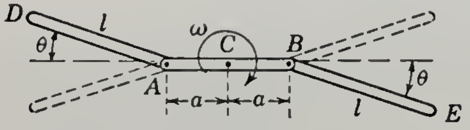


FIG. C

5. Referring again to Fig. C, calculate the maximum outward pull S exerted on each of the hinge pins A and B if the amplitude of relative oscillation of each bar is $\theta_0 = 0.2$ radian. Other numerical data are given as follows: $\omega = 100\pi$ sec^{-1}, $a = 4$ in., $l = 12$ in., and each bar has weight $W = 3$ lb.

Hint. The maximum pull occurs when $\theta = 0$. *Ans.* $S = 9{,}060$ lb.

Appendix One

MOMENTS OF INERTIA OF PLANE FIGURES

A.1. Moment of inertia of a plane figure with respect to an axis in its plane. The moments of inertia of any plane figure (Fig. 1) with respect to x and y axes in its plane are defined, respectively, by the integrals

$$I_x = \int y^2\,dA \qquad I_y = \int x^2\,dA \tag{1}$$

in which each element of area dA is multiplied by the square of its distance from the corresponding axis and integration is extended over the entire area of the figure.

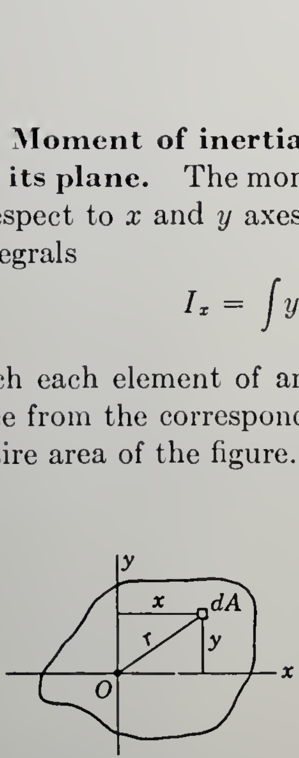


FIG. 1

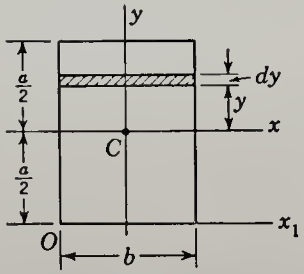


FIG. 2

In simple cases, the integrals (1) can readily be calculated analytically. Take, for example, the case of a rectangle as shown in Fig. 2. In calculating the moment of inertia of this figure with respect to the horizontal axis of symmetry, which is taken as the x axis, we divide the area of the rectangle into infinitesimal elements like the shaded strip shown in the figure. Then $dA = b\,dy$, and we obtain

$$I_x = 2\int_0^{a/2} by^2\,dy = \frac{ba^3}{12} \tag{a}$$

In the same manner, taking the moment of inertia with respect to the

y axis, we find

$$I_y = \frac{ab^3}{12} \tag{b}$$

Formula (a) can also be used for calculating the moment of inertia I_x of the parallelogram shown in Fig. 3. This parallelogram may be considered as obtained from the rectangle shown by dotted lines by a certain displacement parallel to the x axis of each elemental strip, as shown in the figure. Since this transformation changes neither the area of the element nor its distance from the x axis, we conclude that the value of I_x calculated for the rectangle remains unchanged for the parallelogram.

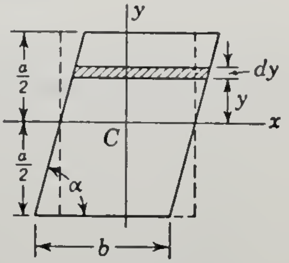


FIG. 3

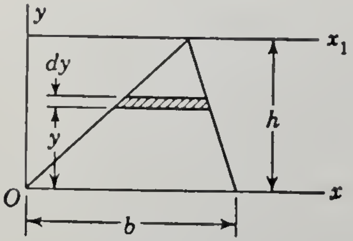


FIG. 4

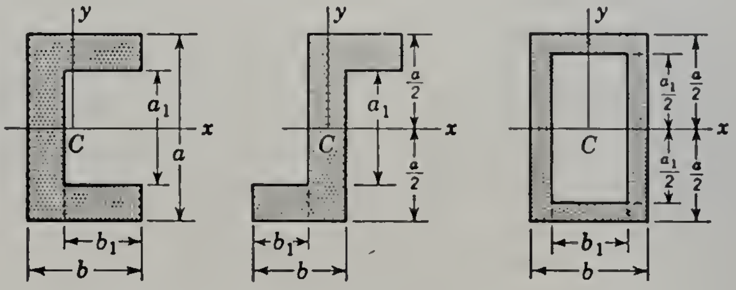
In calculating the moment of inertia of a triangle with respect to its base (Fig. 4), we divide the area into elemental strips, as shown in the figure. Then, for any element at the distance y from the base,

$$dA = \frac{b(h - y)\,dy}{h}$$

and we obtain

$$I_x = \frac{b}{h}\int_0^h y^2(h - y)\,dy = \frac{bh^3}{12} \tag{c}$$

The calculation of moment of inertia with respect to a given axis can often be simplified if the figure can be divided into portions whose



(a) (b) (c)

FIG. 5

moments of inertia about the axis are known. Take, for example, the cross section of a channel, as shown in Fig. 5*a*. Then, with respect to the horizontal axis of symmetry which is taken as the x axis, we evidently obtain the required moment of inertia as the difference of the moments of inertia of the rectangles with the sides ab and a_1b_1, respectively. Hence

$$I_x = \tfrac{1}{12}(ba^3 - b_1a_1^3) \tag{d}$$

This same formula obviously can be applied to the Z section shown in Fig. 5*b* or to the tubular section in Fig. 5*c*.

The method of calculation illustrated by the above examples can be used in the most general case. The moment of inertia of any plane figure, with respect to a given axis, is obtained by dividing its area into infinitesimal strips parallel to the axis and then performing the integration indicated by one of Eqs. (1). If the integration cannot readily be made analytically, an approximate value for the moment of inertia can always be found by dividing the area into a finite number of strips, multiplying the area of each strip by the square of the distance of its centroid from the axis, and then making the summation of such products arithmetically.

From the definitions given by Eqs. (1), it follows that moment of inertia of an area with respect to an axis in its plane has the dimension of length to the fourth power. Hence, dividing the moment of inertia with respect to a certain axis by the cross-sectional area of the figure, a quantity having the dimension of length to the second power is obtained. This length is called the *radius of gyration* of the figure with respect to that axis. Using the notation i for radius of gyration, we have

$$i_x = \sqrt{\frac{I_x}{A}} \qquad i_y = \sqrt{\frac{I_y}{A}} \tag{2}$$

Taking, for example, the rectangle shown in Fig. 2, we find

$$i_x = \sqrt{\frac{ba^3}{12ba}} = \frac{a}{2\sqrt{3}} \qquad i_y = \frac{b}{2\sqrt{3}}$$

PROBLEM SET A.1

1. Find the moment of inertia of the rectangle in Fig. 2 with respect to its base. *Ans.* $I_{x_1} = ba^3/3$.

2. Find the moment of inertia of the parallelogram in Fig. 3 with respect to its base. *Ans.* $I_{x_1} = ba^3/3$.

3. Find the moment of inertia of the triangle in Fig. 4 with respect to the axis x_1. *Ans.* $I_{x_1} = bh^3/4$.

4. Find the moment of inertia of a square with sides of length a with respect to a diagonal. *Ans.* $I_d = a^4/12$.

5. Find the moment of inertia of a rectangle having dimensions a and b with respect to a diagonal. *Ans.* $I_d = a^3b^3/6(a^2 + b^2)$.

A.2. Moment of inertia of a plane figure with respect to an axis perpendicular to the plane of the figure. The moment of inertia of a plane figure with respect to an axis perpendicular to the plane of the figure is defined by the integral

$$J = \int r^2\,dA \tag{3}$$

in which each element of area dA is multiplied by the square of its distance from the axis and integration is extended over the entire area of the figure. If, in Fig. 1, we imagine through point O an axis perpendicular to the xy plane, the moment of inertia with respect to this axis is

$$J = \int r^2\,dA = \int (x^2 + y^2)\,dA = I_y + I_x \tag{4}$$

Since, in this case, each element of the area is multiplied by the square of its distance from point O, the moment of inertia (4) is called the *polar moment of inertia* of the figure with respect to point O. From the relation given by Eq. (4), we see that the polar moment of inertia of any plane figure with respect to a point O in its plane is equal to the sum of the moments of inertia of the figure with respect to two orthogonal axes through that point and also in the plane of the figure.

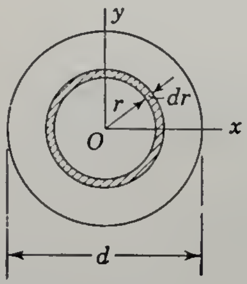


FIG. 6

Let us consider now the calculation of the polar moment of inertia of a circle with respect to its center (Fig. 6). Considering an elemental ring of radius r and width dr, as shown in the figure, we see that the area of this ring is $2\pi r\,dr$ and, by definition, its polar moment of inertia with respect to the center O is $2\pi r^3\,dr$. To obtain the polar moment of inertia of the entire circular area with respect to its center, we have only to make the summation of the polar moments of inertia of the elemental rings. Thus

$$J = \int_0^{d/2} 2\pi r^3\,dr = \frac{\pi d^4}{32} \tag{e}$$

If the circle has a circular hole of diameter d_1 at the center, the lower limit of the integral, in Eq. (e), is $d_1/2$ instead of zero, and we obtain

$$J = \int_{d_1/2}^{d/2} 2\pi r^3 \, dr = \frac{\pi}{32} (d^4 - d_1^4) \tag{f}$$

Having the polar moment of inertia of a circle with respect to its center and using Eq. (4), we can easily find its moment of inertia with respect to a diameter. In the case of a circle, the magnitude of the moment of inertia is the same for all diameters, and hence

$$I_x = I_y = \frac{1}{2} J = \frac{\pi d^4}{64} \tag{g}$$

The moment of inertia of the quarter circle in Fig. 7 with respect to the x axis is

$$I_x = \frac{1}{4} \frac{\pi d^4}{64} = \frac{\pi d^4}{256} = \frac{\pi r^4}{16} \tag{h}$$

The moment of inertia of the shaded area ACB in Fig. 7, with respect to the x axis, is obtained by subtracting from the moment of inertia of the square $OACB$ the moment of inertia of the quarter circle. Thus

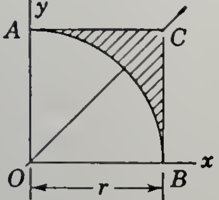


FIG. 7

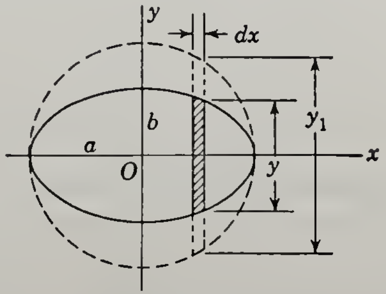


FIG. 8

$$I_x = \frac{r^4}{3} - \frac{\pi r^4}{16} = 0.137 r^4 \tag{i}$$

The moment of inertia of an ellipse with respect to its principal axis x (Fig. 8) can be obtained by comparing the ellipse with the circle shown by a dotted line. The height y of any element of the ellipse, such as the one shown, is obtained by reducing the height y_1 of the corresponding element of the circle in the ratio b/a. From Eq. (a) we conclude that the moments of inertia of these two elements with respect to the x axis are in the ratio b^3/a^3. The moments of inertia of the ellipse and of the circle are evidently in the same ratio. Hence, the moment of inertia of the ellipse is

$$I_x = \frac{\pi(2a)^4}{64}\frac{b^3}{a^3} = \frac{\pi ab^3}{4} \tag{j}$$

In the same manner, for the vertical axis of symmetry,

$$I_y = \frac{\pi ba^3}{4} \tag{k}$$

Finally, by Eq. (4), the polar moment of inertia of the ellipse with respect to its center is

$$J = I_x + I_y = \frac{\pi ab^3}{4} + \frac{\pi ba^3}{4} \tag{l}$$

PROBLEM SET A.2

1. Find the polar moment of inertia of a square with sides of length a with respect to its centroid C. *Ans.* $J_c = a^4/6$.

2. Find the polar moment of inertia of an isosceles triangle having base b and altitude h with respect to its apex A. *Ans.* $J_a = bh^3/4 + hb^3/48$.

3. Find the polar moment of inertia of the shaded area shown in Fig. 7 with respect to point O. *Ans.* $J_0 = 0.274r^4$.

4. Find the polar moment of inertia of the rectangle shown in Fig. 2 with respect to one corner. *Ans.* $J_0 = (a^2 + b^2)\,ab/3$.

5. Find the polar moment of inertia of the area of a circular sector of radius r and central angle α with respect to its center. *Ans.* $J_0 = \alpha r^4/4$.

A.3. Parallel-axis theorem. In Fig. 9, let X, Y be rectangular coordinate axes through any point O in the plane of the figure and x, y correspondingly parallel axes through the centroid C of an area as shown. Then by definition, the moment of inertia of the area with respect to the X axis is

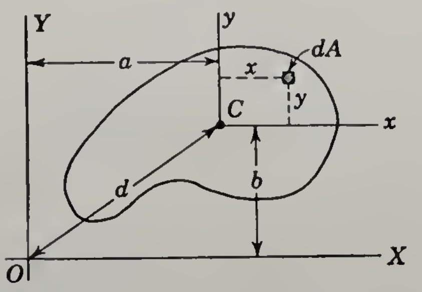


FIG. 9

$$I_X = \int (y + b)^2\,dA = \int y^2\,dA + 2b\int y\,dA + b^2\int dA \tag{m}$$

Noting that $\int y\,dA = 0$, since the x axis passes through the centroid C of the area, Eq. (m) reduces to

$$I_X = \bar{I}_x + Ab^2 \tag{5a}$$

where the notation $\bar{I}_x = \int y^2\,dA$ is used to denote the centroidal moment of inertia of the area with respect to the x axis. Proceeding in the same way, it can be shown that

$$I_Y = \bar{I}_y + Aa^2 \tag{5b}$$

Equations (5*a*) and (5*b*) represent the so-called *parallel-axis theorem* for moments of inertia of plane figures. In words, *the moment of inertia of a plane area with respect to any axis in its plane is equal to the moment of inertia with respect to a parallel centroidal axis plus the product of the total area and the square of the distance between the two axes.* We see that the further an axis is from the centroid of the area, the greater the moment of inertia of the area with respect to that axis.

Adding Eqs. (5*a*) and (5*b*) together and observing from Eq. (4) that $I_X + I_Y = J_0$, while $\bar{I}_x + \bar{I}_y = \bar{J}_c$, and from Fig. 9 that $a^2 + b^2 = d^2$, we obtain

$$J_0 = \bar{J}_c + Ad^2 \tag{6}$$

Thus the parallel-axis theorem holds also for polar moments of inertia.

Using the parallel-axis theorem, various moments of inertia of a plane figure can readily be calculated without integration if the corresponding centroidal moment of inertia is already known. In Fig. 2, for example, we find for the moment of inertia of a rectangle with respect to its base

$$I_{x_1} = \frac{ba^3}{12} + ba\left(\frac{a}{2}\right)^2 = \frac{ba^3}{3} \tag{n}$$

Likewise, the moment of inertia of the circle in Fig. 6 with respect to a tangent is

$$I_T = \frac{\pi d^4}{64} + \frac{\pi d^2}{4}\left(\frac{d}{2}\right)^2 = \frac{5\pi d^4}{64} \tag{o}$$

For the case of the triangle in Fig. 4, we already know the moment of inertia about the base [Eq. (c)]. Then by the parallel-axis theorem, we find the centroidal moment of inertia to be

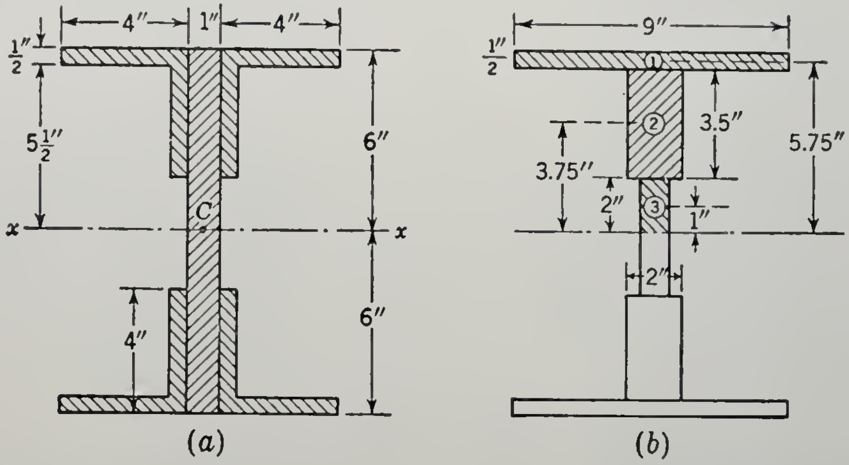
$$\bar{I}_x = \frac{bh^3}{12} - \frac{bh}{2}\left(\frac{h}{3}\right)^2 = \frac{bh^3}{36} \tag{p}$$

The parallel-axis theorem is especially useful in the calculation of moments of inertia of *composite areas* like the one shown in Fig. 10. To illustrate, let us calculate the moment of inertia of this composite area with respect to the x axis through the centroid C, which is an axis of symmetry. The area consists of a narrow rectangle 1 by 12 in. and four identical angles 4 by 4 by $\frac{1}{2}$ in. as shown in Fig. 10*a*. For purposes of calculation, we subdivide the area into rectangles as shown in Fig. 10*b*. Then denoting by A_1, A_2, A_3 the areas of these rectangles and by y_1, y_2, y_3 the y coordinates of their respective centroids and using Eq. (*a*) for centroidal moment of inertia of a rectangle, we have, by reference to Fig. 10*b* for dimensions,

$$\begin{array}{lll} A_1 = 4.5 \text{ in.}^2 & A_2 = 7.0 \text{ in.}^2 & A_3 = 2.0 \text{ in.}^2 \\ y_1 = 5.75 \text{ in.} & y_2 = 3.75 \text{ in.} & y_3 = 1.00 \text{ in.} \\ \bar{I}_1 = 0.09 \text{ in.}^4 & \bar{I}_2 = 7.15 \text{ in.}^4 & \bar{I}_3 = 0.67 \text{ in.}^4 \end{array} \qquad \text{(q)}$$

The required moment of inertia of the entire area with respect to the x axis is

$$I_x = 2(\bar{I}_1 + A_1y_1^2 + \bar{I}_2 + A_2y_2^2 + \bar{I}_3 + A_3y_3^2)$$



(*a*) (*b*)

FIG. 10

Substituting the numerical values from expressions (q), we find $I_x = 514.5$ in.^{4}

There are various ways in which a composite area like that shown in Fig. 10 can be subdivided in calculating its moment of inertia with respect to a given axis. For standard angle sections like those shown in Fig. 10, the positions of the centroids and the moments of inertia with respect to centroidal axes parallel to the edges can be found in handbooks. Then by using the parallel-axis theorem, the moment of inertia of the composite area with respect to any axis can be calculated by using the subdivision, as shown in Fig. 10*a*.

PROBLEM SET A.3

1. Calculate the moment of inertia of the area of the angle section having the dimensions shown in Fig. A with respect to a centroidal axis parallel to the x axis. *Ans.* $I_x = 5.56$ in.^{4}

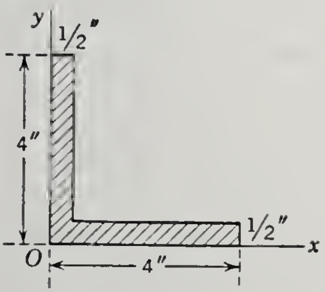


FIG. A

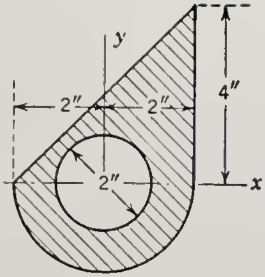


FIG. B

2. Using the result of Prob. 1, calculate the moment of inertia I_x of the composite area shown in Fig. 10*a*. *Ans.* $I_x = 514.5$ in.^{4}

3. Calculate the moment of inertia of the shaded area in Fig. B with respect to the x axis. *Ans.* $I_x = 26.83$ in.^{4}

4. Calculate the moment of inertia of the shaded area in Fig. B with respect to a centroidal axis parallel to the x axis. *Ans.* $I_x = 24.30$ in.^{4}

5. Calculate the moment of inertia of the shaded area in Fig. 7 with respect to a centroidal axis parallel to the x axis. *Ans.* $I_x = 0.0071r^4$.

A.4. Product of inertia; principal axes. Referring to Fig. 11, the integral

$$I_{xy} = \int xy\,dA \tag{7}$$

in which each element of area dA is multiplied by the product of its coordinates and integration is extended over the entire area of a plane figure, is called the *product of inertia* of the figure with respect to the x and y axes. Although moments of inertia with respect to an axis are always positive, as follows from the definitions (1) and (3), the product of inertia (7) may be positive or negative or may vanish, depending on the directions of the x and y axes. Assume, for example, that, for the x and y axes shown in Fig. 11, the integral (7) is positive. If the axes are rotated about the origin O by 90° in a clockwise direction, they take the positions y' and x', as shown, and the new coordinates x', y' of the element dA are in the following relation to its old coordinates:

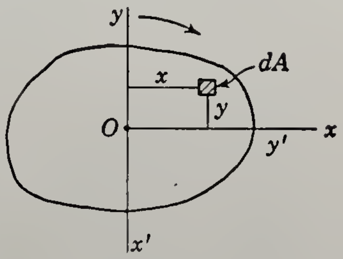


FIG. 11

$$y' = x \qquad x' = -y$$

Hence, the product of inertia for the new axes is

$$I_{x'y'} = \int x'y'\,dA = -\int xy\,dA = -I_{xy}$$

Thus, during this rotation of the axes, the product of inertia changes its sign and becomes negative. Since the product of inertia changes continuously with the rotation of the axes, there must be certain directions of the axes for which this quantity vanishes. The axes taken in these directions are called the *principal axes* of the figure. Usually the centroid of the figure is taken as the origin of coordinates, and the corresponding principal axes are then called the *centroidal principal axes*. If a figure has an axis of symmetry, this axis and any axis perpendicular to it are principal axes of the figure, because the product of inertia with respect to these axes will be equal to zero. To prove this statement, let us consider the plane figure shown in Fig. 12, the y axis of which is an axis of symmetry. Then, for any element dA with a certain positive x, there exists an equal and symmetrically situated element dA with a similar negative x. The corresponding elementary products $xy\,dA$ cancel each other; hence the integral (7) vanishes, and x and y are principal axes of the figure.[1]

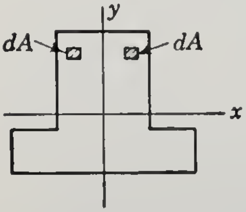


FIG. 12

Now let us consider the change of the product of inertia resulting from a displacement of the axes parallel to themselves (Fig. 13). If the product of inertia $\bar{I}_{xy}$ for the centroidal axes x and y is known, the product of inertia I_{xy} for parallel axes X and Y is given by the expression

$$I_{xy} = \bar{I}_{xy} + Aab \tag{8}$$

in which a and b are the coordinates of the centroid C with respect to the new axes. This is the parallel axis theorem for products of inertia,

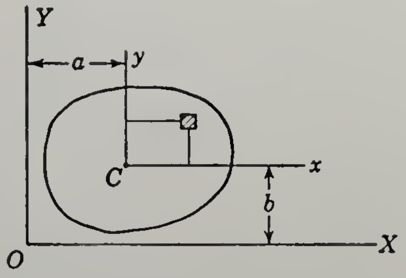


FIG. 13

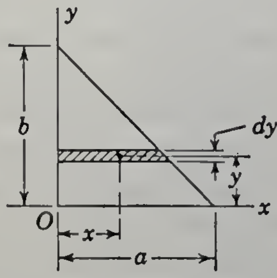


FIG. 14

and it can be proved as follows: The coordinates of an element dA for the new axes X and Y are

$$X = x + a \qquad Y = y + b$$

---

[1] The method of finding the principal axes of a nonsymmetrical figure will be discussed in the next article.

Hence, by definition,

$$I_{XY} = \int (x + a)(y + b)\, dA$$
$$= \int xy\, dA + a \int y\, dA + b \int x\, dA + \int ab\, dA \qquad \text{(r)}$$

The second and third of the four integrals on the right vanish because x and y are centroidal axes. Hence expression (r) reduces to Eq. (8).

As an example of calculation of product of inertia of a plane figure with respect to given axes, let us find the product of inertia I_{xy} of the right triangle shown in Fig. 14. Dividing the area of the triangle into elements as shown, the area of any strip at the distance y from the base is $[a(b - y)/b]\, dy$ and its product of inertia, from Eq. (8), is

$$\frac{a(b - y)\, dy}{b}\,\frac{a(b - y)y}{2b} \qquad \text{(s)}$$

The product of inertia of the triangle is obtained by summation of the elements (s), which gives

$$I_{xy} = \frac{a^2}{2b^2} \int_0^b (b - y)^2 y\, dy = \frac{a^2 b^2}{24} \qquad \text{(t)}$$

In the case of a circular quadrant of radius r, we divide the area into horizontal strips as shown in Fig. 15. Then the area of one element is $dA = x\, dy$ and its product of inertia with respect to the axes x and y is

$$x\, dy \frac{x}{2} y = \frac{1}{2} x^2 y\, dy$$

Since for a circle with center at O, we have

$$x^2 + y^2 = r^2$$

the required product of inertia of the entire area becomes

$$I_{xy} = \frac{1}{2} \int_0^r (r^2 - y^2) y\, dy = \frac{r^4}{8} \qquad \text{(u)}$$

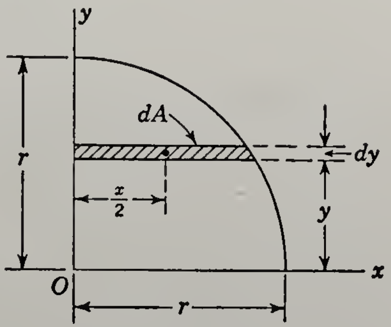


FIG. 15

PROBLEM SET A.4

1. Calculate the product of inertia of a rectangle having sides of lengths a and b with respect to axes x and y coinciding with two adjacent edges. *Ans.* $I_{xy} = a^2b^2/4$.

2. Calculate the product of inertia I_{xy} of the shaded spandrel area in Fig. 7. *Ans.* $I_{xy} = r^4/8$.

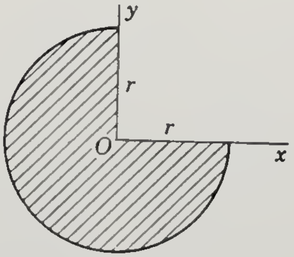


FIG. A

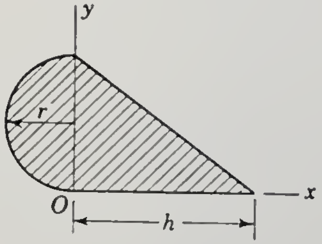


FIG. B

3. Calculate the product of inertia I_{xy} of the area of a three-quarter circular sector as shown in Fig. A. *Ans.* $I_{xy} = -r^4/8$.

4. Referring to Fig. B, find the necessary relation between r and h so that x and y will be principal axes for the composite area. *Ans.* $h = 2r$.

5. Calculate the product of inertia I_{xy} for the angle section shown in Fig. C. Numerical data are given as follows: $a = 4$ in., $h = 1$ in. *Ans.* $I_{xy} = +7.75$ in.^{4}

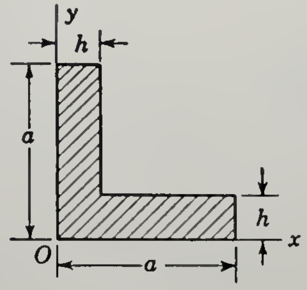


FIG. C

A.5. Principal axes and principal moments of inertia. Assume that for the plane figure shown in Fig. 16 the moments of inertia

$$I_x = \int y^2\, dA \qquad I_y = \int x^2\, dA \tag{v}$$

and the product of inertia

$$I_{xy} = \int xy\, dA \tag{w}$$

are known and that it is required to find $I_{x'}, I_{y'}, I_{x'y'}$ for the new axes x' and y'. Considering an infinitesimal element of area dA, the new coordinates of the element, as can be seen from the figure, are

$$x' = x \cos \varphi + y \sin \varphi \qquad y' = y \cos \varphi - x \sin \varphi$$

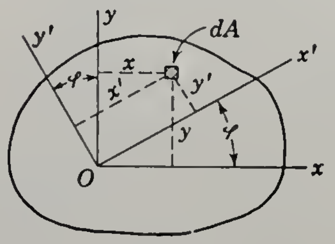


FIG. 16

in which φ is the angle between the axes x', y' and x, y. Then

$$I_{x'} = \int (y \cos \varphi - x \sin \varphi)^2 \, dA$$
$$= \int y^2 \cos^2 \varphi \, dA + \int x^2 \sin^2 \varphi \, dA - 2 \int xy \sin \varphi \cos \varphi \, dA$$

Or, by using expressions (v) and (w),

$$I_{x'} = I_x \cos^2 \varphi + I_y \sin^2 \varphi - 2I_{xy} \sin \varphi \cos \varphi \tag{9a}$$

Substituting in this expression $\pi/2 + \varphi$, instead of φ, we obtain

$$I_{y'} = I_x \sin^2 \varphi + I_y \cos^2 \varphi + 2I_{xy} \sin \varphi \cos \varphi \tag{9b}$$

Taking the sum of expressions (9*a*) and (9*b*), we obtain

$$I_{x'} + I_{y'} = I_x + I_y \tag{10}$$

Thus, we see that as the orthogonal axes x and y are rotated in the plane of the figure around point O, the sum of the moments of inertia with respect to these axes remains constant. Taking the difference between expressions (9*a*) and (9*b*), we obtain

$$I_{x'} - I_{y'} = (I_x - I_y) \cos 2\varphi - 2I_{xy} \sin 2\varphi \tag{11}$$

Formulas (10) and (11) are more convenient for calculating $I_{x'}$ and $I_{y'}$ than Eqs. (9) would be if used directly.

The product of inertia with respect to the new axes is

$$I_{x'y'} = \int (x \cos \varphi + y \sin \varphi)(y \cos \varphi - x \sin \varphi) dA$$
$$= \int y^2 \sin \varphi \cos \varphi \, dA - \int x^2 \sin \varphi \cos \varphi \, dA$$
$$+ \int xy(\cos^2 \varphi - \sin^2 \varphi) dA$$
$$= \tfrac{1}{2}(I_x - I_y) \sin 2\varphi + I_{xy} \cos 2\varphi \tag{12}$$

This expression can be used for finding the directions of the principal axes through point O if we recall that principal axes are those for which the product of inertia vanishes. Thus the axes x' and y' in Fig. 16 are principal axes if $I_{x'y'}$ vanishes, which requires that

$$\tfrac{1}{2}(I_x - I_y) \sin 2\varphi + I_{xy} \cos 2\varphi = 0 \tag{x}$$

which gives

$$\tan 2\varphi = \frac{2I_{xy}}{I_y - I_x} \tag{13}$$

From this equation, two values of the angle 2φ which differ by π are obtained. The corresponding two values of the angle φ, differing from each other by $\pi/2$, define the orthogonal directions of the principal axes. Substituting these values of φ in expressions (10) and (11), the sum and difference of the principal moments of inertia are obtained, from which the quantities themselves can readily be found.

Referring again to Eqs. (9), we see that the magnitude of the moments of inertia $I_{x'}$ and $I_{y'}$ vary continuously with change in the angle φ. To find the values of φ for which these quantities will be a maximum or a minimum, we set the derivatives of expressions (9) with respect to φ equal to zero. In each case, this procedure will result in an equation identical with Eq. (x) above, and from this fact we conclude that the principal axes of the figure represent the two axes for which the moments of inertia are a maximum and a minimum, respectively.

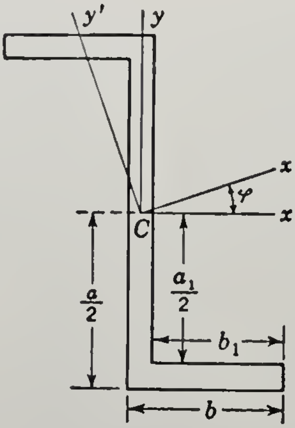


FIG. 17

The method of calculation of the directions of the principal axes and of principal moments of inertia will now be illustrated by an example. We take the Z section shown in Fig. 17 and assume $a = 16$ in., $a_1 = 13.8$ in., $b = 7$ in., $b_1 = 5.9$ in. Treating the figure as made up of three simple rectangles and using Eqs. (5) and (8), we obtain

$$I_x = 1{,}097 \text{ in.}^4 \qquad I_y = 198.4 \text{ in.}^4 \qquad I_{xy} = -338.5 \text{ in.}^4$$

Substituting these values in Eq. (13), we find

$$\tan 2\varphi = +0.753 \qquad 2\varphi = 37^\circ \qquad \text{or} \qquad 2\varphi = 143^\circ$$

Taking the first value of 2φ, we have the axis x' rotated from the x axis in a counterclockwise direction by the angle 18°30′. Substituting $\sin 2\varphi = 0.602$ and $\cos 2\varphi = 0.799$ in Eq. (11), we obtain from Eqs. (10) and (11) the following principal moments of inertia:

$$I_{x'} = 1{,}211 \text{ in.}^4 \qquad I_{y'} = 85.0 \text{ in.}^4$$

PROBLEM SET A.5

1. Calculate the angle φ defining the direction of principal axes x', y' through point O for the angle section shown in Fig. A. Each leg of the angle is 1 in. wide. *Ans.* $\varphi = 13°31'$.

2. Calculate the angle φ defining the direction of principal axes through the centroid C of the angle section shown in Fig. A if each leg of the angle is 1 in. wide. *Ans.* $\varphi = 67°26'$.

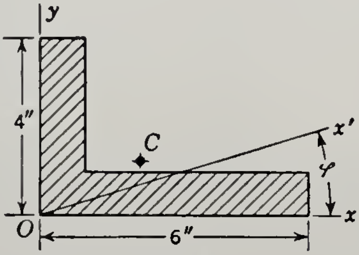


FIG. A

3. Calculate the principal moments of inertia of the angle section shown in Fig. A with respect to centroidal axes in the plane of the figure. Each leg is 1 in. wide. *Ans.* $\bar{I}_{x'} = 34.97$ in.^{4}; $\bar{I}_{y'} = 6.63$ in.^{4}

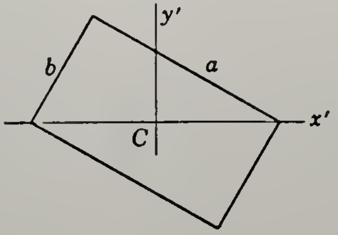
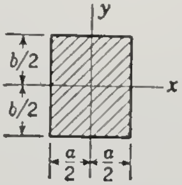
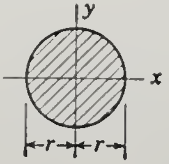
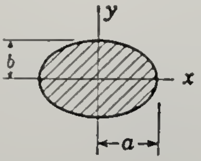
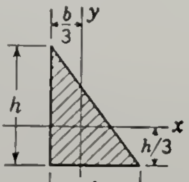
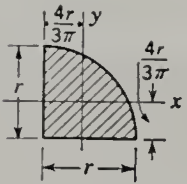


FIG. B

4. Calculate the product of inertia $\bar{I}_{x'y'}$ of the rectangle shown in Fig. B if $a = 10$ in., $b = 6$ in. *Ans.* $\bar{I}_{x'y'} = -141$ in.^{4}

5. Calculate the angle φ defining the direction of principal axes through point O for the right triangle shown in Fig. 14, if $a = 6$ in., $b = 8$ in. *Ans.* $\varphi = 60°07'$.

Various Moments and Products of Inertia of Plane Figures

Figure	$\bar{I}_x$	$\bar{I}_y$	$\bar{I}_{xy}$
 y b/2 b/2 x a/2 a/2	$\frac{ab^3}{12}$	$\frac{ba^3}{12}$	0
	$\frac{\pi r^4}{4}$	$\frac{\pi r^4}{4}$	0
	$\frac{\pi ab^3}{4}$	$\frac{\pi ba^3}{4}$	0
	$\frac{bh^3}{36}$	$\frac{hb^3}{36}$	$-\frac{b^2h^2}{72}$
 4r/3π y 4r/3π r x r	$0.0549r^4$	$0.0549r^4$	$-0.0163r$

Appendix Two

MOMENTS OF INERTIA OF MATERIAL BODIES

A.6. Moment of inertia of a rigid body. The moment of inertia of a body with respect to a given axis is defined by the integral

$$I = \int r^2\, dm \tag{14}$$

in which each element of mass dm of the body is multiplied by the square of its distance r from the axis and the integration is extended over the entire volume of the body. From this definition, it follows that the moment of inertia of a body has the dimension of mass × length² or force × length × time². Thus, we can always express the moment of inertia of a body in the form

$$I = \frac{W}{g} i^2 \tag{15}$$

where W/g is the mass of the body and i is a certain length called the *radius of gyration* of the body with respect to the given axis.

The calculation of the moment of inertia of a body requires the evaluation of the integral (14) above. Various methods of procedure for particular cases will be discussed in the following articles.

A.7. Moment of inertia of a lamina. The calculation of moment of inertia of a material body can be made in a very simple manner if one dimension of the body is very small so that we have the case of a *lamina* or *thin plate.* Assuming that such a plate or lamina is of homogeneous material and of uniform thickness and taking its middle plane as the xy plane (Fig. 1), the mass of an element as shown is $(w/g)t\, dA$, where t is the small thickness of the plate and w/g, the mass per unit volume of the material. Then, by the definition (14), the moments of inertia of the plate with respect to axes x, y, and z, through any point

O as shown, become

$$I_x = \frac{wt}{g}\int y^2\,dA \qquad I_y = \frac{wt}{g}\int x^2\,dA \qquad I_z = \frac{wt}{g}\int r^2\,dA \tag{16}$$

These expressions differ from expressions (1) and (3), previously developed in Appendix I for moments of inertia of a plane figure, only by the constant factor wt/g. From this, it follows that all results obtained in Appendix I for moments of inertia of plane figures can be used also for thin plates, simply by introducing the additional factor wt/g.

Proceeding in such a manner for the case of a rectangular plate of thickness t (Fig. 2), we obtain for the moments of inertia, with respect to the axes of symmetry,

$$I_x = \frac{wt}{g}\frac{ba^3}{12} = \frac{W}{g}\frac{a^2}{12} \qquad I_y = \frac{W}{g}\frac{b^2}{12} \tag{a}$$

where $W = wtab$ is the total weight of the plate. From Eq. (4), the moment of inertia of the plate with respect to the z axis, perpendicular to the plate and passing through its centroid, is

$$I_z = I_x + I_y = \frac{W}{g}\frac{a^2 + b^2}{12} \tag{b}$$

Similarly, the moments of inertia of a thin circular plate or disk with respect to centroidal axes (Fig. 6) are

$$I_x = I_y = \frac{wt}{g}\frac{\pi d^4}{64} = \frac{W}{g}\frac{a^2}{4} \qquad I_z = \frac{W}{g}\frac{a^2}{2} \tag{c}$$

where $a = d/2$ is the radius of the disk.

Comparing expressions (a) and (b) with expression (15) in Art. A.6, we conclude that the radii of gyration of a rectangular plate with respect to the axes x, y, and z, are, respectively,

$$i_x = \frac{a}{2\sqrt{3}} \qquad i_y = \frac{b}{2\sqrt{3}} \qquad i_z = \sqrt{\frac{a^2 + b^2}{12}} \tag{d}$$

which are seen to be identical with the radii of gyration of a plane rectangular figure (see page A.3). In general, the radius of gyration of any homogeneous material lamina, with respect to a given axis, is identical with the radius of gyration of its middle surface considered as a plane figure.

Since the moments of inertia of a lamina as given by Eqs. (16) are simply equal to the factor wt/g times the corresponding moments of inertia of the area of its middle surface, it follows that the parallel-axis theorem [Eqs. (5) and (6)] must hold also for such material laminae. Thus, referring to Fig. 18, we have for axes X, Y, Z, parallel to the centroidal axes x, y, z,

$$\begin{aligned} I_X &= \frac{W}{g}(\bar{\imath}_x^2 + b^2) \\ I_Y &= \frac{W}{g}(\bar{\imath}_y^2 + a^2) \\ I_Z &= \frac{W}{g}(\bar{\imath}_z^2 + d^2) \end{aligned} \tag{17}$$

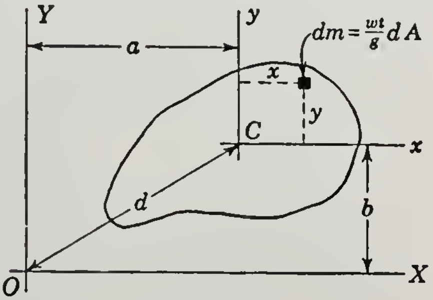


FIG. 18

where $\bar{\imath}_x$, $\bar{\imath}_y$, $\bar{\imath}_z$ are radii of gyration of the body with respect to centroidal axes x, y, z, respectivly.

Using the parallel-axis theorem together with the last of Eqs. (c), we find, for example, that the polar moment of inertia of a thin circular plate of weight W and radius a with respect to an axis Z through a point O on its circumference is

$$I_Z = \frac{W}{g}\left(\frac{a^2}{2} + a^2\right) = \frac{3W}{2g}\, a^2 \tag{e}$$

PROBLEM SET A.7

1. A slender piece of wire weighing 0.14 lb is bent in the shape shown in Fig. A. Calculate its moments of inertia I_x and I_y. *Ans.* $I_x = 466 \times 10^{-6}$ lb-sec²-in.; $I_y = 3{,}590 \times 10^{-6}$ lb-sec²-in.

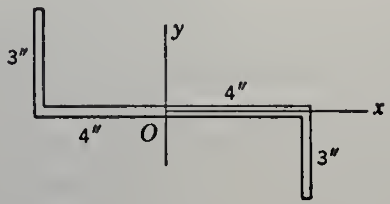


FIG. A

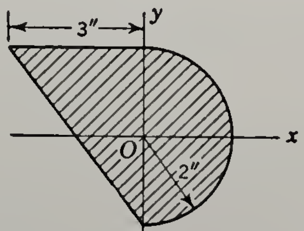


FIG. B

2. Calculate the moment of inertia I_y of a homogeneous thin plate having the dimensions shown in Fig. B. The total weight of the plate is $W = 0.2456$ lb. *Ans.* $I_y = 792 \times 10^{-6}$ lb-sec²-in.

3. Find the moment of inertia of a homogeneous triangular plate of weight W with respect to its base by using formula c, page A.2 for a plane triangular figure. *Ans.* $I_x = Wa^2/6g$.

4. Determine the moment of inertia of a homogeneous regular hexagonal lamina having weight W and sides of length a, with respect to a diagonal. *Ans.* $I_d = 5Wa^2/24g$.

5. Determine the moment of inertia I_x of a thin plate having the Z shape shown in Fig. 17 if the total weight of the plate is $W = \frac{1}{2}$ lb and $a = 5$ in., $b = 3$ in., $a_1 = 3$ in., $b_1 = 2$ in. *Ans.* $I_x = 0.00384$ lb-sec²-in.

A.8. Moments of inertia of three-dimensional bodies. In discussing the moments of inertia of bodies, all three dimensions of which are of the same order of magnitude, it is advantageous to use the parallel-axis theorem. Let x, y, z (Fig. 19) be a system of rectangular coordinate axes through the center of gravity C of a body and X, Y, Z a system of corresponding parallel axes with an origin O, a, b, c being the coordinates of the center of gravity C with respect to the axes X, Y, Z. Considering a mass element dm having coordinates $X = a + x$ and $Y = b + y$, we have for the moment of inertia of the body with respect to the Z axis:

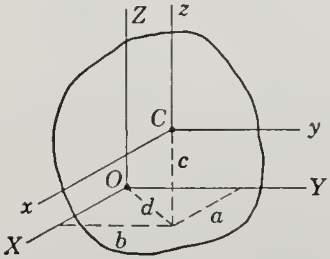


FIG. 19

$$I_Z = \int [(a + x)^2 + (b + y)^2]\,dm = (a^2 + b^2)\int dm + 2a\int x\,dm + 2b\int y\,dm + \int (x^2 + y^2)\,dm \qquad \text{(f)}$$

Since the axes x, y, z pass through the center of gravity C of the body, the second and the third integrals in expression (f) vanish (see page 199). Then, using the notations

$$d^2 = a^2 + b^2 \qquad \text{and} \qquad r^2 = x^2 + y^2$$

we obtain from expression (f)

$$I_Z = \int r^2\,dm + d^2\int dm = \bar{I}_z + \frac{W}{g}d^2 \qquad (18)$$

and we conclude that the moment of inertia of a body with respect to any axis is obtained by adding to its moment of inertia with respect to a parallel centroidal axis, the product of the mass of the body and the square of the distance between the two axes. This represents the parallel-axis theorem for the mass moment of inertia of any material body.

Writing Eq. (18) in the form

$$\frac{W}{g}i_z^2 = \frac{W}{g}\bar{i}_z^2 + \frac{W}{g}d^2$$

we see that

$$i_z^2 = \bar{i}_z^2 + d^2 \tag{19}$$

which states the relationship between radii of gyration for parallel axes.

As an example of the application of the parallel-axis theorem to the calculation of moments of inertia of three-dimensional bodies, let us calculate the moments of inertia of a solid right circular cylinder of uniform density with respect to the centroidal coordinate axes x, y, z, as shown in Fig. 20. We begin with a calculation of the moment of inertia of the cylinder with respect to the x axis. This moment of inertia is obtained by summing up the moments of inertia with respect to the same axis of all such elemental disks of radius a and thickness dz as the one indicated in the figure. Observing that the weight of the disk is $\pi a^2 w\, dz$ and using the first of formulas (c) together with the parallel-axis theorem, we conclude that the moment of inertia of the disk with respect to the x axis is

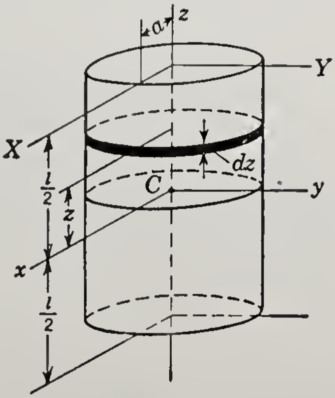


FIG. 20

$$\frac{\pi a^2 w\, dz}{g}\frac{a^2}{4} + \frac{\pi a^2 w\, dz}{g} z^2$$

Hence, for the entire cylinder, we obtain

$$I_x = \frac{\pi a^2 w}{g}\int_{-l/2}^{+l/2}\left(\frac{a^2}{4} + z^2\right) dz = \frac{\pi a^4 w l}{4g} + \frac{\pi a^2 l^3 w}{12g} = \frac{W}{g}\left(\frac{a^2}{4} + \frac{l^2}{12}\right) \tag{g}$$

where $W = \pi a^2 l w$ is the total weight of the cylinder.

From the circular form of the cylinder, it may be concluded that expression (g) gives also the moment of inertia with respect to the y axis as well as for any other centroidal axis perpendicular to the geometric axis of the cylinder.

When the altitude l of the cylinder is small in comparison with its radius a, the term $l^2/12$ in formula (g) can be neglected, and we obtain the same moment of inertia as given by the first of formulas (c) derived for a circular lamina. If a is small compared with l, i.e., if we have a slender cylindrical rod, the term $a^2/4$ in formula (g) can be neglected, and we obtain for the moment of inertia of such a slender rod, with respect to a centroidal axis perpendicular to the axis of the rod, the expression

$$I_x = \frac{W}{g}\frac{l^2}{12} \tag{h}$$

Sometimes we need the moment of inertia of a slender rod with respect to a transverse axis through one end of the rod. Then, using formula (h) and the parallel-axis theorem (18), we obtain

$$I_X = \frac{W}{g}\left(\frac{l^2}{12} + \frac{l^2}{4}\right) = \frac{W}{g}\frac{l^2}{3} \tag{i}$$

In calculating the moment of inertia of the cylinder with respect to the z axis (Fig. 20), we have only to sum up the moments of inertia of all elemental disks, like the one shown in the figure, with respect to the same axis. Thus, by using the second of formulas (e) for a circular lamina, we obtain

$$I_z = \frac{W}{g}\frac{a^2}{2} \tag{j}$$

As a second example, let us calculate the moment of inertia of a homogeneous sphere of radius a with respect to a diameter. Taking rectangular coordinate axes x, y, z, through the center of the sphere, the moments of inertia with respect to these axes are, by definition,

$$I_x = \int (y^2 + z^2)\, dm \qquad I_y = \int (x^2 + z^2)\, dm \qquad I_z = \int (x^2 + y^2)\, dm$$

Adding these expressions together and observing that the moments of inertia of a sphere with respect to all diameters are equal, we obtain

$$I_x = I_y = I_z = \tfrac{2}{3}\int (x^2 + y^2 + z^2)\, dm = \tfrac{2}{3}\int r^2\, dm \tag{k}$$

where r is the distance of the element dm from the center of the sphere. Dividing the sphere into thin shells of radius r and thickness dr, the mass of one such shell is $4\pi r^2\, dr(w/g)$. Substituting this for dm in expression (k), we obtain

$$I_x = I_y = I_z = \frac{2}{3}\int_0^a \frac{4\pi r^4 w\, dr}{g} = \frac{8}{15}\frac{\pi w a^5}{g} = \frac{2}{5}\frac{W}{g}a^2 \tag{l}$$

where $W = \frac{4}{3}\pi a^3 w$ is the total weight of the sphere.

PROBLEM SET A.8

1. Calculate the moment of inertia I_z of a right circular cone of uniform density, radius of base a, and altitude h, with respect to its geometric axis. *Ans.* $I_z = 3Wa^2/10g$.

2. Calculate the moment of inertia I_X of a homogeneous right circular cone with respect to an axis X through the vertex and parallel to the plane of the base. *Ans.* $I_X = W(3a^2 + 12h^2)/20g$.

3. Calculate the moment of inertia I_x of a homogeneous rectangular parallelepiped having dimensions a, b, c, with respect to a centroidal axis parallel to the edges of length a. *Ans.* $I_x = W(b^2 + c^2)/12g$.

4. Using the parallel-axis theorem together with the result of Prob. 3, find the moment of inertia of a homogeneous cube of weight W and dimensions a, with respect to one edge. *Ans.* $I = \frac{2}{3}(W/g)a^2$.

5. Find the moment of inertia, with respect to a diameter, of a hollow sphere if the weight per unit volume of the material is w and the outer and inner radii are a_o and a_i, respectively. *Ans.* $I = \frac{8}{15}(\pi w/g)(a_o^5 - a_i^5)$.

A.9. Product of inertia and principal axes. In our further discussion of moments of inertia of material bodies with respect to various axes, the notion of *product of inertia* will be very useful. With respect to orthogonal coordinate axes x, y, z (Fig. 19), a material body has three products of inertia defined by the integrals

$$I_{xy} = \int xy\,dm \qquad I_{yz} = \int yz\,dm \qquad I_{xz} = \int xz\,dm \tag{20}$$

It is seen that product of inertia of a material body, like moment of inertia, has the dimension of mass × length². However, unlike moment of inertia, which is always a positive quantity, product of inertia may be positive or negative or zero. In general, the calculation of product of inertia requires evaluation of the integrals (20).

In the case of a material lamina, the middle surface of which we take as the xy plane, only the product of inertia I_{xy} will be different from zero, and, for calculating this, we can use directly the formulas developed in Art. A.4. It is necessary only to introduce the additional factor wt/g, as discussed in Art. A.7. Proceeding in this way, for example, with formula (t), page A.11, we find for the product of inertia of a right triangular lamina (Fig. 14), with respect to its two orthogonal edges,

$$I_{xy} = \frac{wt}{g}\frac{a^2b^2}{24} = \frac{W}{g}\frac{ab}{12} \tag{m}$$

where $W = wtab/2$ is the total weight of the lamina.

Similarly, the product of inertia I_{xy} of a lamina having the form of a quarter circle as shown in Fig. 15 is

$$I_{xy} = \frac{wt}{g}\frac{r^4}{8} = \frac{W}{g}\frac{r^2}{2\pi} \tag{n}$$

where $W = wt\pi r^2/4$ is the total weight of the lamina.

In certain cases, the calculation of products of inertia of a three-dimensional body can also be made in a very simple manner. For example, if the orthogonal axes x, y, z in Fig. 19 define planes of symmetry of the body, then the corresponding products of inertia I_{xy}, I_{yz}, I_{xz} are zero. To prove this, assume that the xy plane is a plane of symmetry. Then, for each plus $yz\,dm$ or $xz\,dm$ in expressions (20) there will be a corresponding minus $yz\,dm$ or $xz\,dm$, and, in the process of summation, such pairs of terms cancel each other. Thus, if xy is a plane of symmetry of the body, $I_{xz} = 0$ and $I_{yz} = 0$. Similarly, if xz is a plane of symmetry, we conclude that $I_{xy} = 0$.

Orthogonal axes for which the products of inertia (20) vanish are called *principal axes* of the body. The corresponding moments of inertia I_x, I_y, I_z are called *principal moments of inertia.*

Further calculation of products of inertia for three-dimensional bodies can be greatly simplified by introducing the relation between the products of inertia for correspondingly parallel systems of rectangular axes, one set of which are centroidal axes. Referring again to Fig. 19, where x, y, z are centroidal axes and X, Y, Z are correspondingly parallel axes through any point O, we have for the product of inertia with respect to the axes X, Y

$$\begin{aligned} I_{XY} &= \int (x + a)(y + b)\,dm \\ &= b \int x\,dm + a \int y\,dm + \int xy\,dm + ab \int dm \end{aligned} \tag{o}$$

Since x and y are centroidal axes, the first two integrals in expression (o) vanish, and the next integral is, by definition, the product of inertia with respect to the centroidal axes x and y. Thus, expression (o) becomes

$$I_{XY} = \bar{I}_{xy} + \frac{W}{g}ab \tag{21}$$

Equation (21) is the parallel axis theorem for products of inertia of a material body. Similar relationships hold for the products of inertia I_{YZ}, $\bar{I}_{yz}$ and I_{XZ}, $\bar{I}_{xz}$.

As an example of the application of Eq. (21), let us calculate the products of inertia of a cube of dimensions a with respect to coordinate axes X, Y, Z coinciding with three edges of the cube that meet at one corner. Since correspondingly parallel centroidal axes x, y, z

define planes of symmetry of the cube, the products of inertia $\bar{I}_{xy}$, $\bar{I}_{yz}$, $\bar{I}_{xz}$ are zero, and, by Eq. (21), we have

$$I_{XY} = I_{YZ} = I_{XZ} = \frac{W}{g}\frac{a^2}{4} \tag{p}$$

where W is the weight of the cube.

PROBLEM SET A.9

1. Calculate the product of inertia $\bar{I}_{xy}$ of a triangular lamina having the shape shown in Fig. 14, with respect to centroidal axes parallel to x and y, as shown. *Ans.* $\bar{I}_{xy} = -Wab/36g$.

2. Calculate the product of inertia I_{xy} for an angle-shaped plate as shown in Fig. A, if $W = 1.1$ lb, $a = 6$ in., $h = 1$ in. *Ans.* $I_{xy} = 0.00460$ lb-sec²-in.

3. Calculate the product of inertia I_{xy} of a Z-shaped lamina as shown in Fig. 17. Numerical data are given as follows: $W = \frac{1}{2}$ lb, $a = 5$ in., $b = 3$ in., $a_1 = 3$ in., $b_1 = 2$ in. *Ans.* $I_{xy} = -0.00173$ lb-sec²-in.

4. Calculate the product of inertia $\bar{I}_{xy}$ for the plate in Fig. A with respect to centroidal axes parallel to x and y. Use same data as given in Prob. 2. *Ans.* $\bar{I}_{xy} = -0.00528$ lb-sec²-in.

5. A solid homogeneous cube of weight $6W$ and with edges of length a is cut into two parts through one of its octahedral planes. Find the products of inertia of the tetrahedral corner with respect to axes X, Y, Z, coinciding with its three mutually perpendicular edges. *Ans.* $I_{XY} = I_{YZ} = I_{XZ} = Wa^2/20g$.

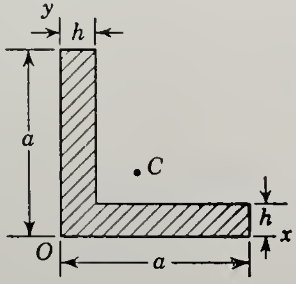


FIG. A

A.10. Change of direction of axes of inertia. If, in Fig. 21, x, y, z are orthogonal axes for which the moments of inertia I_x, I_y, I_z and the products of inertia I_{xy}, I_{yz}, I_{xz} of a body are known, the corresponding quantities I_u, I_v, I_w and I_{uv}, I_{vw}, I_{uw} for another set of orthogonal axes u, v, w through the same point O can be calculated. Let α_1, β_1, γ_1, α_2, β_2, γ_2, α_3, β_3, γ_3 be the direction angles with respect to x, y, z of the new axes u, v, w, respectively. For example, the direction angles α_1, β_1, γ_1, of the u axis are shown in Fig. 21. Then the moment of inertia of the body

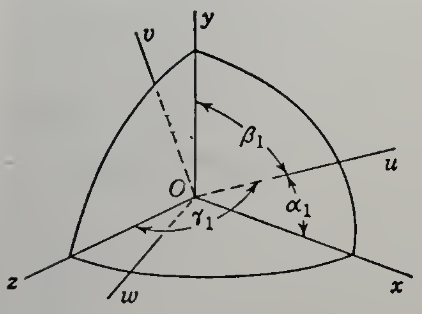


FIG. 21

with respect to the u axis is given by the formula[1]

$$I_u = I_x \cos^2 \alpha_1 + I_y \cos^2 \beta_1 + I_z \cos^2 \gamma_1 \\ - 2I_{xy} \cos \alpha_1 \cos \beta_1 - 2I_{xz} \cos \alpha_1 \cos \gamma_1 - 2I_{yz} \cos \beta_1 \cos \gamma_1 \quad (22)$$

Similar expressions can be written for the moments of inertia I_v and I_w.

In the particular case where x, y, z are principal axes of the body, the products of inertia I_{xy}, I_{xz}, I_{yz} vanish and Eq. (22) takes the simpler form

$$I_u = I_x \cos^2 \alpha_1 + I_y \cos^2 \beta_1 + I_z \cos^2 \gamma_1 \quad (22a)$$

If the body is a lamina and we take the axes x and y in its plane, then for axes u and v also in the plane of the lamina we have $\cos \gamma_1 = 0$ and also the products of inertia I_{yz} and I_{xz} vanish. In such case, Eq. (22) reduces to

$$I_u = I_x \cos^2 \alpha_1 + I_y \cos^2 \beta_1 - 2I_{xy} \cos \alpha_1 \cos \beta_1 \quad (22b)$$

Equation (22b) will be seen to agree with Eq. (9a) on page A.13 if we note that between the two sets of notations used there is the following correspondence (see Fig. 16):

$$\alpha_1 = \varphi \qquad \beta_1 = 90° - \varphi \qquad u = x'$$

Referring again to Fig. 21, the product of inertia of the body with respect to the axes u and v can be calculated from the formula[2]

$$I_{uv} = -I_x \cos \alpha_1 \cos \alpha_2 - I_y \cos \beta_1 \cos \beta_2 - I_z \cos \gamma_1 \cos \gamma_2 \\ + I_{xy}(\cos \alpha_1 \cos \beta_2 + \cos \alpha_2 \cos \beta_1) \\ + I_{xz}(\cos \alpha_1 \cos \gamma_2 + \cos \alpha_2 \cos \gamma_1) \\ + I_{yz}(\cos \beta_1 \cos \gamma_2 + \cos \beta_2 \cos \gamma_1) \quad (23)$$

Similar expressions can be written for the products of inertia I_{vw} and I_{uw}.

If x, y, z are principal axes of the body, the products of inertia I_{xy}, I_{xz}, I_{yz} vanish and Eq. (23) reduces to the simpler form

$$I_{uv} = -I_x \cos \alpha_1 \cos \alpha_2 - I_y \cos \beta_1 \cos \beta_2 - I_z \cos \gamma_1 \cos \gamma_2 \quad (23a)$$

For the case of a lamina coinciding with the xy plane, we have for axes u, v, also in the plane of the lamina,

$$\cos \gamma_1 = \cos \gamma_2 = 0 \qquad \text{and} \qquad I_{xz} = I_{yz} = 0$$

---

[1] The somewhat cumbersome derivation of this formula is omitted. It can be made in a manner analogous to the derivation of Eq. (9a) on page A.13.

[2] The derivation of this formula also is omitted, although it can be made in a manner similar to that used in the derivation of Eq. (12), page A.13, for the case of a plane figure.

Under these conditions, Eq. (23) reduces to

$$I_{uv} = -I_x \cos\alpha_1 \cos\alpha_2 - I_y \cos\beta_1 \cos\beta_2 + I_{xy}(\cos\alpha_1 \cos\beta_2 + \cos\alpha_2 \cos\beta_1) \qquad (23b)$$

Equation (23*b*) will be found to agree with Eq. (12), page A.13, if we note that between our present notations and those used in Fig. 16, there is the following correspondence:

$$\alpha_1 = \varphi \qquad \alpha_2 = 90° + \varphi \qquad \beta_1 = 90° - \varphi \qquad \beta_2 = \varphi$$

Equations (22) and (23) are very helpful in the calculation of moments of inertia and products of inertia of material bodies with respect to various axes for which direct integration would become very difficult. Consider, for example, the solid right circular cylinder of weight W, radius a, and length l, as shown in Fig. 22, and assume that we require the moment of inertia I_u and the product of inertia I_{uv} with respect to inclined axes u and v in an axial plane of symmetry as shown. These quantities, for instance, would be required in discussing rotation of the cylinder about the diagonal axis AB.

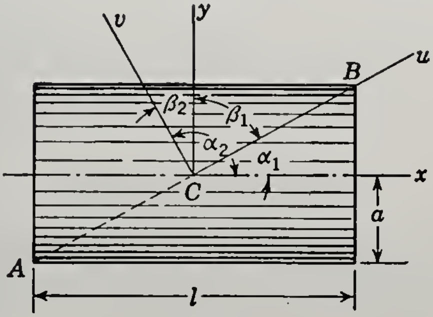


FIG. 22

We begin with a calculation of the moment of inertia I_u. Taking principal axes x, y, z through the center of gravity C of the cylinder, and using Eqs. (g) and (j), we have

$$I_x = \frac{W}{g}\frac{a^2}{2} \qquad I_y = I_z = \frac{W}{g}\left(\frac{a^2}{4} + \frac{l^2}{12}\right) \qquad (q)$$

and since x, y, z are principal axes, $I_{xy} = I_{xz} = I_{yz} = 0$, and we may use Eq. (22*a*). Referring to Fig. 22, we see that

$$\cos\alpha_1 = \frac{l}{\sqrt{l^2 + 4a^2}} \qquad \cos\beta_1 = \frac{2a}{\sqrt{l^2 + 4a^2}} \qquad \cos\gamma_1 = 0 \qquad (r)$$

Substituting the quantities (q) and (r) into Eq. (22*a*), we obtain

$$I_u = \frac{Wa^2(5l^2 + 6a^2)}{6g(l^2 + 4a^2)} \tag{s}$$

To calculate the product of inertia I_{uv} of the cylinder, we use Eq. (23*a*). Then in addition to the quantities (q) and (r) above we have, by reference to Fig. 22,

$$\cos \alpha_2 = -\frac{2a}{\sqrt{l^2 + 4a^2}} \qquad \cos \beta_2 = \frac{l}{\sqrt{l^2 + 4a^2}} \qquad \cos \gamma_2 = 0 \tag{t}$$

Substituting the quantities (q), (r), and (t) into Eq. (23*a*), we obtain

$$I_{uv} = \frac{Wal(3a^2 - l^2)}{6g(4a^2 + l^2)} \tag{u}$$

Attention is called to the fact that this product of inertia vanishes when $l = \sqrt{3}\,a$. This means that for such proportions of the cylinder, u, v, w, as shown in Fig. 22, are also principal axes.

PROBLEM SET A.10

1. Calculate the moment of inertia of a homogeneous rectangular lamina of dimensions a and b with respect to a diagonal of the rectangle. The weight of the lamina is W. *Ans.* $I_d = \frac{1}{6}\frac{W}{g}\frac{a^2b^2}{a^2 + b^2}$.

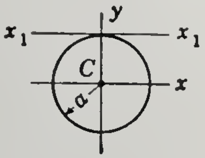
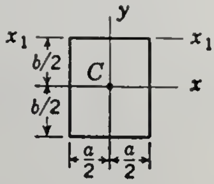
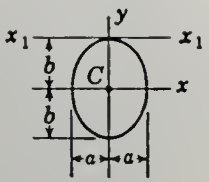
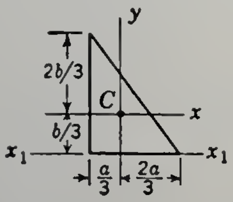
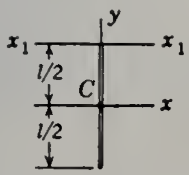
2. Determine the moment of inertia of a homogeneous rectangular parallelepiped of dimensions a, b, c and weight W with respect to a diagonal. *Ans.* $I_d = \frac{1}{6}\frac{W}{g}\frac{a^2b^2 + b^2c^2 + a^2c^2}{a^2 + b^2 + c^2}$.

3. Calculate the moment of inertia of a homogeneous right circular cone of weight W, altitude h, and radius of base a with respect to a generator. *Ans.* $I_g = \frac{3Wa^2}{20g}\frac{a^2 + 6h^2}{a^2 + h^2}$.

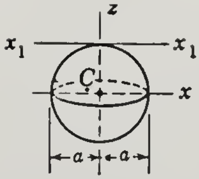
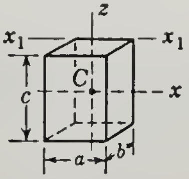
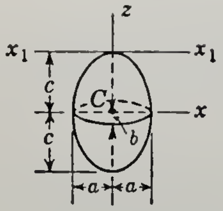
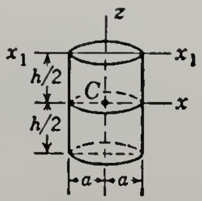
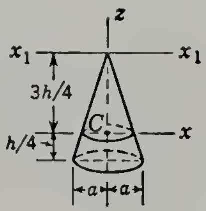
4. Calculate the product of inertia I_{uv} of a homogeneous rectangular lamina of weight W with respect to orthogonal axes u and v through the center of gravity and in the plane of the lamina if the u axis coincides with a diagonal. *Ans.* $I_{uv} = \frac{Wab}{12g}\frac{b^2 - a^2}{b^2 + a^2}$.

5. Calculate the product of inertia I_{uv} of a homogeneous right circular cone of weight W, altitude h, and base radius a with respect to orthogonal axes u, v in an axial plane of symmetry through the vertex of the cone, the u axis coinciding with a generator. *Ans.* $I_{uv} = \frac{3Wah}{20g}\frac{a^2 - 4h^2}{a^2 + h^2}$.

Various Moments of Inertia of Laminas

Lamina	$I_x = \frac{W}{g} i_x^2$	$I_{x1} = \frac{W}{g} i_{x1}^2$	$\bar{I}_z = \frac{W}{g} \bar{i}_z^2$
	$\frac{W}{g}\frac{a^2}{4}$	$\frac{W}{g}\frac{5a^2}{4}$	$\frac{W}{g}\frac{a^2}{2}$
 y x_1 x_1 b/2 C x b/2 $\frac{a}{2}$ $\frac{a}{2}$	$\frac{W}{g}\frac{b^2}{12}$	$\frac{W}{g}\frac{b^2}{3}$	$\frac{W}{g}\frac{a^2+b^2}{12}$
 y x_1 x_1 b C x b a a	$\frac{W}{g}\frac{b^2}{4}$	$\frac{W}{g}\frac{5b^2}{4}$	$\frac{W}{g}\frac{a^2+b^2}{4}$
 y 2b/3 C x b/3 x_1 x_1 $\frac{a}{3}$ $\frac{2a}{3}$	$\frac{W}{g}\frac{b^2}{18}$	$\frac{W}{g}\frac{b^2}{6}$	$\frac{W}{g}\frac{a^2+b^2}{18}$
 y x_1 x_1 l/2 C x l/2	$\frac{W}{g}\frac{l^2}{12}$	$\frac{W}{g}\frac{l^2}{3}$	$\frac{W}{g}\frac{l^2}{12}$

Various Moments of Inertia of Bodies

Body	$I_x = \frac{W}{g} i_x^2$	$I_{x1} = \frac{W}{g} i_{x1}^2$	$\bar{I}_z = \frac{W}{g} \bar{i}_z^2$
	$\frac{W}{g} \frac{2a^2}{5}$	$\frac{W}{g} \frac{7a^2}{5}$	$\frac{W}{g} \frac{2a^2}{5}$
 z x_1 x_1 C x c a b	$\frac{W}{g} \frac{b^2 + c^2}{12}$	$\frac{W}{g} \frac{b^2 + c^2}{3}$	$\frac{W}{g} \frac{a^2 + b^2}{12}$
 z x_1 x_1 c c C x b a a	$\frac{W}{g} \frac{b^2 + c^2}{5}$	$\frac{W}{g} \frac{b^2 + 6c^2}{5}$	$\frac{W}{g} \frac{a^2 + b^2}{5}$
 z x_1 x_1 h/2 h/2 C x a a	$\frac{W}{g} \frac{3a^2 + h^2}{12}$	$\frac{W}{g} \frac{3a^2 + 4h^2}{12}$	$\frac{W}{g} \frac{a^2}{2}$
 z x_1 x_1 3h/4 h/4 C x a a	$\frac{W}{g} \frac{12a^2 + 3h^2}{80}$	$\frac{W}{g} \frac{3a^2 + 12h^2}{20}$	$\frac{W}{g} \frac{3a^2}{10}$

Appendix Three

FORCED VIBRATIONS

A.11. General theory. In Art. 6.6, we considered the *free vibrations* of a spring-suspended mass subjected only to the action of a spring force $-kx$. Such vibrations, as we have seen, occur only if there is some initial disturbance of the mass from its equilibrium position. Let us consider now the case where, in addition to a spring force, we have a force Q varying with time so that the differential equation of motion [Eq. (34)] becomes

$$\frac{W}{g}\ddot{x} = -kx + Q \tag{a}$$

in which the force $Q = F(t)$ is called a *disturbing force.*

Periodic Disturbing Force. In practical cases we very often have a disturbing force that varies with time according to a sine or cosine law. As an example, consider the case of a motor of weight W supported on springs and constrained to move vertically as shown in Fig. 23. If the rotor is perfectly balanced, there will be only the constant gravity force W acting which will produce a deflection $\delta_{st} = W/k$. Thus if the motor is pressed downward from its position of static equilibrium and then suddenly released, free vibrations having the period

$$\tau = 2\pi\sqrt{\frac{\delta_{st}}{g}}$$

will ensue. Suppose now that the rotor has an eccentric mass attached to its rim at A as shown. Such an eccentric rotating mass will produce a centrifugal force acting as shown in the figure. If Q_0 is the magnitude of this force, its vertical component, acting along the line of motion, represents the distributing force in this case and we have[1]

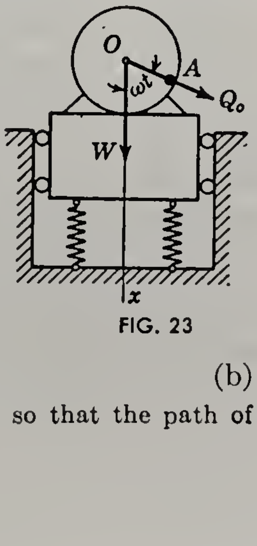


FIG. 23

$$Q = Q_0 \cos \omega t \tag{b}$$

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[1] It is assumed that the amplitude of vibration is small so that the path of point A may be considered to be a circle.

where ω is the uniform angular speed of the rotor in radians per second and time t is measured from the instant when the centrifugal force Q_0 acts vertically downward.

We now replace Q in Eq. (a) by expression (b). Then dividing both sides of the equation by W/g and using the notations

$$\frac{kg}{W} = p^2 \qquad \frac{Q_0 g}{W} = q_0 \tag{c}$$

the latter of which is seen to represent the maximum value of the disturbing force per unit of vibrating mass, we obtain

$$\ddot{x} + p^2 x = q_0 \cos \omega t \tag{24}$$

This is the differential equation of *forced vibrations.*

Periodic Ground Motion. Before proceeding with the solution of Eq. (24), we shall mention another system (Fig. 24), an analysis of which leads to the same differential equation. Consider the weight W suspended from the lower end of a helical spring, the upper end of which is attached to a crosshead A which, by virtue of its connection with the rotating crank OB, performs the simple harmonic motion

$$x_1 = a \cos \omega t \tag{d}$$

where a is the length of the crank OB and ω is the uniform angular speed with which it rotates. This forced motion of the crosshead A is called the *ground motion.* The time t is measured from the instant when the crank is vertically down, and hence the displacement x_1 of the crosshead is given with respect to its middle position. Owing to such motion of the upper end of the spring, the suspended weight W will also move, and we shall measure its displacement x from the position of static equilibrium when the crosshead A is in its middle position. In general x will be different from x_1, and it is evident that the difference $x - x_1$ between these two displacements represents the elongation of the spring over and above that existing for equilibrium conditions. Thus the equation of motion becomes

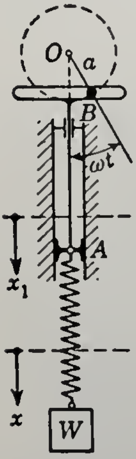

FIG. 24

$$\frac{W}{g}\ddot{x} = W - [W + k(x - x_1)]$$

which, by using expression (d) for x_1, may be written

$$\frac{W}{g}\ddot{x} + kx = ak \cos \omega t \tag{e}$$

Dividing both sides of Eq. (e) by W/g and using the notations

$$\frac{kg}{W} = p^2 \qquad \frac{akg}{W} = q_0 \tag{f}$$

we see that Eq. (e) takes the same form as Eq. (24). The only difference between this and the preceding case is that here the disturbing force is transmitted to the suspended weight through the spring instead of being applied directly to it, as in Fig. 23.

General Solution. We seek now the solution of Eq. (24). An examination of this equation shows that it can be satisfied by taking

$$x = C \cos \omega t \tag{g}$$

where C is a constant. Substituting expression (g) for x in Eq. (24), we obtain

$$-C\omega^2 + Cp^2 = p_0 \tag{h}$$

from which

$$C = \frac{q_0}{p^2 - \omega^2} \tag{i}$$

and the solution (g) becomes

$$x = \frac{q_0}{p^2} \frac{1}{1 - \omega^2/p^2} \cos \omega t \tag{25}$$

This is only a *particular solution* of Eq. (24). To obtain the *general solution* which can be adapted to any initial conditions of motion, we add to this particular solution the general solution for free vibrations, as given by Eq. (41), page 280. Then

$$x = C_1 \cos pt + C_2 \sin pt + \frac{q_0 \cos \omega t}{p^2\,(1 - \omega^2/p^2)} \tag{26}$$

By substitution it can be proved that this expression satisfies Eq. (24), and since it has two constants of integration, we can always adjust these constants so as to satisfy any initial conditions regarding displacement and velocity. The first two terms in Eq. (26) represent *free vibrations* having the period

$$\tau = \frac{2\pi}{p}$$

For a given system this period is definite and independent of the disturbing force or ground motion. The last term represents *forced vibrations,* which are seen to have the period

$$\tau_1 = \frac{2\pi}{\omega}$$

Thus the period of forced vibrations is the same as that of the disturbing force or ground motion which produces them and does not depend at all upon the period of natural vibration of the system.

The motion represented by Eq. (26) is, then, the sum of two harmonic vibrations having, in general, two distinct periods. The result of a summation of two such harmonic functions is illustrated in Fig. 25. Figure 25*a* shows the displacement-time diagram for the forced vibrations as represented by the last term of Eq. (26), while Fig. 25*b* shows that of free vibrations as represented by the first two terms, assuming $C_2 = 0$. Figure 25*c* shows the displacement-time diagram resulting from a summation of these two harmonic motions.

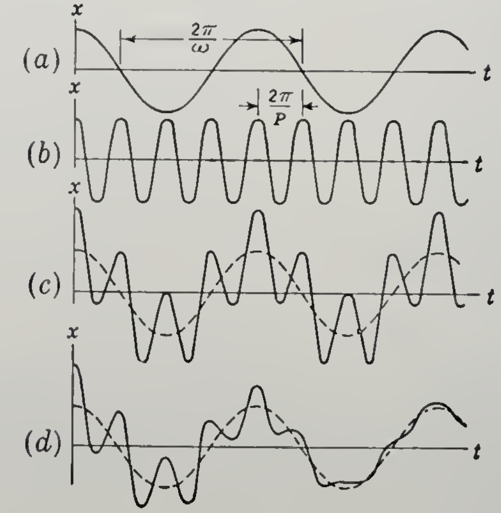

FIG. 25

Owing to the fact that some friction will always be present, even though it has not been taken into account in the development of Eq. (26), the amplitude of free vibrations as represented by the first two terms of the equation will diminish with time and thus these vibrations will gradually disappear while the forced vibrations sustained by the continuously acting disturbing force or ground motion remain and we have the condition represented by Fig. 25*d* in which the irregularity in the curve gradually disappears. That is, the motion soon reaches a state of steady forced vibrations completely defined by Eq. (25). These forced vibrations are maintained indefinitely by the disturbing force or ground motion and therefore are of great practical importance.

Forced Vibration Phenomena. Let us now consider in detail the forced vibrations as represented by Eq. (25). If, for the case of the spring-mounted motor shown in Fig. 23, we replace p^2 and q_0 by their values (c), we obtain for the equation of forced vibration

$$x = \frac{Q_0 \cos \omega t}{k} \frac{1}{1 - \omega^2/p^2} \tag{25a}$$

For the case of the weight suspended from the agitated spring (Fig. 24) Eq. (25) becomes, by using notations (f),

$$x = a \cos \omega t \frac{1}{1 - \omega^2/p^2} \tag{25b}$$

The factor $(Q_0 \cos \omega t)/k$ in Eq. (25*a*) is seen to represent the deflection of the springs that would be produced by the static action of the disturbing force Q_0. It should not be confused with the deflection $\delta_{st} = W/k$ produced by the gravity force W. In Eq. (25*b*) we note that the factor $a \cos \omega t$ represents the motion of the crosshead to which the upper end of the spring is attached i.e., the ground motion. In an analogous manner then this may be considered as the static effect on the suspended weight of the forced motion of the upper end of the spring. Thus, in either case, we conclude that the dynamic effect of the disturbing force or ground motion on the system is represented by the factor

$$\frac{1}{1 - \omega^2/p^2} \tag{j}$$

the absolute value of which is called the *magnification factor.* The magnitude of this factor is seen to depend solely upon the ratio ω/p, that is, upon the ratio between the frequency $f_1 = \omega/2\pi$ of the disturbing force or ground motion and the frequency $f = p/2\pi$ of free vibrations of the system. Hereafter we shall refer to these two frequencies, respectively, as the *impressed frequency* f_1 and the *natural frequency* f.

Since the amplitude of forced vibration is directly proportional to the magnification factor, let us investigate, now, the variation of this factor with the ratio $\omega/p = f_1/f$. This variation has been represented graphically in Fig. 26, by plotting the magnification factor as ordinates and the ratio ω/p as abscissa. When ω is small compared with p, that is, when the impressed frequency is very low compared with the natural frequency, the magnification factor is not greatly different from unity and we conclude that under such conditions the dynamical deflections do not differ much from those that would be produced by the static action of the same disturbing force. This conclusion has important practical applications as we shall see later.

As ω approaches p in value, that is, as the impressed frequency approaches the natural frequency, the magnification factor increases very rapidly until when $\omega = p$ it becomes infinite. This condition is known as *resonance.* Thus we might conclude that at the resonance condition the amplitude of forced vibration could be gradually built up and grow without limit. However, owing to the effect of friction forces (not taken into account), the amplitude at resonance does not become indefinitely large although it may become dangerously large.

As ω is further increased above resonance, the magnification factor begins to decrease, and when ω/p is very large, that is, when the impressed frequency is many times greater than the natural frequency, it approaches zero. Thus we conclude that, for impressed frequencies well above the resonance condition, the weight W tends to stand practically still in space. This conclusion also is of great practical importance.

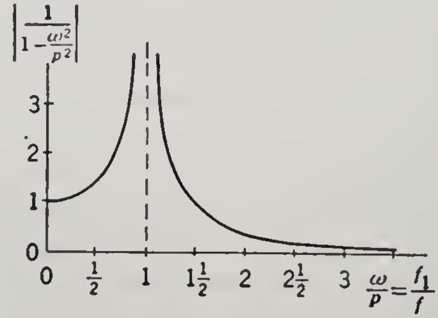

FIG. 26

For all values of $\omega > p$, that is, for all conditions where the impressed frequency is higher than the natural frequency, the factor (j) is seen to be negative. This means that the displacement of the weight W is opposite to the direction of the disturbing force or ground motion, whereas below resonance it is in the same direction. Thus, above resonance, the forced vibration of the weight W is said to be *out of phase* with the disturbance while below resonance, it is said to be *in phase* with the disturbance.

PROBLEM SET A.11

1. The spring-mounted motor shown in Fig. 23 operates at 1,800 rpm, and the eccentric mass sets up at this speed a centrifugal force $Q_0 = W/10$. The weight W of the motor and the spring constant k for the system of springs are such that $\delta_{st} = W/k = 0.003$ in. Calculate the ratio of the amplitude x_m of forced vibration to δ_{st}. *Ans.* $x_m/\delta_{st} = 0.138$.

2. For the system shown in Fig. 24, the following numerical data are given: $a = 1$ in., $\omega = 180\ \text{sec}^{-1}$, $\delta_{st} = W/k = 3$ in. Calculate the amplitude x_m of forced vibration of the suspended weight W. *Ans.* $x_m = 0.004$ in.

A.12. Technical applications. For each of the three rather distinct regions of the magnification-factor diagram in Fig. 26, representing the conditions (1) well below resonance, (2) well above resonance, and (3) at resonance, the phenomena of forced vibrations have many important technical applications, some of which we shall now discuss.

Pressure Indicator. For the condition of forced vibration well below resonance, we note from Fig. 26 that the chief characteristic of the motion is that the amplitude does not differ much from the static deflection produced by the maximum value of the disturbing force. This fact is of importance in the design of various types of *pressure indicators* used for measuring variable forces such as steam pressure in the cylinder of an engine. Such an instrument usually consists of a small piston attached to a spring, as shown in Fig. 27. Thus we have a system, the natural frequency of which depends upon the spring constant k and the piston weight W. The instrument is connected to the cylinder of an engine by the tube B so that the piston is at all times subjected to the same pressure existing in the engine cylinder. Since this pressure is variable, it produces forced vibrations of the piston which are recorded on a rotating cylinder by the pencil A. In order that the displacement of the piston will always be approximately the same as would be produced by the same pressure acting statically, it is necessary that the natural frequency of the indicator system be many times greater than the frequency of fluctuations in the variable pressure that is to be measured. Under such conditions the ratio ω/p will be small and the value of the magnification factor will differ but little from unity. Thus the ordinates of the record can be taken as proportional to the pressure. It is seen that the general requirements for the instrument are a lightweight piston and a stiff spring, as this combination results in a high natural frequency.

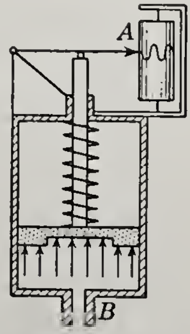

FIG. 27

Vibrograph. Let us consider now the condition of forced vibration well above resonance, where the ratio ω/p is large. This condition, as we have already seen, is characterized by very small amplitudes and a half-cycle phase difference between the disturbance and the forced vibration. These facts are utilized in the design of such instruments as the *vibrograph* and *seismograph* for measuring vibrations. An instrument of this kind is shown in Fig. 28. It consists essentially of a frame from which a heavy weight W is suspended by flexible springs.

A recording dial A is arranged, as shown, to register any relative motion between the suspended weight and the frame. When the frame is fastened to any vertically vibrating body, say the bearing of a large turbine or generator, forced vibrations of the suspended weight W will be produced. If the natural frequency of the instrument is very low compared with the impressed frequency, represented in this case by the rps of the turbine or generator, the suspended weight W will practically stand still in space and hence the dial will show with good accuracy the absolute value of the vertical motion of the bearing to which the frame is bolted. The instrument can easily be adapted to the measurement of horizontal vibrations by using horizontal springs.

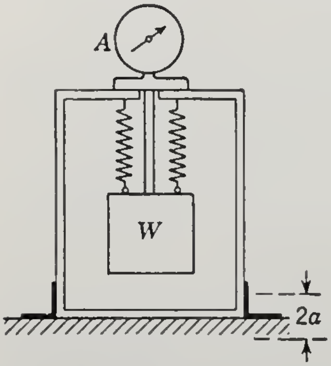

FIG. 28

Spring Mountings. The fact that a high-frequency disturbance practically does not produce vibrations of a system of very low natural frequency is also utilized in the design of various kinds of flexible supports for instruments that we should like to isolate from vibrations produced in the streets by heavy traffic and transmitted to our laboratory buildings. For example, if the natural frequency of vibration of the supporting platform shown in Fig. 29 is very low, the instruments will be practically unaffected even though the building may vibrate considerably. Since only disturbances of frequencies near those of free vibrations of the building will be transmitted to it to any appreciable extent and further since most buildings are of rigid construction and therefore have fairly high natural frequencies of vibration, it is not difficult to arrange such a platform to work satisfactorily.

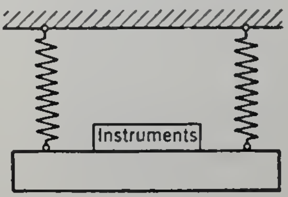

FIG. 29

The same principle is also applied in the use of flexible mountings placed between an engine and its foundation. If a motor bedplate AB, for instance, is rigidly bolted to the foundation (Fig. 30a) and there is some unbalance in the rotor, the corresponding centrifugal force Q_0 will be transmitted to the foundation and will produce undesirable disturbances. Imagine now that the bedplate is mounted on springs in such a way that only vertical motion is possible (Fig. 30b). In this manner we obtain a flexible system, the natural frequency of which can be calculated if we know the spring constant k for the group of springs and the weight W of the motor together with the bedplate.

If the springs are so chosen that the natural frequency of the system is small compared with the impressed frequency, equal to the rps of the motor, the ratio ω/p is large and the amplitude of forced vibration is small compared with the deflections that the centrifugal force Q_0 would produce statically. In the same proportion, the fluctuating forces transmitted through the springs to the foundation will be smaller than the force Q_0. In this way undesirable vibrations of the foundation produced by unbalance in the rotor will be substantially reduced.

Frahm's Tachometer. We come now to a consideration of the condition of resonance. Although our solution in which the effect of friction has been neglected, does not give an exact picture of what happens at resonance, it will be sufficient, for the present, to observe that this condition is characterized chiefly by very large amplitudes of forced vibration. Owing to this fact, the condition of resonance is often a dangerous one because of the corresponding large stresses produced in the spring or springs of the system. However, certain instruments, used for measuring the frequency of vibrations, are designed to operate at or very near resonance.

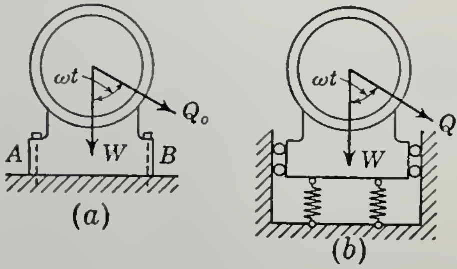

FIG. 30

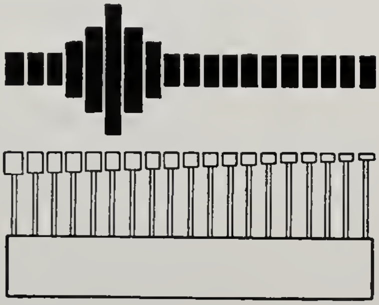

FIG. 31

Figure 31 represents an instrument called Frahm's tachometer, which is used for determining the frequency of any vibrating body. It consists essentially of a number of thin steel strips, each built into a base at one end and carrying a weight at the other. Thus each unit is a small cantilever beam carrying a load at one end and having a definite natural frequency of lateral vibration. The weights are so adjusted that their frequencies represent a series of numbers over some definite range. Now when the base is attached to a vibrating body, each weight undergoes forced vibrations having the frequency of the body to which the base is fastened. If this frequency is within the range of the tachometer, the weight that has most nearly the same natural frequency will vibrate with a larger amplitude than any of the others (see Fig. 31) and thus the frequency is determined.

Vibrator. Another example of a practical application of resonance is the so-called *vibrator* used for finding the natural frequency of vibration of bridges, buildings, and other large structures. This instrument (Fig. 32) consists essentially of two similar rotating masses purposely unbalanced so that, when they are rotated in opposite directions, they set up two equal centrifugal forces Q_0. Further, they are so adjusted that these forces always make the same angle with the vertical, thus causing their horizontal projections to balance each other while their vertical projections add together to give a vertical pulsating force, the frequency of which can be controlled by varying the speed of rotation. Suitably mounted in a rigid frame, the vibrator can be stationed on the bridge and gradually speeded up until the resonance condition is reached. This speed will easily be recognized owing to the large amplitude of forced vibration of the bridge. It is only necessary then to note the rps of the vibrator to have the natural frequency of the bridge.

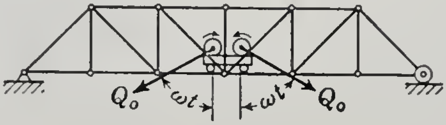

FIG. 32

The same instrument has also been utilized for testing large structural models as well as some obsolete structures. By means of the vibrator operating at the resonance speed, the structure is brought to complete destruction after which the types of structural failure are carefully studied. The very fact that it is possible by means of a comparatively small vibrator to bring a large structure to complete destruction emphasizes the great danger involved in operating any machine at or near a resonance condition.

PROBLEM SET A.12

1. A steam-pressure indicator like the one illustrated Fig. 27 uses a spring for which $k = 100$ lb/in. Determine the correct weight W for the piston if the maximum frequency of fluctuating pressure to be measured is 600 cycles/min and a limit of 2 per cent error is imposed. *Ans.* $W = 0.192$ lb.

2. Determine the spring constant k for the system of springs used for the mounting of the motor shown in Fig. 30*b* so that the variable force transmitted to the foundation will be only one-tenth as much as when the motor is mounted as shown in Fig. 30*a*. The motor operates at 1,800 rpm, and has the weight $W = 200$ lb. *Ans.* $k = 1,680$ lb/in.

Appendix Four

BALANCING OF RIGID ROTORS

A.13. Reactions on a fixed axis. Let us consider a body of any shape which is free to rotate about a fixed axis AB (Fig. 33) supported in bearings at A and B. If there is no friction in the bearings and only the gravity force is acting on the body, it will take the position of stable equilibrium for which its center of gravity has the lowest position. The pressures transmitted to the bearings for this position of equilibrium can be determined by using equations of statics. By applying a torque to the axle, the body can be turned and put in any other position provided the moment of the gravity force is balanced by the applied torque. It is evident that the applied torque does not change the pressures on the bearings and they remain the same as for the case of stable equilibrium mentioned above.

Assume now that the body rotates with respect to the axis AB with constant angular velocity ω, which requires that the moment of all forces acting on the body with respect to this axis be zero. Owing to this rotation, additional pressures on the bearings will usually be produced. In calculating these additional pressures, we shall use D'Alembert's principle and apply to each element of the rotating body its inertia force. Then, during rotation, the pressures on the bearings produced by these inertia forces can be calculated by using equations of statics. In writing these equations, we shall use a system of rectangular coordinate axes x, y, z, the origin of which coincides with the center of the bearing A and the z axis of which coincides with the axis of rotation and shall assume that during rotation the x and y axes rotate together with the body. Taking at any point, distance r from the axis of rotation, a particle of mass dm and observing that the body is rotating uniformly, we conclude that the element has only radial acceleration $\omega^2 r$ and that the inertia force acting on it is $\omega^2 r\, dm$, directed as shown in the figure. The projections of this inertia force on the x and y axes are, respectively,

$$\omega^2 x\, dm \qquad \text{and} \qquad \omega^2 y\, dm \tag{a}$$

and its moments with respect to the same axes are, respectively,

$$-\omega^2 yz\, dm \qquad \text{and} \qquad \omega^2 xz\, dm \tag{b}$$

where the signs of moments are determined in accordance with the right-hand rule. The unknown reactions at the bearings can be resolved into components X_a, Y_a, and X_b, Y_b, as shown in the figure.

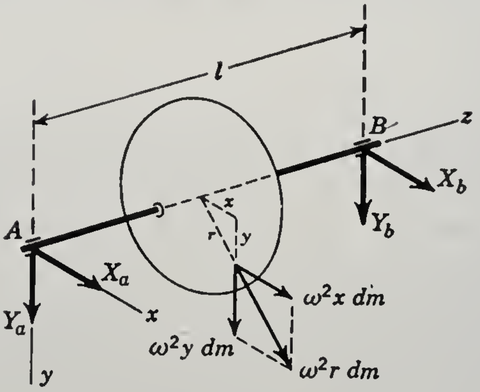

FIG. 33

For determining these components of the reactions, we write four equations of statics, by equating to zero the sums of the projections of all forces on the x and y axes and likewise the sums of moments of all forces with respect to the same axes. Thus, by using expressions (a) and (b), we obtain

$$\begin{aligned} X_a + X_b + \omega^2 \textstyle\int x\, dm &= 0 \\ Y_a + Y_b + \omega^2 \textstyle\int y\, dm &= 0 \\ -Y_b\, l - \omega^2 \textstyle\int yz\, dm &= 0 \\ X_b\, l + \omega^2 \textstyle\int xz\, dm &= 0 \end{aligned} \tag{c}$$

Introducing the notations[1]

$$\int x\, dm = \frac{W}{g} x_c, \quad \int y\, dm = \frac{W}{g} y_c, \quad \int yz\, dm = I_{yz}, \quad \int xz\, dm = I_{xz}$$

where W/g is the total mass of the body and x_c, y_c, the coordinates of its center of gravity, this system of equations can be put in the form

---

[1] The first two integrals are simply the *statical moments* of the body with respect to the yz and xz planes, respectively, and are well known from statics. The second two integrals are called *products of inertia* of the body with respect to the y, z and x, z axes, respectively. They are dimensionally similar to *moments of inertia* and are fully discussed in Appendix II, p. A.23.

$$
\begin{aligned}
X_a + X_b &= -\omega^2 \frac{W}{g} x_c \\
Y_a + Y_b &= -\omega^2 \frac{W}{g} y_c \\
Y_b &= -\frac{\omega^2}{l} I_{yz} \\
X_b &= -\frac{\omega^2}{l} I_{xz}
\end{aligned}
\tag{27}
$$

From these equations the four components of the reactions can be calculated provided the coordinates of the center of gravity and the products of inertia of the body with respect to the x, z and y, z axes are known.

In the particular case where the center of gravity of the body is on the axis of rotation, $x_c = y_c = 0$ and from the first two of Eqs. (26) we conclude that

$$X_a = -X_b \qquad \text{and} \qquad Y_a = -Y_b$$

Thus the components X_a, X_b and Y_a, Y_b, of the bearing reactions represent a pair of couples in the xz and yz planes, respectively, and we conclude that there is a resultant couple in some intermediate axial plane of the rotating body. During rotation of the body, the plane of this resultant couple rotates also and the bearings are submitted to the action of uniformly rotating forces. Such rotating forces, as we have seen in Appendix III, can produce forced vibrations of the bearing pedestals and are, generally speaking, very undesirable. To eliminate them it is necessary, as is seen from the last two of Eqs. (26), to make the products of inertia of the body with respect to the y, z and x, z axes zero. Then the reactions at the bearings due to inertia forces completely vanish. This means that the axis of rotation must go through the center of gravity of the body and must coincide with one of its principal axes of inertia. In short, a body which rotates about a principal central axis will not produce fluctuating pressures on the bearings.

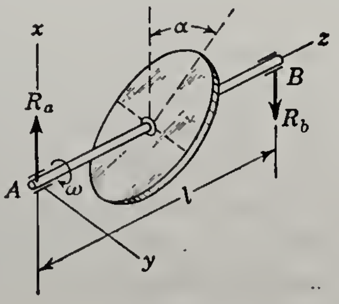

FIG. 34

As an example of calculating bearing reactions for a rigid body rotating about a fixed axis AB which is not a principal axis, let us consider the case shown in Fig. 34. Here we have a homogeneous solid right circular disk of weight W and radius r obliquely mounted on a shaft so that the middle plane of the disk makes an angle α with a plane normal to the axis of the shaft. Taking coordiate axes x, y, z, as shown, we see that $x_c = y_c = 0$ and also, since xz is a plane of symmetry, we have $I_{yz} = 0$. Then for this case, Eqs. (26) reduce to

$$R_a = -R_b = \frac{\omega^2}{l} I_{xz} \tag{d}$$

where ω is the angular velocity around the z axis and I_{xz} is the product of inertia of the disk with respect to the x and z axes. To calculate I_{xz} we first use the parallel-axis theorem for product of inertia (see page A.24) and write

$$I_{xz} = \bar{I}_{xz} + \frac{W}{g} x_c z_c = \bar{I}_{xz}$$

since $x_c = 0$. Then using Eq. (23*a*), page A.26, we find

$$I_{xz} = \bar{I}_{xz} = \frac{W}{g}\frac{r^2}{8} \sin 2\alpha \tag{e}$$

Substituting this value into Eq. (*d*), we obtain for the reactions

$$R_a = -R_b = \frac{W}{g}\frac{r^2}{8}\frac{\omega^2}{l} \sin 2\alpha \tag{f}$$

Thus R_a is in the positive direction of the x axis while R_b is in the negative direction and together they represent a couple as shown in Fig. 34. During rotation, the xz plane containing this couple rotates with the disk and the bearings themselves are subjected to rotating forces equal but opposite to the forces shown. We see from expression (*f*) that the dynamic bearing reactions have their maximum value when $\alpha = 45°$ and that they vanish when $\alpha = 0$, that is, when the z axis is a principal central axis of the disk.

PROBLEM SET A.13

1. A homogeneous thin rectangular plate having dimensions a and b and weight W rotates in fixed bearings with uniform angular velocity ω about a diagonal AB as shown in Fig. A. Calculate the dynamical bearing reactions R_a and R_b as shown.

Hint. See page A.13 for product of inertia of the plate with respect to non-principal axes. *Ans.* $R_a = R_b = \dfrac{W\omega^2 ab(a^2 - b^2)}{12g(a^2 + b^2)^{3/2}}$.

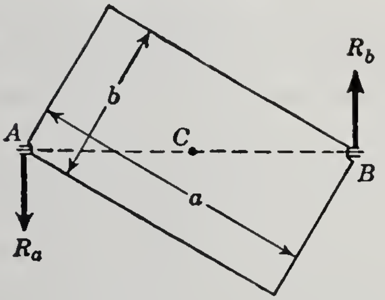

FIG. A

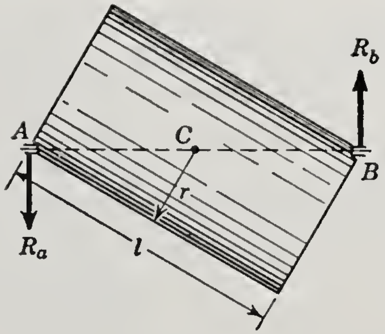

FIG. B

2. A homogeneous right circular cylinder of length l, radius r, and weight W rotates in fixed bearings with uniform angular velocity ω about a diagonal AB as shown in Fig. B. Calculate the dynamical bearing reactions R_a and R_b as shown.

Hint. See page A.28 for product of inertia of the cylinder with respect to nonprincipal axes. *Ans.* $R_a = R_b = \dfrac{W\omega^2 rl(l^2 - 3r^2)}{6g(l^2 + 4r^2)^{3/2}}$.

A.14. Balancing of rigid rotating bodies. It was shown in the preceding article that due to uniform rotation of a rigid body about a fixed axis, there will generally be some dynamical pressures exerted by the axle on the bearings and that these pressures rotate with the body. Assuming the axis of rotation to be horizontal and resolving the rotating force on each bearing into a horizontal and a vertical component, we find that each of these components represents a periodically varying force with a period equal to the time of one revolution of the body. Such forces, as we have seen before, are likely to produce forced vibrations of the foundation or supporting structure. These forced vibrations may become very severe if the frequency of the disturbing force is the same as the natural frequency of vibration of the supporting structure for, in such case, a condition of resonance exists.

To eliminate such periodical disturbing forces in the case of rotating machine parts, the body is usually so designed that its axis of rotation represents a principal central axis. In such case, as we have already seen, uniform rotation does not produce dynamical pressures and the reactions at the bearings are constant and the same as under static conditions. However, it is not always possible, for practical reasons, to have the axis of rotation of a body or system of bodies coincide with a principal central axis, and in such cases pulsating forces will act on the bearings during rotation. To eliminate these forces, it is necessary to *balance* the rotating body or system, i.e., to make the axis of rotation a principal central axis of inertia by adding some appropriate mass or masses to the system. In the following discussion we shall consider some typical cases of unbalance and discuss methods of balancing.

Static Unbalance. As a first example take the case where several masses (1, 2, 3, . . .) in the same plane are rigidly connected to an axle perpendicular to this plane and are uniformly rotating about this axle (Fig. 35). In this case the inertia forces represent a system of forces intersecting in one point O on the axis of rotation. This rotating system will be perfectly balanced if the free vectors representing the several inertia forces build a closed polygon of forces, for in such case the resultant inertia force is zero. If the polygon of inertia forces is not closed, the closing side represents the resultant inertia force which produces pressure on the bearings. Knowing the magnitude and direction of this resultant, the magnitude and location of a weight which will bring the system into perfect balance can be calculated easily.

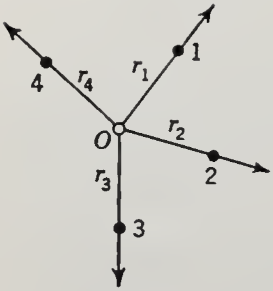

FIG. 35

In the case of rotation of a rigid system about a principal axis, the resultant inertia force can be obtained by assuming the entire mass of the system to be concentrated at the center of gravity and considering the acceleration of this point. From this, it follows that the rotating system shown in Fig. 35 will be perfectly balanced if the center of gravity of the masses 1, 2, 3, . . . is on the axis of rotation, since in such a case acceleration of the center of gravity is zero, and consequently the resultant inertia force is zero. If the center of gravity of the system does not coincide with the axis of rotation, the system is unbalanced. Such unbalance is seen to represent the case where the axis of rotation is a principal axis but not a central axis of the system. This kind of unbalance is called *static unbalance,* since it can be detected by putting the ends of the axle on horizontal rails. Under the action of the gravity forces, the axle will rotate until the center of gravity of the system occupies the lowest position. As soon as the radius on which the center of gravity lies is determined by this test, we can bring the center of gravity of the system to the axis of rotation by adding a proper *correction weight* on the opposite radius. In this way the static unbalance is removed.

In the case of thin rotating disks and flywheels, it can be assumed with sufficient accuracy that all mass is distributed in the middle plane of the disk or flywheel perpendicular to the axis of rotation.

Then static balancing, as described above, is entirely satisfactory unless the plane of the disk or flywheel happens not to be exactly perpendicular to the axis of its shaft (see Fig. 34).

Dynamic Unbalance. In the case where several eccentric masses are distributed along the length of the shaft so that the inertia forces set up owing to rotation are not all in one plane, the problem of balancing becomes more complicated since static balancing is usually insufficient to eliminate dynamic pressures on the bearings.

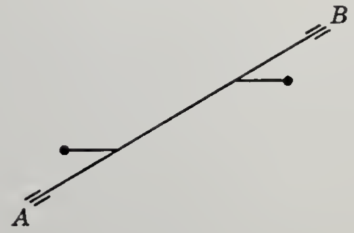

FIG. 36

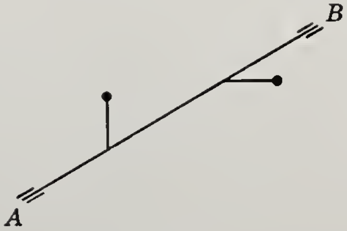

FIG. 37

Let us consider first the case where we have two equal masses in the same axial plane and at equal distances from the axis of rotation but on opposite sides (Fig. 36). In such a case the center of gravity of the system is on the axis of rotation and a static test would show the system to be in balance. However, dynamic pressures will be produced on the bearings as soon as the system is put in rotation and these pressures form a couple rotating with the body. We see that this condition corresponds to the case where the axis of rotation of the system is a central axis but not a principal axis of inertia. Because such unbalance can be detected only when the system is in rotation, it is called *dynamic unbalance.*

General Case. In a more general case we may have two masses in two perpendicular axial planes, and at different distances from the axis of rotation, as shown in Fig. 37. In this case the axis of rotation is neither a principal axis nor a central axis for the system and we have a combination of static and dynamic unbalance.

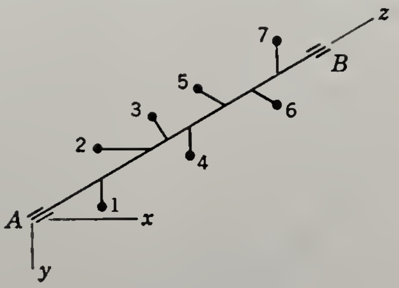

FIG. 38

Considering finally the completely general case of several eccentric masses 1, 2, 3, . . . rigidly attached to, and rotating about, an axis AB with uniform angular velocity (Fig. 38), we see that during rotation we obtain a system of radially directed inertia forces which are scattered along the axis of the shaft. Resolving each of these forces into two components parallel to the x and y axes[1] and denoting by P the resultant of the forces in the xz plane and by Q the resultant of the forces in the yz plane, we find that the resultant action of such a system of inertia forces can be represented by the action of two perpendicular nonintersecting forces P and Q and the system is equivalent to that represented in Fig. 37.

We shall show now that the action of these resultant inertia forces P and Q can be completely balanced, and the dynamic pressures on the bearings during rotation eliminated, by putting two correction weights in two planes 1 and 2, perpendicular to the axis of rotation and intersecting this axis at arbitrarily chosen points m and n in Fig. 39. These planes are called *correction planes.* Considering the force P in the xz plane, we see that it can be completely balanced by forces P_1 and P_2 acting along the lines of intersection of the correction planes 1 and 2 with the xz plane. In calculating the magnitudes of these forces, we use the two equations of statics:

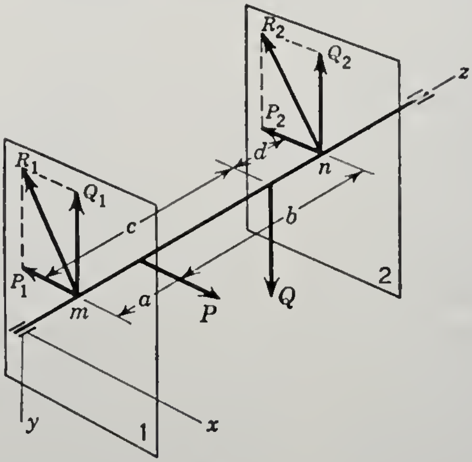

FIG. 39

$$\begin{aligned} P_1 + P_2 &= P \\ P_1a &= P_2b \end{aligned} \tag{k}$$

In the same manner the force Q can be balanced by forces Q_1 and Q_2 of such magnitudes that

$$\begin{aligned} Q_1 + Q_2 &= Q \\ Q_1c &= Q_2d \end{aligned} \tag{l}$$

Considering the resultant R_1 of the forces P_1 and Q_1 in plane 1, we can produce this force by putting a correction weight W_1 on the radius defined by the direction of R_1 and at such a distance r_1 from the axis of rotation that

---

[1] These axes are assumed to rotate with the system.

$$\frac{W_1}{g}\,\omega^2 r_1 = R_1 \tag{m}$$

In the same manner the resultant R_2 in plane 2 can be produced by a weight W_2 at such a distance r_2 from the z-axis that

$$\frac{W_2}{g}\,\omega^2 r_2 = R_2 \tag{n}$$

Thus the two weights W_1 and W_2, put, as described, in two arbitrarily chosen planes 1 and 2 perpendicular to the axis of rotation, completely balance the given system of bodies rigidly connected to this axis.

If the inertia forces in one of the planes, say the xz plane, are in equilibrium, we obtain $P = 0$ and have to balance only the force Q. If the forces P and Q represent a couple, this couple can be balanced by two correction weights in planes 1 and 2, such that their inertia forces represent an equal and opposite couple. We see that in all cases where the unbalance is known it is not difficult to eliminate it by introducing two properly chosen correction weights.

In the case of rotating bodies such as rotors, flywheels, etc., which are designed to rotate about a principal central axis, there should be no problem of balancing. However, due to nonhomogeneity of the material and, owing to small inaccuracies in machining, there is always likely to be some small deviation of the axis of rotation from the position of a principal central axis of inertia and owing to this fact some pulsating forces act on the bearings during rotation. To eliminate these forces completely, it is necessary to balance such rotors by adding proper correction weights after the body has been cast and machined. Consider, for example, the rotor shown in Fig. 40. If we imagine this rotor to be divided into a number of thin disks, as shown, the center of gravity of each disk, owing to slight imperfections, will not be exactly on the axis of rotation and hence we obtain, during rotation, a system of radial inertia forces analogous to the case represented in Fig. 38. It follows then that such unbalance can be completely equilibrated by two properly placed correction weights, as discussed above. In the case of large rotors, as in electric machinery, the ends of the rotors usually have special holes along the circumference in which correction weights can conveniently be placed. In such a case

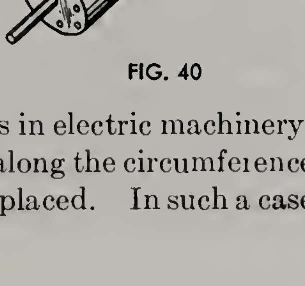
FIG. 40

these end planes of the rotor would be chosen as the correction planes. In dealing with unbalance due to imperfections, however, we cannot calculate the proper correction weights since the unbalance is of an unknown nature, so it is necessary to accomplish balancing by a method of trial and error. This is usually done with the aid of special *balancing machines.*

Balancing Machines. Figure 41 shows a horizontal bed supported by a fulcrum C and a spring S allowing small rotational oscillations about the axis through C normal to the plane of the figure. Such a system has a definite natural frequency of vibration, as discussed in Art. 8.6. The bed carries two adjustable bearing pedestals A and B in which the rotor to be balanced can be mounted and connected to a variable-speed drive D as shown. The rotor is placed in these bearings with one correction plane directly over the fulcrum, and the drive speed is adjusted until the unbalance in the rotor comes into resonance with the natural frequency of the system. This condition will be recognized by the rather large amplitude of vibration of the bed. A correction weight W_2 is now placed in correction plane 2 and adjusted in both amount and angular position[1] until the rotor runs without producing any vibrations of the bed. This indicates that the proper correction in plane 2 has been made. That is, the vertical projection of the rotating inertia force R_2 set up by the chosen correction weight is exactly counteracting the effect of the system of inertia forces set up by the rotor proper. This done, the rotor is turned end for end and the same procedure repeated until the proper correction weight W_1 is found. The fact that the previously placed correction weight W_2 is now exerting its inertia force R_2 in the plane containing the fulcrum prevents it from having any influence on the selection of the weight W_1.

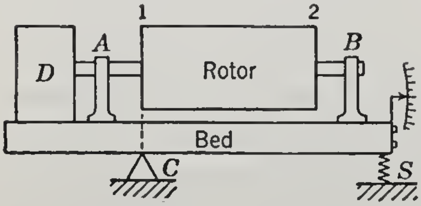

FIG. 41

In the foregoing discussion it was always assumed that the body rotates with a constant angular velocity ω. It can be shown, however, that, if a rotor is perfectly balanced at constant speed, there will be no dynamic pressures on the bearings if the speed is nonuniform. Considering an element dm of a rotating body, the radial and tangential components of the inertia force in the case of nonuniform rotation are $\omega^2 r\,dm$ and $\dot{\omega} r\,dm$, respectively. Since ω and $\dot{\omega}$ are the same for all elements of the rotating body, it can be concluded that the system of inertia forces due to any angular acceleration $\dot{\omega}$ is obtained from the system of inertia forces due to angular velocity ω simply by multiplying the latter system of forces by a constant factor $\dot{\omega}/\omega^2$ and by rotating this system through 90° around the axis of rotation. Now, owing to the balancing described above, the system of forces $\omega^2 r\,dm$ is in equilibrium; hence, the system of forces $\dot{\omega} r\,dm$ is also in equilibrium and does not produce pressures on the bearings.

---

[1] This can be done by trial and error, although most balancing machines have more or less elaborate devices to aid the operator in a more rapid selection of the proper weight and its angular position.

PROBLEM SET A.14

1. A homogeneous circular steel disk rotates about its geometric axis and has two holes drilled through it at A and B, as shown in Fig. A. Determine the diameter d and the angular position φ of a hole which should be drilled at C in order to balance the disk. *Ans.* $d = 5\frac{1}{2}$ in.; $\varphi = 118°$.

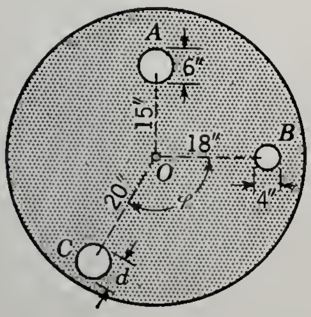

FIG. A

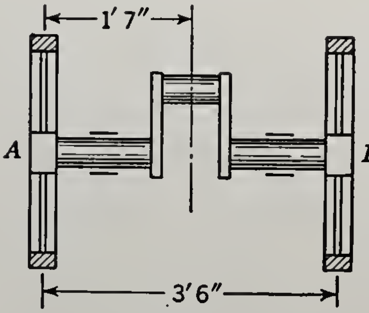

FIG. B

2. A wheel weighing 2,100 lb has its center of gravity $\frac{4}{10}$ in. from its geometric axis. The wheel is mounted on a shaft which rotates in two bearings A and B, 5 ft apart and on opposite sides of the wheel, bearing A being 2 ft from the plane of revolution of the wheel. Find the bearing reactions due to the centrifugal force of the unbalanced wheel when the latter has an angular speed $n = 200$ rpm. What correction weight W placed at a radius of 3 ft 6 in. in the plane of revolution of the wheel will balance it? *Ans.* $R_a = 573$ lb; $R_b = 382$ lb; $W = 20$ lb.

3. The crankshaft of a gas engine carries two flywheels A and B, as shown in Fig. B. The crank arms and crankpin are equivalent to a weight of 108 lb at a radius of 10 in. in the plane of revolution of the crank. What correction weights W_a and W_b placed at a radius of 2 ft, one on each flywheel, will balance the system? *Ans.* $W_a = 24.64$ lb; $W_b = 20.36$ lb.

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